(K, E)-SOFT TOPOLOGIES AND L-FUZZY (K, E)-SOFT NEIGHBORHOOD SYSTEMS

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Abstract: In this paper, we define a (K, E)-soft topology in stsc-quantales. We investigate the relations between (K, E)-soft topologies and L-fuzzy (K, E)-soft neighborhood systems. We give their examples.

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1. Introduction

Molodtsov [14] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. Maji et al. [11,12] gave the first practical application of soft sets in decision making problems. Many researchers have contributed towards the algebraic structure of soft set theory [1-5,7]. In 2011, Shabir and Naz [22] initiated the study of
soft topological spaces. They defined soft topology on the collection of soft sets over $X$ and established their several properties. Aygünoglu et.al [2] introduced the concept of $(K,E)$-soft topology in the sense of Šostak [9]. Cetkin et.al [3] studied $(K,E)$-soft proximities and discuss their properties.

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices. Höhle [9] introduced $L$-fuzzy topologies with algebraic structure $L$(cqm, quantales, $MV$-algebra). It has developed in many directions [17-19]. Ramadan et al. [18] define the the concept of $L$- fuzzy soft topogenous orders, $L$-fuzzy soft uniform spaces, $L$- fuzzy soft topological spaces in strictly two sided commutative quantales and investigated the relation between them.

In this paper, we define a $(K,E)$-soft topology in stsc-quantales. We investigate the relations between $(K,E)$-soft topologies and $L$-fuzzy $(K,E)$-soft neighborhood systems. We give their examples.

2. Preliminaries

Let $L = (L, \leq, \lor, \land, 0, 1)$ be a completely distributive lattice with the least element $0$ and the greatest element $1$ in $L$.

**Definition 1.** [8,9,18] A complete lattice $(L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \odot)$ is a commutative semigroup,
(L2) $x = x \odot 1$, for each $x \in L$ and $1$ is the universal upper bound,
(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i) \odot y = \bigvee_i (x_i \odot y)$.

There exists a further binary operation $\rightarrow$ (called the implication operator or residuated) satisfying the following condition

$$x \rightarrow y = \bigvee \{z \in L | x \odot z \leq y\}.$$

Then it satisfies Galois correspondence; i.e., $(x \odot z) \leq y$ iff $z \leq (x \rightarrow y)$.

In this paper, we always assume that $(L, \leq, \odot, \rightarrow, ^*)$ is a stsc-quantales with an order reversing involution $^*$ which is defined $x^* = x \rightarrow 0$ unless otherwise specified.

**Remark 2.** Every completely distributive lattice $(L, \leq, \land, \lor, ^*)$ with order reversing involution $^*$ is a stsc-quantale $(L, \leq, \odot = \land, ^*)$ with a strong negation $^*$. 
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Lemma 3. [8,9,18] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

1. $1 \rightarrow x = x$, $0 \odot x = 0$,
2. If $y \leq z$, then $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,
3. $x \leq y$ iff $x \rightarrow y = 1$.
4. $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $\bigvee_i y_i = \bigwedge_i (x \rightarrow y_i)$,
5. $\bigvee_i x_i \rightarrow y = \bigwedge_i (x_i \rightarrow y_i)$,
6. $x \rightarrow (\bigvee_i x_i) \geq \bigvee_i (x \rightarrow x_i)$,
7. $\bigwedge_i x_i \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,
8. $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
9. $(x \odot y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,
10. $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
11. $x \rightarrow y = y^* \rightarrow x^*$.
12. $\bigvee_{i\in I} x_i \rightarrow \bigvee_{i\in I} y_i \geq \bigwedge_{i\in I} (x_i \rightarrow y_i)$ and $\bigwedge_{i\in I} x_i \rightarrow \bigwedge_{i\in I} y_i \geq \bigwedge_{i\in I} (x_i \rightarrow y_i)$.

Throughout this paper, $X$ refers to an initial universe, $E$ and $K$ are the sets of all parameters for $X$, and $L^X$ is the set of all $L$-fuzzy sets on $X$.

Definition 4. [4] A map $f$ is called an $L$-fuzzy soft set on $X$, where $f$ is a mapping from $E$ into $L^X$, i.e., $f_e := f(e)$ is an $L$-fuzzy set on $X$, for each $e \in E$. The family of all $L$-fuzzy soft sets on $X$ is denoted by $(L^X)^E$. Let $f$ and $g$ be two $L$-fuzzy soft sets on $X$.

1. $f$ is an $L$-fuzzy soft subset of $g$ and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. $f$ and $g$ are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.
2. The intersection of $f$ and $g$ is an $L$-fuzzy soft set $h = f \cap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.
3. The union of $f$ and $g$ is an $L$-fuzzy soft set $h = f \cup g$, where $h_e = f_e \lor g_e$, for each $e \in E$.
4. An $L$-fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.
5. The complement of an $L$-fuzzy soft sets on $X$ is denoted by $f^*$, where $f^* : E \rightarrow L^X$ is a mapping given by $f_e^* = (f_e)^*$, for each $e \in E$.
6. $f$ is called a null $L$-fuzzy soft set and is denoted by $0_X$, if $f_e(x) = 0$, for each $e \in E$, $x \in X$.
7. $f$ is called an absolute $L$-fuzzy soft set and is denoted by $1_X$, if $f_e(x) = 1$, for each $e \in E$, $x \in X$ and $(1_x)_e(x) = 1$.

Definition 5. [4] Let $\varphi : X \rightarrow Y$ and $\psi : E_1 \rightarrow E_2$ be two mappings, where $E_1$ and $E_2$ are parameters sets for the crisp sets $X$ and $Y$, respectively.
Then \( \varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2} \) is called a fuzzy soft mapping.

(1) For \( f \in (L^X)^{E_1} \), the image of \( f \) under the fuzzy soft mapping \( \varphi_\psi \) defined by, \( \forall k \in K, \forall y \in Y, \)

\[
\varphi_\psi(f)_{e_2}(y) = \bigvee_{x \in \varphi^{-1}(\{y\})} \left( \bigvee_{e_1 \in \psi^{-1}(\{e_2\})} f_{e_1}(x) \right)
\]

(2) For \( f \in (L^X)^{E_1} \), the pre-image of \( g \) defined by

\[
\varphi_\psi^{-1}(g)_{e}(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.
\]

(3) The soft mapping \( \varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2} \) is called injective (resp. surjective, bijective) if \( f \) and \( \phi \) are both injective (resp. surjective, bijective).

**Lemma 6.** [10] Let \( \varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2} \) be a soft mapping. Then we have the following properties. For \( f, f_i \in (L^X)^{E_1} \) and \( g, g_i \in (L^Y)^{E_2} \),

(1) \( g \supseteq \varphi_\psi(\varphi_\psi^{-1}(g)) \) with equality if \( \varphi_\psi \) is surjective,

(2) \( f \supseteq \varphi_\psi^{-1}(\varphi_\psi(f)) \) with equality if \( \varphi_\psi \) is injective,

(3) if \( \varphi_\psi \) is injective,

\[
\varphi_\psi(f)_{e_2}(y) = \begin{cases} 
  f_{e_1}(x), & \text{if } x \in \varphi^{-1}(y), e_1 \in \psi^{-1}(e_2) \\
  0, & \text{otherwise,}
\end{cases}
\]

(4) \( \varphi_\psi^{-1}(g^*) = (\varphi_\psi^{-1}(g))^* \),

(5) \( \varphi_\psi^{-1}\left(\bigvee_{i \in I} g_i\right) = \bigvee_{i \in I} \varphi_\psi^{-1}(g_i) \),

(6) \( \varphi_\psi^{-1}\left(\bigwedge_{i \in I} g_i\right) = \bigwedge_{i \in I} \varphi_\psi^{-1}(g_i) \),

(7) \( \varphi_\psi\left(\bigvee_{i \in I} f_i\right) = \bigvee_{i \in I} \varphi_\psi(f_i) \),

(8) \( \varphi_\psi\left(\bigwedge_{i \in I} f_i\right) \subseteq \bigwedge_{i \in I} \varphi_\psi(f_i) \) with equality if \( \varphi_\psi \) is injective,

(9) \( \varphi_\psi^{-1}(g_1 \odot g_2) = \varphi_\psi^{-1}(g_1) \odot \varphi_\psi^{-1}(g_2) \),

(10) \( \varphi_\psi(f_1 \odot f_2) \subseteq \varphi_\psi(f_1) \odot \varphi_\psi(f_2) \) with equality if \( \varphi_\psi \) is injective.

**Definition 7.** [2,16] A set \( \tau\{\tau_k \subset P((L^X)^E) \mid k \in K\} \) is called a \((K, E)\)-soft topology on \( X \) if it satisfies the following conditions for each \( k \in K \).

- **(O1)** \( 0_X, 1_X \in \tau_k \),
- **(O2)** If \( f, g \in \tau_k \), then \( f \odot g \in \tau_k \).
- **(O3)** If \( f_i \in \tau_k \), then \( \bigcup_{i \in I} f_i \in \tau_k \).

The pair \((X, \tau)\) is called a \((K, E)\)-soft topological space. Let \((X, \tau^1)\) be a \((K_1, E_1)\)-soft topological space and \((Y, \tau^2)\) be a \((K_2, E_2)\)-soft topological space. Let \( \varphi : X \to Y, \psi : E_1 \to E_2 \) and \( \eta : K_1 \to K_2 \) be mappings. Then \( \varphi_\psi, \eta \) from \((X, \tau^1)\) into \((Y, \tau^2)\) is called soft continuous if \( \varphi_\psi^{-1}(f) \in (\tau^1)_k \) \( \forall f \in (\tau^2)_{\eta(k)}, k \in K_1 \).
**Definition 8.** [19] An $L$-fuzzy $(K, E)$-soft neighborhood system on $X$ is a set $N = \{N^x \mid x \in X\}$ of mappings $N^x : K \to L^{(L^X)^E}$ such that for each $k \in K$:

- (SN1) $N^x_k(1_X) = 1$ and $N^x_k(0_X) = 0$,
- (SN2) $N^x_k(f \circ g) \geq N^x_k(f) \circ N^x_k(g)$ for each $f, g \in (L^X)^E$,
- (SN3) If $f \subseteq g$, then $N^x_k(f) \leq N^x_k(g)$,
- (SN4) $N^x_k(f) \leq f_e(x)$ for all $f \in (L^X)^E$ and $e \in E$.
- (SN5) $N^x_k(f) \leq \bigvee\{N^y_k(g) \mid g_e(y) \supseteq N^y_k(f), \forall y \in X, e \in E\}$.

The previous axiom can be reformulated in the following way:

$\forall f \in (L^X)^E$ and $x \in X$, $N^x_k(f) \leq N^x_k(N^{-1}_k(f))$, where $N^{-1}_k(f) \in (L^X)^E$ is defined by

$$(N^{-1}_k(f))_{e}(y) = N^y_k(f) \quad \forall y \in Y, e \in E.$$ 

An $L$-fuzzy $(K, E)$-soft neighborhood system is called stratified if

- (SR) $N^x_k(\alpha \circ f) \geq \alpha \circ N^x_k(f)$ for all $f \in (L^X)^E$ and $\alpha \in L$.

The pair $(X, N)$ is called an $L$-fuzzy $(K, E)$-soft neighborhood space.

Let $(X, N)$ be an $L$-fuzzy $(K_1, E_1)$-soft neighborhood space and $(Y, M)$ be an $L$-fuzzy $(K_2, E_2)$-soft neighborhood space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from $(X, N)$ into $(Y, M)$ is called $L$-fuzzy soft $N$-continuous at every $x \in X$ if $M^\psi_{\eta(k)}(f) \leq N^x_k(\varphi^{-1}_\psi(f)) \quad \forall f \in (L^Y)^{E_2}, k \in K_1$.

**Lemma 9.** [19] Define a binary mapping $S : (L^X)^E \times (L^X)^E \to L$ by

$$S(f, g) = \bigwedge_{x \in X} \bigwedge_{e \in E} (f_e(x) \to g_e(x)) \quad \forall f, g \in (L^X)^E, \forall e \in E.$$ 

Then $\forall f, g, h, m, n \in (L^X)^E$ the following statements hold.

1. $f \sqsubseteq g$ iff $S(f, g) = 1$.
2. If $f \sqsubseteq g$, then $S(h, f) \leq S(h, g)$ and $S(f, h) \geq S(g, h)$.
3. $S(f, h) \circ S(h, g) \leq S(f, g)$. Moreover, $\bigvee_{h \in (L^X)^E} (S(f, h) \circ S(h, g)) = S(f, g)$
4. $S(f, g) \circ S(m, n) \leq S(f \circ m, g \circ n)$.
5. If $\varphi_\psi : (X, E) \to (Y, F)$ is a fuzzy soft mapping, then $S(p, q) \leq S(\varphi_\psi^{-1}(p), \varphi_\psi^{-1}(q))$, for each $p, q \in (L^Y)^F$. 

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3. \((K, E)\)-Soft Topologies and \(L\)-Fuzzy \((K, E)\)-Soft Neighborhood Systems

Theorem 10. Let \((X, \tau)\) be a \((K, E)\)-soft topological space. Define a map \(N^\tau : X \to L^{(K,E)}\) by

\[
(N^\tau)_k^x(f) = \bigvee \{ \bigwedge_{e \in E} g_e(x) \mid g \sqsubseteq f, \ g \in \tau_k \}.
\]

Then the following properties hold.

1. \((X, N^\tau)\) is a \(L\)-fuzzy \((K, E)\)-soft neighborhood space.
2. If \(\tau\) is enriched, then \(N^\tau\) is stratified and

\[
(N^\tau)_k^x(f) = \bigvee_{g \in \tau} \bigwedge_{e \in E} (g_e(x) \odot S(g, f)).
\]

Proof. (1) (SN1) Since \(1_X, 0_X \in \tau_k\), \((N^\tau)_k^x(1_X) = 1\) and \((N^\tau)_k^x(0_X) = 0\).

(SN2)

\[
(N^\tau)_k^x(f) \odot (N^\tau)_k^x(g) = \bigvee \{ \bigwedge_{e \in E}(f_1)_e(x) \mid f_1 \sqsubseteq f, \ f_1 \in \tau_k \}
\]

\[
\odot \bigvee \{ \bigwedge_{e \in E}(g_1)_e(x) \mid g_1 \sqsubseteq g, \ g_1 \in \tau_k \}
\]

\[
\leq \bigvee \{ \bigwedge_{e \in E}(f_1 \odot g_1)_e(x) \mid f_1 \odot g_1 \sqsubseteq f \odot g, \ f_1 \odot g_1 \in \tau_k \}
\]

\[
\leq \bigvee (N^\tau)_k^x(f \odot g).
\]

(SN3-5) follow from the definition of \(N^\tau\).

(SN6) Put \(N^\tau_\land(f) = \bigvee \{ g \mid g \sqsubseteq f, \ g \in \tau_k \}\) with \(N^\tau_\land(x) = (N^\tau)_k^x\). Then \(N^\tau_\land(f) \in \tau_k\). By (SN3) and the definition of \(N^\tau\),

\[
(N^\tau)_k^x(N^\tau_\land(f)) = (N^\tau)_k^x(f).
\]

\[
(N^\tau)_k^x(f) = (N^\tau)_k^x(N^\tau_\land(f))
\]

\[
\leq \bigvee \{(N^\tau)_k^x(g) \mid g \in \tau_k \} \leq N^\tau_y(f).\]

Thus \((X, N^\tau)\) is an \(L\)-neighborhood space.

(2)

\[
\alpha \odot (N^\tau)_k^x(f) = \alpha \odot \bigvee \{ \bigwedge_{e \in E} g_e(x) \mid g \sqsubseteq f, \ g \in \tau_k \}
\]

\[
\leq \bigvee \{ \bigwedge_{e \in E} (\alpha \odot g_e(x)) \mid g \sqsubseteq \alpha \odot f, \ \alpha \odot g \in \tau_k \} \leq (N^\tau)_k^x(\alpha \odot f).
\]

Put \(\gamma(x) = \bigvee_{g \in \tau_k} \{ \bigwedge_{e \in E} (g_e(x) \odot S(g, f)) \}.\) Let \(g\) with \(g \sqsubseteq f\) and \(g \in \tau_k\). Then \(g_e(x) \odot S(g, f) = g_e(x) \odot 1 = g_e(x)\). Thus \(\bigwedge_{e \in E} g_e(x) \leq \gamma(x)\). Therefore \((N^\tau)_k^x(f) \leq \gamma(x)\).

Let \(g_e(x) \odot S(g, f)\) with \(g \in \tau_k\). Since \(\tau\) is enriched, \(g \odot S(g, f) \in \tau_k\) and \(g_e(x) \odot S(g, f) \leq g_e(x) \odot (g_e(x) \rightarrow f_e(x)) \leq f_e(x)\). Then \(\gamma(x) \leq (N^\tau)_k^x(f)\).
Theorem 11. Let \((X, N)\) be an \(L\)-fuzzy \((K, E)\)-soft neighborhood space. Define \(\tau^N_k \subset (L^X)^E\) as follows
\[
\tau^N_k = \{ f \in (L^X)^E \mid f_e(x) = N^x_k(f), \forall x \in X, e \in E \}.
\]
Then,
1. \(\tau^N\) is a \((K, E)\)-soft topology on \(X\),
2. If \(N\) is stratified, then \(\tau^N\) is an enriched \((K, E)\)-soft topology.
3. \(N \leq N^\tau_k\). If \(E = \{e\}\), then \(N = N^\tau_k\).
4. If \((X, \tau)\) is a \((K, E)\)-soft topological space and \(E = \{e\}\), then \(\tau = \tau^N_k\).

Proof. (1) \((O1)\) Since \(N^x_k(1_X) = 1\) and \(N^x_k(0_X) = 0\), we have \(1_X, 0_X \in \tau^N_k\).

\((O2)\) Let \(f, g \in \tau^N_k\). Since \(N^x_k(f \circ g) \geq N^x_k(f) \circ N^x_k(g) = (f \circ g)_e(x)\) and \((SN4)\), then \(f \circ g \in \tau^N_k\).

\((O3)\) Let \(f_i \in \tau^N_k\) for all \(i \in \Gamma\). Since \(N^x_k(\bigcup_{i \in \Gamma} f_i) \geq \bigvee_{i \in \Gamma} N^x_k(f_i) = \bigcup_{i \in \Gamma} (f_i)_e(x)\) and \((SN4)\), then \(\bigcup_{i \in \Gamma} f_i \in \tau^N_k\).

(2) \((R)\) Let \(f \in \tau^N_k\). Since \(N^x_k(\alpha \circ f) \geq \alpha \circ N^x_k(f) = \alpha \circ f_e(x)\) and \((SN4)\), then \(\alpha \circ f \in \tau^N_k\).

(3) Since \(N^x_k(f) \leq N^x_k(N^x_k(f)) \leq N^x_k(f)\) from \((SN3)\) and \((SN5)\), \(N^x_k(f) = N^x_k(N^x_k(f))\) for all \(x \in X\). Since \(N^x_k(f) \in \tau^N_k\), by the definition of \(N^\tau_k\), \(N^x_k(f) \leq (N^\tau_k)^x_k(f)\).

Let \(E = \{e\}\) be given. Since \(N^\tau_k(x) = \bigvee\{(g_i)_e(x) \mid g_i \subseteq f, g_i \in \tau^N\}\) and \((g_i)_e(x) = N^x_k(g_i)\), then
\[
\bigvee_i (g_i)_e(x) = \bigvee_i N^x_k(g_i)_e(x) \leq N^x_k(N^\tau_k^\Gamma(x)) = N^x_k(\bigvee_i g_i) \leq \bigvee_i (g_i)_e(x).
\]

Hence \(N^x_k((N^\tau_k^\Gamma)^x_k(f)) = (N^\tau_k)^x_k(f)\). Since \((N^\tau_k)^x_k(f) \subseteq f\), by \((SN3)\),
\[
(N^\tau_k)^x_k(f) = N^x_k((N^\tau_k)^x_k(f)) \leq N^x_k(f).
\]

Thus \((N^\tau_k)^x_k = N^x_k\) for all \(x \in X\).

(4) Let \(f \in \tau^N_k\). Then \(f = N^\tau_k(\Gamma) = \bigcup\{g \mid g \subseteq f, g \in \tau_k\} \in \tau_k\). Hence \(\tau^N_k \subseteq \tau_k\).

Let \(g \in \tau_k\). Then \(g_e(x) = (N^\tau_k)^x_k(g)\) for all \(x \in X\). Then \(g \in \tau^N_k\).

Theorem 12. Let \((X, \tau_X)\) be a \((K_1, E_1)\)-soft topological space and \((Y, \tau_Y)\) be a \((K_2, E_2)\)-soft topological space. Let \(\varphi : X \to Y\), \(\psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) be mappings. If \(\varphi, \psi, \eta : (X, \tau_X) \to (Y, \tau_Y)\) is soft continuous, then \(\varphi, \psi, \eta : (X, N^\tau_X) \to (Y, N^\tau_Y)\) is soft \(N\)-continuous.
We obtain a \( \eta f \) for each \( g \in (\tau_Y)_{\eta(k)} \), we have

\[
(N^{\tau_Y})_{\eta(k)}^f(x)(f) = \bigvee \{g_{\psi(e)}(\varphi(x)) \mid g \subseteq f, g \in (\tau_Y)_{\eta(k)}\} \\
\leq \bigvee \{\varphi^{-1}_\psi(g.e(x)) \mid \varphi^{-1}_\psi(g) \subseteq \varphi^{-1}_\psi(f), \varphi^{-1}_\psi(g) \in (\tau_X)_k\} \\
\leq (N^{\tau_Y})_k^x(\varphi^{-1}_\psi(f)).
\]

**Theorem 13.** Let \( (X, N_X) \) be a \((K_1, E_1)\)-soft topological space and \((Y, N_Y)\) be a \((K_2, E_2)\)-soft topological space. Let \( \varphi : X \to Y \), \( \psi : E_1 \to E_2 \) and \( \eta : K_1 \to K_2 \) be mappings. If \( \varphi, \eta : (X, N_X) \to (Y, N_Y) \) is soft \( N \)-continuous, then \( \varphi, \eta : (X, \tau^{N_X}) \to (Y, \tau^{N_Y}) \) is soft continuous.

**Proof.** Let \( f \in (\tau_Y)_{\eta(k)} \). Since \( \tau_Y = \tau^{N_Y} \) from Theorem 11(4),

\[
f_{\psi(e)}(\varphi(x)) = (N^{\tau_Y})_{\eta(k)}^f(x)(f) \leq (N^{\tau_Y})_k^x(\varphi^{-1}_\psi(f)).
\]

Hence \( \phi^{-1}(f) \in \tau_Y \).

**Example 14.** Let \( X = \{h_i \mid i = \{1, 2, 3\}\} \) with \( h_i \)-house and \( E = \{e, b\} \) with \( e \)-expensive, \( b \)-beautiful. Define a binary operation \( \odot \) on \([0, 1]\) by

\[
x \odot y = \max\{0, x + y - 1\}, \quad x \to y = \min\{1 - x + y, 1\}
\]

Then \(([0, 1], \land, \to, 0, 1)\) is a stsc-quantile (ref [8,9]). Put \( f, g \in (L^X)^E \) such that

\[
\begin{align*}
f_e(h_1) &= 0.5, f_e(h_2) = 0.5, f_e(h_3) = 0.6 \\
f_b(h_1) &= 0.6, f_b(h_2) = 0.3, f_b(h_3) = 0.6 \\
g_e(h_1) &= 0.3, g_e(h_2) = 0.2, g_e(h_3) = 0.5 \\
g_b(h_1) &= 0.4, g_b(h_2) = 0.4, g_b(h_3) = 0.1
\end{align*}
\]

Put \( K = \{k_1, K_2\} \). We define \( \tau_{k_1}, \tau_{k_2} \subset [0, 1]^{([0, 1]^X)^E} \) as follows:

\[
\tau_{k_1} = \{1_X, 0_X, f, f \odot f\}, \quad \tau_{k_2} = \{1_X, 0_X, g\}.
\]

We obtain a \([0, 1]\)-fuzzy \((K, E)\)-soft neighborhood system \( N^{\tau} \) on \( X \) as:

\[
(N^{\tau})_{k_1}^{h_1}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.5, & \text{if } f \subseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases} \quad (N^{\tau})_{k_1}^{h_1}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.3, & \text{if } g \subseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases}
\]

\[
(N^{\tau})_{k_2}^{h_2}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.3, & \text{if } f \subseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases} \quad (N^{\tau})_{k_2}^{h_2}(h) = \begin{cases} 1, & \text{if } h = 1_X \\ 0.3, & \text{if } g \subseteq h \neq 1_X, \\ 0, & \text{otherwise.} \end{cases}
\]
Then

$$
(N^\tau)^{h_3}_{k_1}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.6, & \text{if } f \sqsubseteq h \neq 1_X, \\
0.2, & \text{if } f \odot f \sqsubseteq h \nsubseteq f, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
(N^\tau)^{h_3}_{k_2}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.1, & \text{if } g \sqsubseteq h \neq 1_X, \\
0, & \text{otherwise.}
\end{cases}
$$

Then

$$
\tau^N_{k_1} = \{1_X, 0_X, f_1, f_1 \odot f_1\}, \quad \tau^N_{k_2} = \{1_X, 0_X, g_1\}.
$$

We obtain a [0, 1]-fuzzy \((K, E)\)-soft neighborhood system \(N^{\tau^N}_{\tau^N}\) on \(X\) as:

$$
(N^{\tau^N})^{h_1}_{k_1}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.5, & \text{if } f_1 \sqsubseteq h \neq 1_X, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
(N^{\tau^N})^{h_1}_{k_2}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.3, & \text{if } g_1 \sqsubseteq h \neq 1_X, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
(N^{\tau^N})^{h_2}_{k_1}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.3, & \text{if } f_1 \sqsubseteq h \neq 1_X, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
(N^{\tau^N})^{h_2}_{k_2}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.3, & \text{if } g_1 \sqsubseteq h \neq 1_X, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
(N^{\tau^N})^{h_3}_{k_1}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.6, & \text{if } f_1 \sqsubseteq h \neq 1_X, \\
0.2, & \text{if } f_1 \odot f_1 \sqsubseteq h \nsubseteq f, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
(N^{\tau^N})^{h_3}_{k_2}(h) = \begin{cases} 
1, & \text{if } h = 1_X \\
0.1, & \text{if } g_1 \sqsubseteq h \neq 1_X, \\
0, & \text{otherwise.}
\end{cases}
$$

References


