ON THE UNISOLVENT NONCONFORMING
FINITE ELEMENT METHOD

Dibyendu Adak\textsuperscript{1}, E. Natarajan\textsuperscript{2} \textsuperscript{§}
\textsuperscript{1,2}Department of Mathematics
Indian Institute of Space Science and Technology
Thiruvananthapuram, Kerala, INDIA

Abstract: In this paper, we introduce a new finite element on the triangle using $P_1$ nonconforming element enriched with incomplete $P_2$ conforming element. The resultant finite element approximation is piecewise quadratic discontinuous along the edges except at the mid points of the edges. The new finite element is unisolvent and satisfies the interpolation estimates.

AMS Subject Classification: 65L60, 65M60, 65M30
Key Words: NC1-C2 element, $P_1$ element

1. Introduction

Nonconforming finite elements has been studied extensively for solving partial differential equations. A brief description of some of the conforming and nonconforming finite elements are given below.

$P_1$ nonconforming element [1, 2]. This is a triangular element which is not $C^0$. This element converges for second order elliptic problem with optimal rate. Another nonconforming piecewise quadratic finite element on triangles has been discussed in [3]. This element satisfy patch test of Irons and Razzaque [4]. This implies that on element interfaces, one should ensure the continuity

\textsuperscript{§}Correspondence author
of the approximation at the Gauss-Legendre quadrature points needed for the exact integration of third-degree polynomials. Recent studies on topics such as simulation of electromagnetic wave interaction [5], time domain electromagnetics [6, 7], elastodynamics [8], micromagnetics [9], incompressible fluid flows [10], polymer flows [11], linear elasticity [12] shows the importance of applications of non conforming methods.

In this paper we will propose a new finite element which is a bridge between conforming and nonconforming finite element. This new element is piecewise quadratic and quasi conforming. Though the element is piecewise quadratic but it does not require two point continuity restriction on each interface of $\tau^h$, which is needed for the above mentioned piecewise quadratic nonconforming element [3]. We shall show here that these elements are in fact very simple to use and they are nothing but combination of $P^1$ nonconforming element with incomplete $P^2$ conforming element.

2. Definition and Properties of NC1-C2 Element

In order to define a nonconforming space, we introduce some further notation.

$$V_h^1 = \{ v \in L^2(\Omega) : v|_K \text{ is linear } \forall K \in \tau_h, \quad v \text{ is continuous at the mid points of the edges of } \tau_h \}$$

The above space is basically $P^1$ nonconforming space.

We define $D^2_K = \text{span}\{\phi_1, \phi_2, \phi_3\}$; where $\phi_i = \hat{\phi}_i \circ F_K^{-1}$ and $F_K$ is affine mapping from $\hat{K}$ to $K$ (Figure 1). $\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3$ are basis functions on reference triangle $\hat{K}$ corresponding to the vertices $\hat{b}_1, \hat{b}_2, \hat{b}_3$ respectively, which is defined by

$$\hat{\phi}_1 = (-1 + 2x + 2y)(-1 + x + y)$$
$$\hat{\phi}_2 = (2x - 1)x$$
$$\hat{\phi}_3 = (2y - 1)y$$

$\phi_i, \ 1 \leq i \leq 3$ is continuous along edge on each element. In this paper we use the following finite element space

$$V_h := V_h^1 \oplus V_h^2, \quad V_h^2 := \{ v_h \in L^2(\Omega) : v_h|_K \in D^2_K \}$$

Finite element space $V_h$ consists of piecewise quadratic function which is discontinuous along edge of each triangle except at mid points of edges.

A typical element $\omega \in V_h$ is demonstrated below
\[ \omega = \omega^1 + \omega^2 \text{ where } \omega^1 \in \mathcal{V}_h^1 \text{ and } \omega^2 \in \mathcal{V}_h^2 \]

since \( \omega^1 \) is discontinuous along edges except at the midpoints and \( \omega^2 \) is continuous along edge, \( \omega \) is discontinuous along edges except at the midpoints.

On interior edges jump of typical element is reduced to \( P^1 \) polynomial. Let \( e \) be an interior edge which is shared by two triangle \( K^1 \) and \( K^2 \). Let \( \omega|_{K^1} \) and \( \omega|_{K^2} \) are restrictions of \( \omega \) on \( K^1 \) and \( K^2 \) respectively.

\[
[\omega] = \omega|_{K^1} - \omega|_{K^2} = (\omega^1|_{K^1} + \omega^2|_{K^1}) - (\omega^1|_{K^2} + \omega^2|_{K^2}) = (\omega^1|_{K^1} - \omega^1|_{K^2}) + (\omega^2|_{K^1} - \omega^2|_{K^2})
\]

This space contains the space of continuous piecewise quadratic and space of nonconforming piecewise linear, since

\[ V_h = V_h^1 + P^2_h \]

where \( P^2_h \) is piecewise quadratic conforming finite element space.

### 3. Interpolation Error Estimation

**Lemma 1.** Assume \( \dim(V_h) < \infty \). Let \( a_h(.,.) \) be a symmetric positive definite bilinear form on \( V + V_h \) which reduces to \( a(.,.) \) on \( V \). Let \( u \in V \) solve

\[ a(u,v) = F(v) \ \forall \ v \in V \]

where \( F \in V' \cap V'_h \). Let \( u_h \in V_h \) solve

\[ a_h(u_h,v) = F(v) \ \forall \ v \in V \]

Then

\[ \|u - u_h\|_h \leq \inf_{v \in V_h} \|u - v\|_h + \sup_{\omega \in V_h \setminus \{0\}} \frac{|a_h(u - u_h, \omega)|}{\|\omega\|_h} \]

where \( \|\|_h = \sqrt{a_h(.,.)} \)

**Proof.** Let \( \bar{u}_h \in V_h \) satisfies

\[ a_h(\bar{u}_h,v) = a_h(u,v) \ \forall \ v \in V_h \]
which implies that
\[ a_h(\tilde{u}_h - u, v) = 0 \quad \forall \quad v \in V_h \]
\[ \Rightarrow \|u - \tilde{u}_h\|_h = \inf_{v \in V_h} \|u - v\|_h \]

Then
\[ \|u - u_h\|_h \leq \|u - \tilde{u}_h\|_h + \|\tilde{u}_h - u_h\|_h \]
\[ \leq \|u - \tilde{u}_h\|_h + \sup_{\omega \in V_h \setminus \{0\}} \frac{|a_h(\tilde{u}_h - u_h, \omega)|}{\|\omega\|_h} \]

Lemma 2. Let \( K \) be an arbitrary element of conforming triangulation \( \tau_h \). Then the following inequality holds
\[ |e|^{-1} \|\zeta\|_{L^2(e)}^2 \leq C (h_K^{-2} \|\zeta\|_{L^2(K)}^2 + |\zeta|_{H^1(K)}^2) \quad \forall \quad \zeta \in H^1(K) \quad (3) \]
where \(|e|\) denotes the length of edge \( e \subset \partial K \), \( h_K = \text{diam}(K) \), and the positive constant depends only on the chunkiness parameter of \( K \).

Proof. See the details in [2].

Lemma 3. Let all assumptions of Lemma 2 hold and \( \omega \) be an arbitrary element of \( V_h \). Then
\[ |e| \|[\omega]\|_{L^2(e)}^2 \leq C \sum_{K \in \tau_e} h_K^2 |\omega|_{H^1(K)}^2 \quad (4) \]
where \( [\omega] \) denotes jump of \( \omega \) along edge \( e \in \varepsilon^h \).

Proof. Using lemma 2 we can write
\[ |e|^{-1} \|[\omega]\|_{L^2(e)}^2 \leq C \sum_{K \in \tau_e} (h_K^{-2} \|\omega\|_{L^2(K)}^2 + |\omega|_{H^1(K)}^2) \]
\[ |e| \|[\omega]\|_{L^2(e)}^2 \leq C \sum_{K \in \tau_e} (|e|^2 h_K^{-2} \|\omega\|_{L^2(K)}^2 + |e|^2 |\omega|_{H^1(K)}^2) \]
where \( \tau_e \) is the set of triangles in \( \tau_h \) containing \( e \) on their boundaries. Again \( [\omega] = 0 \) at midpoint of each edge of \( K \) hence first part of right hand side will be vanished. Therefore we have
\[ |e| \|[\omega]\|_{L^2(e)}^2 \leq C \sum_{K \in \tau_e} |e|^2 |\omega|_{H^1(K)}^2 \]
\[ \leq C \sum_{K \in \tau_e} h_K^2 |\omega|_{H^1(K)}^2 \]
Lemma 4. Let $B$ be a ball in $\Omega$ such that $\Omega$ is star shaped with respect to $B$ and such that its radius $\rho > \left(\frac{1}{2}\right)\rho_{\text{max}}$. Let $Q^m u$ be the Taylor polynomial of order $m$ of $u$ averaged over $B$ where $u \in W^m_p(\Omega)$ and $p \geq 1$. Then

$$|u - Q^m u|_{W^k_p(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W^m_p(\Omega)} \quad k = 0, 1, \ldots, m,$$

where $d = \text{diam}(\Omega)$ and $\rho_{\text{max}} = \sup \{ \rho : \Omega \text{ is star shaped with respect to a ball of radius } \rho \}$

Proof. See the details in [2]

The important ingredient in the error analysis is a bound on the approximation error $\|u - u_I\|$ where $u_I \in V_h$ is a suitable interpolation which agrees with $u$ at mid point of each edges of $\epsilon^h$ of exact solution $u$. The interpolation operator is defined at the element level. We just require the local approximation property

$$|u - u_I|_{H^s(K)} \leq C h_K^{p+1-s} |u|_{H^{p+1}(K)} \quad \forall K \in \tau_h, s = 0, 1, 2$$

It will be useful to define it explicitly. It is defined in two steps. Let $\hat{K}$ be the reference triangle with vertices $\hat{b}_1, \hat{b}_2, \hat{b}_3$ whose coordinates are $(0, 0), (1, 0), (0, 1)$ respectively and $\hat{m}_i$ be the midpoint of the side joining $i$ and $i + 1$(modulo 3) vertices. We define

$$\hat{I}^1(\hat{u}) = \hat{u}(\hat{m}_1)\hat{\phi}_4 + \hat{u}(\hat{m}_2)\hat{\phi}_5 + \hat{u}(\hat{m}_3)\hat{\phi}_6$$

Figure 1
\[ \hat{I}^2(\hat{u}) = (\hat{u}(\hat{b}_1) - \hat{I}^1(\hat{u})(\hat{b}_1))\hat{\phi}_1 + (\hat{u}(\hat{b}_2) - \hat{I}^1(\hat{u})(\hat{b}_2))\hat{\phi}_2 + (\hat{u}(\hat{b}_3) - \hat{I}^1(\hat{u})(\hat{b}_3))\hat{\phi}_3 \]

Finally we define interpolation as
\[ \hat{I}(\hat{u}) = \hat{I}^1(\hat{u}) + \hat{I}^2(\hat{u}) = \sum_{j=1}^{6} \hat{L}_j(\hat{u})\hat{\phi}_j \]

where \( \hat{L}_i \) for \( i = 1, 2, 3, 4, 5, 6 \) continuous linear functional.

Now we will show that \( P_2(\hat{K}) \) is unisolvent with respect to these functionals, i.e. for an arbitrary polynomial \( \hat{p} \in P_2(\hat{K}) \), \( \hat{L}_i(\hat{p}) = 0 \) implies \( \hat{p} = 0 \).

Since \( \hat{p} \in P_2(\hat{K}) \) implies that \( \hat{p} \) can be written as linear combination of basis of \( P_2(\hat{K}) \). Then
\[ \hat{p} = \sum_{i=1}^{6} C_i \hat{\phi}_i. \]

\[ C_4 = \hat{p}(\hat{m}_1) = \hat{L}_4(\hat{p}) = 0 \]
\[ C_5 = \hat{p}(\hat{m}_2) = \hat{L}_5(\hat{p}) = 0 \]
\[ C_6 = \hat{p}(\hat{m}_3) = \hat{L}_6(\hat{p}) = 0 \]
\[ \Rightarrow \hat{p} = C_1\hat{\phi}_1 + C_2\hat{\phi}_2 + C_3\hat{\phi}_3 \]

again
\[ C_1 = \hat{p}(\hat{b}_1) = \hat{L}_1(\hat{p}) = 0 \]
\[ C_2 = \hat{p}(\hat{m}_2) = \hat{L}_2(\hat{p}) = 0 \]
\[ C_3 = \hat{p}(\hat{m}_3) = \hat{L}_3(\hat{p}) = 0 \]
\[ \Rightarrow \hat{p} = 0 \]

Similarly it can be shown that for arbitrary \( \hat{p} \in P_2(\hat{K}) \), \( \hat{I}(\hat{p}) = \hat{p} \). Let \( (K, P_2(K), \Sigma) \) be an affine finite element of \( (\hat{K}, P_2(\hat{K}), \hat{\Sigma}) \) where \( \Sigma = \{L_1, L_2, L_3, L_4, L_5, L_6\} \) and \( \hat{\Sigma} = \{\hat{L}_1, \hat{L}_2, \hat{L}_3, \hat{L}_4, \hat{L}_5, \hat{L}_6\} \).

**Theorem 5.** Let \( u \in H^{m+1}(K) \) be an arbitrary element. Then we have
\[ \|D^s(u - I_h u)\|_{L^2(K)} \leq C h^{m+1-s}_K \|D^{m+1}u\|_{L^2(K)} \] (7)

where \( s \leq m + 1 \) and \( m = 0, 1, 2 \) and \( C \) is a positive constant does not depend on \( h_K \).

**Proof.** Using Bramble-Hilbert lemma and properties of affine transformation. \( \square \)
4. Conclusions

In this paper we have introduced a new finite element method with satisfies all the finite element properties. Also we have shown the interpolation estimates in the standard sobolev norm.

References
