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SOLUTION TO WEIGHTED NON-LOCAL FRACTIONAL DIFFERENTIAL EQUATION

Mohammed Mazhar-Ul-Haque[1 §], Tarachand L. Holambe[2], Govind P. Kamble[3]

[1]Dr. B.A.M. University
Aurangabad, Maharashtra, INDIA
[2]Department of Mathematics
Kai Shankarrao Gutte ACS College
Dharmapuri, Beed, Maharashtra, INDIA
[3]Department of Mathematics
P.E.S. College of Engineering
Nagsenvana, Aurangabad, (M.S.) INDIA

Abstract: In this paper we prove the nature and existence of the solutions for a weighted nonlinear fractional differential equation with nonlocal condition. Given a bounded interval $J = (0, T]$ of the real line $\mathbb{R}$ for some $T > 0$ and $T < \infty$, we consider the fractional differential equation

$$A_0 v(t) + \sum_{i=1}^{n} A_i D^{\beta_i}(v(t)u(t)) = D^{\alpha}(v(t)u(t)),$$

$$\lim_{t\to 0^+} t^{1-\alpha} v(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j),$$

where D^{α} and D^{β_i} are Riemann Liouville fractional derivatives of order $0 < \alpha, \beta_i \leq 1$.

Under some assumptions the nonlocal weighted Cauchy type fractional differential equation and result on its solution will be discussed in nonlinear fractional differential equation.

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§Correspondence author

1. Introduction

Fractional Calculus is the branch of mathematics related to the derivatives and integral of integer as well as noninteger order. This branch of mathematics has been applied to almost every field of Mathematics, Science, Engineering and Technology etc.

Nonlinear fractional differential equation with weighted initial data has been studied by several authors. The weighted Cauchy-type problem

$$\begin{aligned} D^{\alpha}(u(t)) &= f(t,u(t)), \\ t^{1-\alpha}u(t)|_{t=0} &= b, \end{aligned} \tag{1}$$

studied by Khaled et al in [9].

The solution of the periodic boundary value problem for a fractional differential equation involving a RiemannLiouville fractional derivative

$$\begin{aligned} &D^{\alpha}(u(t)) = f(t,u(t)), \\ &t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}, \end{aligned} \tag{2}$$

studied by Weia et al in [10]. Also the existence of solutions of fractional equations of Volterra type with the RiemannLiouville derivative,

$$\begin{aligned} &D^{\alpha}(u(t)) = f(t,u(t),\int_0^t k(t,s)u(s)ds) \\ &t^{1-\alpha}u(t)|_{t=0} = r \end{aligned} \tag{3}$$

Studied by Jankowski [11] etc. and the references therein.Problems in nonlinear fractional differential equation were studied by various researchers[12-18].
Now here we consider the weighted nonlocal fractional differential equation

$$\begin{aligned} &A_0v(t) + \sum_{i=1}^{n} A_i D^{\beta_i}(v(t)u(t)) = D^{\alpha}(v(t)u(t)), \\ &\lim_{t\to 0^+} t^{1-\alpha}v(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j), \end{aligned} \tag{4}$$

where D^{α} and D^{β_i} are Riemann Liouville fractional derivatives of order $0 < \alpha, \beta_i \leq 1$ and $0 < t, \tau_j \leq T < \infty$.

2. Auxiliary Results

We will use the following definitions, lemmas and theorems in the next sections.

The extensively studied fractional derivative and integral are Riemann-Liouville fractional derivative and integral, defined by [1]-[7].

Definition 2.1. The Riemann-Liouville fractional integral of order α of a real valued function f is defined by

$$I_a^\alpha f(z) = \frac{1}{\Gamma\alpha} \int_a^z (z-t)^{\alpha-1} f(t)dt \tag{5}$$

and when $a = 0$ this becomes

$$I^\alpha f(z) = \frac{1}{\Gamma\alpha} \int_0^z (z-t)^{\alpha-1} f(t)dt \tag{6}$$

The property studied by T.L. Holambe and Mohammed Mazhar-Ul-Haque (see [8]) is: Let $\alpha > 0, \beta > 0$ and f be any function. Then

$$I_k^\alpha (I_k^\beta f(z)) = I_k^\beta (I_k^\alpha f(z)), \tag{7}$$

$$\begin{aligned} I_k^\alpha (I_k^\beta f(z)) = I_k^\beta (I_k^\alpha f(z)) = I_k^{\alpha+\beta} f(z) \\ = \frac{1}{k\Gamma_k(\alpha+\beta)} \int_0^z (z-t)^{\frac{\alpha+\beta}{k}-1} f(t)dt, \end{aligned} \tag{8}$$

for the $k-$generalized fractional integral.

Definition 2.2. The Riemann-Liouville fractional derivative of order $0 < \alpha \leq 1$ of a real valued function f is defined by

$$D^\alpha f(z) = \frac{d}{dz} I^{1-\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z (z-t)^{-\alpha} f(t)dt. \tag{9}$$

Definition 2.3. A real valued function f from $I \times \mathbb{R}$ is called Caratheodary function if:

1. f is measurable on $\mathbb{R}$.
2. f is continuous on I.
3. There exist Lebesgue function h on I such that h is an Upper bound of f on I.

Theorem 2.4. *(Kolmogorov Compactness Criterion, see [19]) Let* $\Omega \subseteq L^p(0,T)$, $1 \leq p < \infty$ *if:*

(i) Ω *is bounded in* $L^p(0,T)$, *and*

(ii) $u_h \to u$ *as* $h \to 0$ *uniformly with respect to* $u \in \Omega$,

then Ω *is relatively compact in* $L^p(0,T)$ *where* $u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds$.

3. Main Result

For the fractional differential equation (4) with nonlocal condition we will use the following assumptions:

1. The function from $(0,T] \times \mathbb{R}$ to $\mathbb{R}$ is Caratheodary function.
2. $v(t)$ is positive and continuous with $inf_{(0,T]} \mid v(t) \mid = v$.
3. $\sum_j^m \frac{a_j}{\tau_j^{1-\alpha} v(\tau_j)} \neq 1$.

Theorem 3.1. *The solution of nonlocal weighted fractional differential equation (4)*

$$A_0 v(t) + \sum_{i=1}^{n} A_i D^{\beta_i}(v(t)u(t)) = D^{\alpha}(v(t)u(t)),$$
$$\lim_{t \to 0^+} t^{1-\alpha} v(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j),$$

$0 < \alpha, \beta_i \leq 1$ *and* $0 < t, \tau_j \leq T < \infty$ *can be considered as*

$$\begin{aligned} u(t) =& \frac{Kt^{\alpha-1}}{v(t)} \sum_{j=1}^{m} \frac{a_j A_0}{v(\tau_j)\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} v(s) ds \\ &+ \frac{Kt^{\alpha-1}}{v(t)} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{a_j}{v(\tau_j)} \frac{A_i}{\Gamma(\alpha - \beta_i)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-\beta_i-1}(v(s)u(s)) ds \\ &+ A_0 \frac{1}{v(t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds \\ &+ \frac{1}{v(t)} \sum_{i=1}^{n} A_i \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1}(v(s)u(s)) ds \end{aligned} \tag{10}$$

where $K = \left(1 - \sum_{j=1}^{m} \frac{a_j}{\tau_j^{1-\alpha} v(\tau_j)}\right)^{-1}$.

Proof. Let the non-local weighted fraction differential equation 4

$$A_0 v(t) + \sum_{i=1}^{n} A_i D^{\beta_i}(v(t)u(t)) = D^{\alpha}(v(t)u(t)),$$
$$\lim_{t\to 0^+} t^{1-\alpha} v(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j),$$

$0 < \alpha, \beta_i \le 1$ and $0 < t, \tau_j \le T < \infty$.

By the Reimann Liouville fractional derivative of order $\alpha > 0$:

$$D_a^{\alpha} f(z) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int_a^z (z-t)^{n-\alpha-1} f(t)dt, \quad n-1 < \alpha \le n, \ n \in \mathbb{N},$$

for $n = 1$, $a = 0$:

$$D^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z (z-t)^{-\alpha} f(t)dt.$$

By the definition of Reimann liouville fractional integral

$$D^{\alpha} f(z) = \frac{d}{dz} I^{1-\alpha} f(z),$$

we obtain

$$A_0 v(t) + \sum_{i=1}^{n} A_i D^{\beta_i}(v(t)u(t)) = D^{\alpha}(v(t)u(t)),$$
$$A_0 v(t) + \sum_{i=1}^{n} A_i \frac{d}{dt} I^{1-\beta_i}(v(t)u(t)) = \frac{d}{dt} I^{1-\alpha}(v(t)u(t)).$$

Integrating both side from 0 to t, we receive

$$A_0 \int_0^t v(s)ds + \sum_{i=1}^{n} A_i I^{1-\beta_i}(v(t)u(t)) - \sum_{i=1}^{n} A_i I^{1-\beta_i}(v(t)u(t)) \mid_{t=0}$$
$$= I^{1-\alpha}(v(t)u(t)) - I^{1-\alpha}(v(t)u(t)) \mid_{t=0},$$

$$A_0 \int_0^t v(s)ds + \sum_{i=1}^{n} A_i I^{1-\beta_i}(v(t)u(t)) - C_1 = I^{1-\alpha}(v(t)u(t)) - C_2,$$

$$A_0\int_0^t v(s)ds+\sum_{i=1}^{n}A_iI^{1-\beta_i}(v(t)u(t))+C=I^{1-\alpha}(v(t)u(t)).$$

Using Reimann Liouville integral I^{α} on bothside

$$\begin{aligned}A_0I^{\alpha+1}v(t)+\sum_{i=1}^{n}A_iI^{\alpha}I^{1-\beta_i}(v(t)u(t))+\frac{Ct^{\alpha}}{\Gamma(\alpha+1)}&=I(v(t)u(t)),\\A_0I^{\alpha+1}v(t)+\sum_{i=1}^{n}A_iI^{\alpha-\beta_i+1}(v(t)u(t))+\frac{Ct^{\alpha}}{\Gamma(\alpha+1)}&=I(v(t)u(t)).\end{aligned}$$

Taking differentiation on both side

$$\begin{aligned}A_0I^{\alpha}v(t)+\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(t)u(t))+\frac{Ct^{\alpha-1}}{\Gamma(\alpha)}&=(v(t)u(t)),\\t^{1-\alpha}A_0I^{\alpha}v(t)+t^{1-\alpha}\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(t)u(t))+\frac{C}{\Gamma(\alpha)}&=t^{1-\alpha}(v(t)u(t)),\\\frac{C}{\Gamma(\alpha)}&=\lim_{t\to0^+}t^{1-\alpha}(v(t)u(t)),\end{aligned}\tag{11}$$

since, by the equation (4), we have

$$\begin{aligned}\lim_{t\to0^+}t^{1-\alpha}(v(t)u(t))&=\sum_{j=1}^{m}a_ju(\tau_j),\\\frac{C}{\Gamma(\alpha)}&=\sum_{j=1}^{m}a_ju(\tau_j).\end{aligned}\tag{12}$$

Setting $t=\tau_j$ in equation (11), we obtain

$$\begin{aligned}&\tau_j^{1-\alpha}A_0\frac{1}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds\\&\qquad+\tau_j^{1-\alpha}\sum_{i=1}^{n}A_i\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds+\frac{C}{\Gamma(\alpha)}\\&\qquad\qquad=\tau_j^{1-\alpha}(v(\tau_j)u(\tau_j)),\end{aligned}$$

$$\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds$$

$$+\sum_{j=1}^{m}\frac{a_j}{v(\tau_j)}\sum_{i=1}^{n}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds$$
$$+\sum_{j=1}^{m}\frac{Ca_j}{\Gamma(\alpha)\tau_j^{1-\alpha}v(\tau_j)}=\sum_{j=1}^{m}a_ju(\tau_j)),$$

$$\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds$$
$$+\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_j}{v(\tau_j)}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds$$
$$+\sum_{j=1}^{m}\frac{Ca_j}{\Gamma(\alpha)\tau_j^{1-\alpha}v(\tau_j)}=\frac{C}{\Gamma(\alpha)},$$

$$\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds$$
$$+\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_j}{v(\tau_j)}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds$$
$$=\frac{C}{\Gamma(\alpha)}-\sum_{j=1}^{m}\frac{Ca_j}{\Gamma(\alpha)\tau_j^{1-\alpha}v(\tau_j)},$$

$$\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds$$
$$+\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_j}{v(\tau_j)}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds$$
$$=\frac{C}{\Gamma(\alpha)}\left(1-\sum_{j=1}^{m}\frac{a_j}{\tau_j^{1-\alpha}v(\tau_j)}\right),$$

$$K\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds$$

$$+ K\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_j}{v(\tau_j)}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds = \frac{C}{\Gamma(\alpha)},$$

where $K = \left(1-\sum_{j=1}^{m}\frac{a_j}{\tau_j^{1-\alpha}v(\tau_j)}\right)^{-1}$.

Thus, the solution from equation (11) is

$$\begin{aligned}u(t) =& \frac{Kt^{\alpha-1}}{v(t)}\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds\\ &+\frac{Kt^{\alpha-1}}{v(t)}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_j}{v(\tau_j)}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds\\ &+A_0\frac{1}{v(t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\\ &+\frac{1}{v(t)}\sum_{i=1}^{n}A_i\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t(t-s)^{\alpha-\beta_i-1}(v(s)u(s))ds.\end{aligned}$$ □

Theorem 3.2. *Let the assumptions 1-3 are satisfied by the non-local problem* (4)

$$A_0v(t)+\sum_{i=1}^{n}A_iD^{\beta_i}(v(t)u(t)) = D^{\alpha}(v(t)u(t)),$$

$$\lim_{t\to 0^+} t^{1-\alpha}v(t)u(t) = \sum_{j=1}^{m}a_ju(\tau_j),$$

$$0<\alpha,\beta_i\le 1 \text{ and } 0<t,\tau_j\le T<\infty.$$

Then the non-local problem has atleast one L_1 solution.

Proof. Let T be an operator defined by

$$\begin{aligned}(Tu)(t) =& \frac{Kt^{\alpha-1}}{v(t)}\sum_{j=1}^{m}\frac{a_jA_0}{v(\tau_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}v(s)ds\\ &+\frac{Kt^{\alpha-1}}{v(t)}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_j}{v(\tau_j)}\frac{A_i}{\Gamma(\alpha-\beta_i)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-\beta_i-1}(v(s)u(s))ds\\ &+A_0\frac{1}{v(t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\\ &+\frac{1}{v(t)}\sum_{i=1}^{n}A_i\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t(t-s)^{\alpha-\beta_i-1}(v(s)u(s))ds.\end{aligned}$$

We can write it as,

$$
\begin{aligned}
(Tu)(t) =&\frac{Kt^{\alpha-1}}{v(t)}\sum_{j=1}^{m}\frac{a_j}{v(\tau_j)}\left\{A_0 I^{\alpha}v(\tau_j)+\sum_{i=1}^{n}A_i I^{\alpha-\beta_i}(v(\tau_j)u(\tau_j))\right\}\\
&+\frac{A_0}{v(t)}I^{\alpha}v(t)+\frac{1}{v(t)}\sum_{i=1}^{n}A_i I^{\alpha-\beta_i}(v(t)u(t)).
\end{aligned}
$$

Let $\eta < min\{\alpha, \alpha-\beta_i\}$, then we receive

$$
\begin{aligned}
(Tu)(t) =&\frac{Kt^{\alpha-1}}{v(t)}\sum_{j=1}^{m}\frac{a_j}{v(\tau_j)}\left\{A_0 I^{\alpha-\eta}I^{\eta}v(\tau_j)+\sum_{i=1}^{n}A_i I^{\alpha-\beta_i-\eta}I^{\eta}(v(\tau_j)u(\tau_j))\right\}\\
&+\frac{A_0}{v(t)}I^{\alpha-\eta}I^{\eta}v(t)+\frac{1}{v(t)}\sum_{i=1}^{n}A_i I^{\alpha-\beta_i-\eta}I^{\eta}(v(t)u(t)),
\end{aligned}
$$

$$
\begin{aligned}
\mid (Tu)(t)\mid \leq &\frac{\mid K\mid t^{\alpha-1}}{inf\mid v(t)\mid}\sum_{j=1}^{m}\frac{\mid a_j\mid}{inf\mid v(\tau_j)\mid}\{\mid A_0\mid I^{\alpha-\eta}I^{\eta}\mid v(\tau_j)\mid\\
&+\sum_{i=1}^{n}\mid A_i\mid I^{\alpha-\beta_i-\eta}I^{\eta}\mid (v(\tau_j)u(\tau_j))\mid\Big\}\\
&+\frac{\mid A_0\mid}{inf\mid v(t)\mid}I^{\alpha-\eta}I^{\eta}\mid v(t)\mid\\
&+\frac{1}{inf\mid v(t)\mid}\sum_{i=1}^{n}\mid A_i\mid I^{\alpha-\beta_i-\eta}I^{\eta}\mid (v(t)u(t))\mid .
\end{aligned}
$$

Let $M=\max_I\{I^{\eta}v(t)u(t)\}$, then

$$
\begin{aligned}
\mid (Tu)(t)\mid\leq&\frac{\mid K\mid t^{\alpha-1}}{inf\mid v(t)\mid}\sum_{j=1}^{m}\frac{\mid a_j\mid}{inf\mid v(\tau_j)\mid}\\
&\Big\{\mid A_0\mid M\int_0^{\tau_j}\frac{(\tau_j-s)^{\alpha-\eta-1}}{\Gamma(\alpha-\eta)}ds\\
&+\sum_{i=1}^{n}\mid A_i\mid M\int_0^{\tau_j}\frac{(\tau_j-s)^{\alpha-\beta_i-\eta-1}}{\Gamma(\alpha-\beta_i-\eta}ds\Big\}\\
&+\frac{\mid A_0\mid}{inf\mid v(t)\mid}M\int_0^{t}\frac{(t-s)^{\alpha-\eta-1}}{\Gamma(\alpha-\eta}ds\\
&+\frac{1}{inf\mid v(t)\mid}\sum_{i=1}^{n}\mid A_i\mid M\int_0^{t}\frac{(t-s)^{\alpha-\beta_i-\eta-1}}{\Gamma(\alpha-\beta_i-\eta}ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{| K | t^{\alpha-1}}{v} \sum_{j=1}^{m} \frac{| a_j |}{v} \left\{ | A_0 | M \frac{\tau_j^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} | A_i | M \frac{\tau_j^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\} \\
&\quad + \frac{| A_0 |}{v} M \frac{t^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \frac{1}{v} \sum_{i=1}^{n} | A_i | M \frac{t^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \\
&\leq \frac{| K | t^{\alpha-1}}{v^2} \sum_{j=1}^{m} | a_j | \left\{ | A_0 | M \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} | A_i | M \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\} \\
&\quad + \frac{| A_0 |}{v} M \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \frac{1}{v} \sum_{i=1}^{n} | A_i | M \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \\
&\leq \left(\frac{| K | t^{\alpha-1}}{v} \sum_{j=1}^{m} | a_j | +1 \right) \\
&\quad \left\{ \frac{| A_0 | M}{v} \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} \frac{| A_i | M}{v} \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|Tu\|_{L_1} &= \int_0^T | (Tu)(t) | \, dt \\
&\leq \int_0^T \left(\frac{| K | t^{\alpha-1}}{v} \sum_{j=1}^{m} | a_j | +1 \right) \\
&\quad \left\{ \frac{| A_0 | M}{v} \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} \frac{| A_i | M}{v} \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\} dt \\
&\leq \left\{ \frac{| A_0 | M}{v} \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} \frac{| A_i | M}{v} \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\} \\
&\quad \int_0^T \left(\frac{| K | t^{\alpha-1}}{v} \sum_{j=1}^{m} | a_j | +1 \right) dt \\
&\leq \left\{ \frac{| A_0 | M}{v} \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} \frac{| A_i | M}{v} \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\} \\
&\quad \left(\frac{| K | \int_0^T t^{\alpha-1} dt}{v} \sum_{j=1}^{m} | a_j | + \int_0^T dt \right)
\end{aligned}$$

$$\leq \left\{ \frac{|\ A_0\ |\ M}{v} \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} \frac{|\ A_i\ |\ M}{v} \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\}$$
$$\left(\frac{|\ K\ |\ T^{\alpha}}{v} \sum_{j=1}^{m} |\ a_j\ | + T \right).$$

Let

$$r = \left\{ \frac{|\ A_0\ |\ M}{v} \frac{T^{\alpha-\eta}}{\Gamma(\alpha-\eta)} + \sum_{i=1}^{n} \frac{|\ A_i\ |\ M}{v} \frac{T^{\alpha-\beta_i-\eta}}{\Gamma(\alpha-\beta_i-\eta)} \right\}$$
$$\left(\frac{|\ K\ |\ T^{\alpha}}{v} \sum_{j=1}^{m} |\ a_j\ | + T \right).$$

Define the subset $B_r \subset L_1(I)$ by $B_r = \{u(t), t \in I : \|u\|_{L_1} \leq r, r > 0\}$ the set B_r is nonempty, closed and convex. So $\|Tu\|_{L_1} \leq r$, which implies that the operator T maps B_r into itself. Implies that T is continuous. Now, to apply Theorem 2.4, we will show that T is compact. So, let Q_r be a bounded subset of B_r. Then $T(Q_r)$ is bounded in $L_1(I)$. It remains to show that $(Tu)_h \to (Tu)$ in $L_1(I)$ as $h \to 0$, uniformly with respect to $Tu \in T(Q_r)$. Now, we have

$$\|(Tu)_h - (Tu)\| = \int_0^T |(Tu)_h - (Tu)| dt$$
$$= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Tu)(s) ds - (Tu) \right| dt$$
$$\leq \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Tu)(s) - (Tu)| ds \right) dt$$
$$\leq \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Tu)(s) - (Tu)| ds \right) dt$$
$$\leq \int_0^T \frac{1}{h} \int_t^{t+h} \left| \left(\frac{K s^{\alpha-1}}{v(s)} \sum_{j=1}^{m} \frac{a_j}{v(\tau_j)} \left\{ A_0 I^{\alpha} v(\tau_j) + \sum_{i=1}^{n} A_i I^{\alpha-\beta_i} (v(\tau_j) u(\tau_j)) \right\} \right. \right.$$
$$+ \left. \frac{A_0}{v(s)} I^{\alpha} v(s) + \frac{1}{v(s)} \sum_{i=1}^{n} A_i I^{\alpha-\beta_i} (v(s) u(s)) \right)$$
$$- \left(\frac{K t^{\alpha-1}}{v(t)} \sum_{j=1}^{m} \frac{a_j}{v(\tau_j)} \left\{ A_0 I^{\alpha} v(\tau_j) + \sum_{i=1}^{n} A_i I^{\alpha-\beta_i} (v(\tau_j) u(\tau_j)) \right\} \right.$$

$$+\frac{A_0}{v(t)}I^{\alpha}v(t)+\frac{1}{v(t)}\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(t)u(t))\Bigg)\Bigg|dsdt$$

$$\leq\sum_{j=1}^{m}\frac{a_j}{v(\tau_j)}\left\{A_0I^{\alpha}v(\tau_j)+\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(\tau_j)u(\tau_j))\right\}$$

$$\int_0^T\frac{1}{h}\int_t^{t+h}\left|\left(\frac{Ks^{\alpha-1}}{v(s)}\right)-\left(\frac{Kt^{\alpha-1}}{v(t)}\right)\right|dsdt$$

$$+\int_0^T\frac{1}{h}\int_t^{t+h}\Bigg|\frac{A_0}{v(s)}I^{\alpha}v(s)+\frac{1}{v(s)}\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(s)u(s))$$

$$-\frac{A_0}{v(t)}I^{\alpha}v(t)-\frac{1}{v(t)}\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(t)u(t))\Bigg|dsdt,$$

since:

$$\frac{1}{h}\int_t^{t+h}\left|\left(\frac{Ks^{\alpha-1}}{v(s)}\right)-\left(\frac{Kt^{\alpha-1}}{v(t)}\right)\right|ds\to 0$$

and

$$\frac{1}{h}\int_t^{t+h}\Bigg|\frac{A_0}{v(s)}I^{\alpha}v(s)+\frac{1}{v(s)}\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(s)u(s))$$

$$-\frac{A_0}{v(t)}I^{\alpha}v(t)-\frac{1}{v(t)}\sum_{i=1}^{n}A_iI^{\alpha-\beta_i}(v(t)u(t))\Bigg|ds\longrightarrow 0.$$

Here, we used that the sum belongs to $L_1(I)$ and $\frac{Kt^{\alpha-1}}{v(t)}\in L_1(I)$.

Therefore, by Theorem 2.4, we have that $T(Q_r)$ is relatively compact, that is, T is a compact operator, then the operator T has a fixed point in B_r, which proves the existence of solution in $L_1(I)$. □

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