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ON UNIQUE FACTORIZATION MODULES

Murat Alan[1 §], Elif Özbulur[2]
[1,2]Department of Mathematics
Faculty of Arts and Sciences
Yildiz Technical University
Davutpasa Campus, 34210, Esenler, Istanbul, TURKEY

Abstract: Let R be a commutative ring with identity and M be, not necessarily torsion-free, R-module. Unique factorization module (UFM) is introduced via U-decomposition and it is shown that M is a cyclic R-module is necessary but not sufficient condition for M to be a UFM.

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Let D be an integral domain. It is well known that D is a Unique Factorizatition Domain (UFD) if and only if every nonzero non unit of D is a product of irreducibles, and this factorization into irreducibles is unique up to order and associates. There are several generalizations of this notion UFD to unitary modules over commutative rings [2, 3, 6, 7, 8]. In [7] the author defined various type *unique factorization* modules for torsion-free modules over integral domains. Results in [7] are extended in [6]. In [1] the authors presented a detailed study of factorization in commutative rings with zero divisors, and

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§Correspondence author

in [2] among other things, they generalized these results to modules. In [8] the authors continued to study on factorization in modules and gave various generalizations of the notion UFD to modules assuming considered modules as *présimplifiable* module [8, Definition 2.5]. In here we give a new type of unique factorization modules over commutative rings by using U-decomposition and we do not impose an extra condition such as torsion-free or *présimplifiable* on considered unitary modules.

Let R be a commutative ring with identity. Any elements $a, b \in R$ are *associate*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, or equivalently $(a) = (b)$. A nonunit $a \in R$ is an *irreducible* if $a = bc \Rightarrow a \sim b$ or $a \sim c$ and 0 is irreducible $\Leftrightarrow R$ is an integral domain. R is called *atomic* if each nonzero, nonunit is a finite product of irreducible elements. R is said to be *présimplifiable* if $x = xy \Rightarrow x = 0$ or $y \in U(R)$. Let M be a unitary $R-$module and $m, n \in M$. m and n are called *associates*, denoted $m \sim n$, $\Leftrightarrow (m) = (n)$, as cyclic submodules of M. M is *présimplifiable* if for every $r \in R$, $m \in M$, $m = rm \Rightarrow m = 0$ or $r \in U(R)$. For more details on this basic definitions, we refer to [1, 2]

Throughout this paper R denotes a commutative ring with unity and M denotes unitary R-modules.

Definition 1. Let R be a commutative ring and M be an $R-$module and $m \in M$. Then we say that m is irreducible in M if for $0 \neq r \in R$ and $n \in M$, $m = rn$ implies $m \sim n$.

If $R = M$ then $0 \neq m \in M$ is an irreducible in M if and only if $m \in U(R)$ since $m = m1$. So for the case $M = R$ irreducible elements of M and units of R are coincide with each other.

In Definition 1 we allow $0 \in M$ to be an irreducible in M and $0 \in M$ is an irreducible in $M \Longleftrightarrow M$ is a torsion-free $R-$module.

Proposition 2. *Let R be a commutative ring and M be an $R-$module. $0 \in M$ is irreducible in M if and only if M is a torsion-free $R-$module.*

Proof. Suppose that 0 is irreducible in M and $0 = rn$ for some $r \in R$, $n \in M$. If $r = 0$ is the only value of $r \in R$ then M is torsion-free module. If $r \neq 0$ then $0 \sim n$ in M. So $n = 0$ and hence M is a torsion-free module. Now assume that M is a torsion-free module and $0 = rn$ for some $0 \neq r \in R$ and $n \in M$. Then $n = 0$ and hence $0 \sim n$. So 0 is irreducible in M. □

Proposition 3. *Let M be a torsion-free module and $m \in M$ be a nonzero element. Then following two conditions are equivalent.*

i) m is an irreducible in M.

ii) $m = rn$ for some $r \in R$ and $n \in M$ implies $r \in U(R)$.

Proof. (i)$\Rightarrow$ (ii): Suppose that $m = rn$. $r \neq 0$, since $m \neq 0$. So $m \sim n \Rightarrow n = sm$ for some $s \in R \Rightarrow (1-rs)m = 0 \Rightarrow 1-rs = 0 \Rightarrow r \in U(R)$. (ii) $\Rightarrow$ (i): $m = rn \Rightarrow r \in U(R) \Rightarrow n = r^{-1}m \Rightarrow m \sim n$. □

So by Proposition 3, the Definition 1 agrees with the Definition 1.1 in [6] for nonzero elements in torsion-free modules.

In [4] Fletcher gave a generalization of UFD for commutative rings with unity by introducing U-decomposition. A U-decomposition of $r \in R$ is a factorization $r = (q_1 \dots q_k)(p_1 \dots p_n)$ such that (i) the q_i's and p_j's are irreducible (ii) $q_i(p_1 \dots p_n)=(p_1 \dots p_n)$ for $i = 1, \dots, k$ and (iii) $p_i(p_1 \dots \hat{p}_i \dots p_n) \neq (p_1 \dots \hat{p}_i \dots p_n)$ for $i = 1, \dots, n$, where $\hat{p}_i$ means the omission of this element from the product. The product $p_1 \dots p_n$ is called the *relevant part* and the other is the *irrelevant part* in this U-decomposition. Two U-decompositions $r = (q_1 \dots q_k)(p_1 \dots p_n)=(q_1' \dots q_{k'}')(r_1 \cdots r_s)$ are *associates* if (i) $n = s$ and (ii) after a suitable change in the order of the factors in the relevant parts, we have p_i and r_i are associates for $i = 1, \dots, n$. R is called *unique factorization ring* (UFR) if each nonunit has a U-decomposition, (or equivalently R is atomic) and any two U-decompositions of a nonunit element of R are associates. In [5] Fletcher shows that R is a UFR if and only if R is a finite direct product of UFD's and special principal ideal rings.

Definition 4. A *U-decomposition* of $n \in M$ is a factorization $n = (q_1 \dots q_k)(p_1 \dots p_r m)$ such that

(i) $q_i \neq 0$, $p_j \neq 0$ and q_i's and p_j's are irreducible in R and $m \in M$ is an irreducible in M for all i and j,

(ii) $q_i(p_1 \dots p_r m) = (p_1 \dots p_r m)$ as cyclic submodule of M for $i = 1, \dots, k$

(iii) $p_i(p_1 \dots \hat{p}_i \dots p_r m) \neq (p_1 \dots \hat{p}_i \dots p_r m)$ for $i = 1, \dots, r$, where $\hat{p}_i$ means the omission of this element from the product. As in the rings, the product $p_1 \dots p_r m$ is called the *relevant part* and the other is the *irrelevant part.*

Let $m \in M$. We say that m has an irreducible factorization if m is of the form $m = p_1 \dots p_r m_1$ where each p_i is irreducible in R and m_1 is irreducible in M. This factorization into irreducibles is convertible to a U-decomposition as an analogous result of [4, Proposition 2].

Proposition 5. *If $m \in M$ has an irreducible factorization then m has a U-decomposition.*

Proof. Suppose $m = p_1 \dots p_k n$, where each p_i is irreducible in R and n is irreducible in M. If $m = ()(p_1 \dots p_k n)$ then m has a U-decomposition. If

$$p_i(p_1 \dots \hat{p}_i \dots p_k n) = (p_1 \dots \hat{p}_i \dots p_k n)$$

for some p_i, say p_1, then we have

$$p_1(p_2 \dots p_k n) = (p_2 \dots p_k n).$$

Now without lost of generality choose p_2 say, where $p_2(p_3 \dots p_k n)$=$(p_3 \dots p_k n)$ and perform this operation repeatedly, choosing $p_1, \dots, p_t$ until no further p_i has this property. We claim that $m = (p_1 \dots p_t)(p_{t+1} \dots p_k n)$ is a U-decomposition of $m \in M$. From the construction

$$p_j(p_{t+1} \dots \hat{p}_j \dots p_k n) \neq (p_{t+1} \dots \hat{p}_j \dots p_k n)$$

for all $j = t+1, \dots, k$ and so condition (iii) of the Definition 4 is satisfied. Condition (ii) is a direct result of the general fact: If $a(n) = (n)$ and $b(n) = (n)$ then $ab(n) = (n)$ and if $a(bn) = (bn)$ and $b(n) = (n)$ then $a(n) = (n)$ for $a, b \in R$, $n \in M$ □

Proposition 6. *Let $n = (q_1 \dots q_k)(p_1 \dots p_r m)$ be a U- decomposition of $n \in M$. Then $(n) = (p_1 \dots p_r m)$, as cyclic submodule of M.*

Proof. From part (ii) of the Definition 4, we have that

$$(n) = (q_1 \dots q_k p_1 \dots p_r m) = q_1 \dots q_k (p_1 \dots p_r m) = (p_1 \dots p_r m).$$ □

We call two U-decompositions

$$n = (q_1 \dots q_k)(p_1 \dots p_r m_1) = (q_1^{'} \dots q_{k'}^{'})(r_1 \cdots r_s m_2)$$

are *associates* if (i) $r = s$ and (ii) after a suitable change in the order of the factors in the relevant parts, we have $p_i \sim r_i$ in R for $i = 1, \dots, r$ and $m_1 \sim m_2$ in M.

Definition 7. Let R be a commutative ring and M be an $R-$module. M is a *Unique Factorization Module* (UFM) if and only if the following two conditions are satisfied :

i) Every element of M has a U-decomposition, that is M is atomic,

ii) Any two U-decompositions of every $n \in M$ are associates.

Note that if M is a torsion-free R-module then irrelevant part of any U-decomposition of a nonzero element of M is empty. So U-decomposition is identical to standart irreducible factorization for nonzero elements. Since 0 is irreducible when M is torsion-free, 0 has always unique (up to associate) U-decomposition as $0 = ()(0)$. Therefore our definition of UFM is the same as in [6] for torsion-free modules. Since UFM is detaily studied in [6] for torsion-free modules, from now on we are interested in non torsion-free modules.

Proposition 8. *Let R be a commutative ring and M be a non torsion-free $R-$module. Suppose that M is a UFM and $0 = (q_1 \dots q_l)(p_1 \dots p_r m_1)$ is the unique U-decomposition of $0 \in M$. Then m_1 is the only irreducible element of M up to associates.*

Proof. Let $m_2 \in M$ be another irreducible element of M. If $p_1 \dots p_r m_2 = 0$ then $m_1 \sim m_2$ in M. Assume that $p_1 \dots p_r m_2 \neq 0$. Now $n = p_1 \dots p_r m_2 = p_1 \dots p_r m_2 + p_1 \dots p_r m_1 = p_1 \dots p_r(m_1 + m_2)$. Since M is a UFM, the element $m_1 + m_2 \in M$ has an irreducible factorization in M, say $m_1 + m_2 = t_1 \dots t_k m_3$. So taking the U-decomposition of the both sides of two irreducible factorization $p_1 \dots p_r m_2 = p_1 \dots p_r t_1 \dots t_k m_3$, we have that $m_2 \sim m_3$ in M since M is a UFM. Therefore $m_3 = s m_2$ for some $s \in R$. Hence $m_1 + m_2 = t_1 \dots t_k s m_2 \Rightarrow m_1 = (t_1 \dots t_k s - 1) m_2$. So we get $m_1 \sim m_2$ in M since m_1 is an irreducible element in M. □

Corollary 9. *Let M be a non torsion-free UFM and*

$$0 = (q_1 \dots q_k)(p_1 \dots p_r m_1)$$

be the U-decomposition of $0 \in M$. Then $M = (m_1)$.

Corollary 10. *Let R be an atomic ring and M be a non torsion-free $R-$module. If M is a UFM and $0 = (q_1 \dots q_k)(p_1 \dots p_r m_1)$ is the U-decomposition of $0 \in M$ then $Ann(M) = (p_1 ... p_r)$.*

Proof. By Corollary 9 $p_1 \dots p_r m_1 = 0 \Rightarrow p_1 \dots p_r M = 0 \Rightarrow (p_1 \dots p_r) \subseteq Ann(M)$. Now assume that $s \in Ann(M)$. Then $sM = 0 \Rightarrow s m_1 = 0 = p_1 \dots p_r m_1$. Since R is atomic, s has an irreducible factorization in R, say $s = t_1 \dots t_l$. From uniqueness, without lost of generality, we have

$$(t_{r+1} \dots t_l)(t_1 \dots t_r m_1) = (q_1 \dots q_k)(p_1 \dots p_r m_1)$$

and hence $t_i \sim p_i$ in R for $i = 1, \dots, r$. So $s \in (p_1 \dots p_r) \Rightarrow Ann(M) \subseteq (p_1 \dots p_r)$. □

Example 11. *(i) Let R be a commutative ring and $R = M$. Then $M = (1)$ and M is a UFM $\Leftrightarrow$ R is a UFR. In particular if we take $R = M = \mathbb{Z}_6$ then M is neither présimplifiable nor torsion-free as R-module but a UFM.*

(ii) Let R be a UFR and M be a cyclic R-module. Then M need not be a UFM as an R-module: Let $R = \mathbb{Z} \times \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}_3$. Then R is a UFR and $M = ((1,1))$ is a cyclic R-module. Write $p_1 = (0,1)$, $p_2 = (1,0)$, $p_3 = (1,3)$ and $m = (1,1)$ such that p_1, p_2 and p_3 are irreducible in R and m is irreducible

in M. Now $0 = ()(p_1p_2m) = ()(p_1p_3m)$ are two non associate U-decompositions of $0 \in M$. Hence M is not a UFM as an R-module.

It would be interesting to know for any commutative ring R and a cyclic non torsion-free R-module M whether M is a UFM implies R is a UFR.

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