International Journal of Pure and Applied Mathematics
Volume 108 No. 2 2016, 387-393
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v108i2.13

ijpam.eu

ON THE UNISOLVENT NONCONFORMING FINITE ELEMENT METHOD

Dibyendu Adak[1], E. Natarajan[2 §]
[1,2]Department of Mathematics
Indian Institute of Space Science and Technology
Thiruvananthapuram, Kerala, INDIA

Abstract: In this paper, we introduce a new finite element on the triangle using $\boldsymbol{P^1}$ nonconforming element enriched with incomplete $\boldsymbol{P^2}$ conforming element. The resultant finite element approximation is piecewise quadratic discontinuous along the edges except at the mid points of the edges. The new finite element is unisolvent and satisfies the interpolation estimates.

AMS Subject Classification: 65L60, 65M60, 65M30
Key Words: NC1-C2 element, $\boldsymbol{P^1}$ element

1. Introduction

Nonconforming finite elements has been studied extensively for solving partial differential equations. A brief description of some of the conforming and nonconforming finite elements are given below.

P^1 *nonconforming element* [1, 2]. This is a triangular element which is not C^0. This element converges for second order elliptic problem with optimal rate. Another nonconforming piecewise quadratic finite element on triangles has been discussed in [3]. This element satisfy patch test of Irons and Razzaqque [4]. This implies that on element interfaces, one should ensure the continuity

Received: February 11, 2016
Published: June 20, 2016

§Correspondence author

of the approximation at the Gauss-Legendre quadrature points needed for the exact integration of third-degree polynomials. Recent studies on topics such as simulation of electromagnetic wave interaction [5], time domain electromagnetics [6, 7], elastodynamics [8], micromagnetics [9], incompressible fluid flows [10], polymer flows [11], linear elasticity [12] shows the importance of applications of non conforming methods.

In this paper we will propose a new finite element which is a bridge between conforming and nonconforming finite element. This new element is piecewise quadratic and quasi conforming. Though the element is piecewise quadratic but it does not require two point continuity restriction on each interface of τ^h, which is needed for the above mentioned piecewise quadratic nonconforming element [3]. We shall show here that these elements are in fact very simple to use and they are nothing but combination of P^1 nonconforming element with incomplete P^2 conforming element.

2. Definition and Properties of NC1-C2 Element

In order to define a nonconforming space, we introduce some further notation.

$$\begin{aligned} V_h^1 = \{v \in L^2(\Omega) \ : v|_K & \text{ is linear } \forall\ K \in \tau_h, \\ & v \text{ is continuous at the mid points of the edges of } \tau_h\} \end{aligned}$$

The above space is basically P^1 nonconforming space.

We define $D_K^2 = \text{span}\{\phi_1, \phi_2, \phi_3\}$; where $\phi_i = \hat{\phi}_i \circ F_K^{-1}$ and F_K is affine mapping from $\hat{K}$ to K (Figure 1). $\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3$ are basis funtions on reference triangle $\hat{K}$ corresponding to the vertices $\hat{b}_1, \hat{b}_2, \hat{b}_3$ respectively, which is defined by

$$\begin{aligned} \hat{\phi}_1 &= (-1 + 2x + 2y)(-1 + x + y) \\ \hat{\phi}_2 &= (2x - 1)x \\ \hat{\phi}_3 &= (2y - 1)y \end{aligned}$$

$\phi_i,\ 1 \le i \le 3$ is continuous along edge on each element. In this paper we use the following finite element space

$$V_h := V_h^1 \oplus V_h^2, \quad V_h^2 := \{v_h \in L^2(\Omega) : v_h|_K \in D_K^2\}$$

Finite element space V_h consists of piecewise quadratic function which is discontinuous along edge of each triangle except at mid points of edges.

A typical element $\omega \in V_h$ is demonstrated below

$$\omega = \omega^1 + \omega^2 \text{ where } \omega^1 \in V_h^1 \text{ and } \omega^2 \in V_h^2$$

since ω^1 is discontinuous along edges except at the midpoints and ω^2 is continuous along edge, ω is discontinuous along edges except at the mid points.
On interior edges jump of typical element is reduced to P^1 polynomial. Let e be an interior edge which is shared by two triangle K^1 and K^2. $\omega|_{K^1}$ and $\omega|K^2$ are restrictions of ω on K^1 and K^2 respectively.

$$\begin{aligned}
[\omega] &= \omega|K^1 - \omega|K^2 \\
&= (\omega^1|K^1 + \omega^2|K^1) - (\omega^1|K^2 + \omega^2|K^2) \\
&= (\omega^1|K^1 - \omega^1|K^2) + (\omega^2|K^1 - \omega^2|K^2) \\
&= (\omega^1|K^1 - \omega^1|K^2)
\end{aligned} \tag{1}$$

This space contains the space of continuous piecewise quadratic and space of nonconforming piecewise linear, since

$$V_h = V_h^1 + P_h^2$$

where P_h^2 is piecewise quadratic conforming finite element space.

3. Interpolation Error Estimation

Lemma 1. *Assume $dim(V_h) < \infty$. Let $a_h(..)$ be a symmetric positive definite bilinear form on $V + V_h$ which reduces to $a(.,.)$ on V. Let $u \in V$ solve*

$$a(u, v) = F(v) \ \forall \ v \in V$$

where $F \in V' \cap V_h'$. Let $u_h \in V_h$ solve

$$a_h(u_h, v) = F(v) \ \forall \ v \in V$$

Then

$$\|u - u_h\|_h \leq \inf_{v \in V_h} \|u - v\|_h + \sup_{\omega \in V_h \setminus \{0\}} \frac{|a_h(u - u_h, \omega)|}{\|\omega\|_h} \tag{2}$$

where $\|.\|_h = \sqrt{a_h(.,.)}$

Proof. Let $\tilde{u_h} \in V_h$ satisfies

$$a_h(\tilde{u_h}, v) = a_h(u, v) \ \forall \ v \in V_h$$

which implies that

$$\begin{aligned} a_h(\tilde{u_h} - u, v) &= 0 \ \forall \ v \in V_h \\ \Rightarrow \ \|u - \tilde{u_h}\|_h &= \inf_{v \in V_h} \|u - v\|_h \end{aligned}$$

Then

$$\begin{aligned} \|u - u_h\|_h &\leq \|u - \tilde{u_h}\|_h + \|\tilde{u_h} - u_h\|_h \\ &\leq \|u - \tilde{u_h}\|_h + \sup_{\omega \in V_h \setminus \{0\}} \frac{|a_h(\tilde{u_h} - u_h, \omega)|}{\|\omega\|_h} \end{aligned}$$

□

Lemma 2. *Let K be an arbitrary element of conforming triangulation τ_h. Then the following inequality holds*

$$|e|^{-1}\|\zeta\|^2_{L^{\mathbf{2}}(e)} \leq \ C(h_K^{-2}\|\zeta\|^2_{L^{\mathbf{2}}(K)} + |\zeta|^2_{H^{\mathbf{1}}(K)}) \ \forall \ \zeta \in H^1(K) \tag{3}$$

where $|e|$ denotes the length of edge $e \subset \partial K$, $h_K = diam(K)$, and the positive constant depends only on the chunkiness parameter of K.

Proof. See the details in [2]. □

Lemma 3. *Let all assumptions of Lemma 2 hold and ω be an arbitrary element of V_h.Then*

$$|e|\|[\omega]\|^2_{L^{\mathbf{2}}(e)} \leq \ C \sum_{K \in \tau_e} h_K^2 |\omega|^2_{H^{\mathbf{1}}(K)} \tag{4}$$

where $[\omega]$ denotes jump of ω along edge $e \in \varepsilon^h$.

Proof. Using lemma 2 we can write

$$\begin{aligned} |e|^{-1}\|[\omega]\|^2_{L^{\mathbf{2}}(e)} \ &\leq \ C \sum_{K \in \tau_e} (h_K^{-2}\|\omega\|^2_{L^{\mathbf{2}}(K)} + |\omega|^2_{H^{\mathbf{1}}(K)}) \\ |e|\|[\omega]\|^2_{L^{\mathbf{2}}(e)} &\leq \ C \sum_{K \in \tau_e} (|e|^2 h_K^{-2}\|\omega\|^2_{L^{\mathbf{2}}(K)} + |e|^2|\omega|^2_{H^{\mathbf{1}}(K)}) \end{aligned}$$

where τ_e is the set of triangles in τ_h containing e on their boundaries. Again $[\omega]$ =0 at midpoint of each edge of K hence first part of right hand side will be vanished. Therefore we have

$$\begin{aligned} |e|\|[\omega]\|^2_{L^{\mathbf{2}}(e)} &\leq \ C \sum_{K \in \tau_e} |e|^2|\omega|^2_{H^{\mathbf{1}}(K)} \\ &\leq C \sum_{K \in \tau_e} h_K^2 |\omega|^2_{H^{\mathbf{1}}(K)} \end{aligned}$$

□

Lemma 4. *Let B be a ball in Ω such that Ω is star shaped with respect to B and such that its radius $\rho > (\frac{1}{2})\rho_{max}$. Let $Q^m u$ be the Taylor polynomial of order m of u averaged over B where $u \in W_p^m(\Omega)$ and $p \geq 1$. Then*

$$|u - Q^m u|_{W_p^k(\Omega)} \leq C_{m,n,\gamma}\ d^{m-k}|u|_{W_p^m(\Omega)}\ k = 0, 1, .\because, m, \tag{5}$$

where d =diam(Ω) and ρ_{max}=sup $\{\ \rho : \Omega$ is star shaped with respect to a ball of radius $\rho\ \}$

Proof. See the details in [2] □

The important ingredient in the error analysis is a bound on the approximation error $\|u - u_I\|$ where $u_I \in V_h$ is a suitable interpolation which agrees with u at mid point of each edges of ε^h of exact solution u. The interpolation operator is defined at the element level. We just require the local approximation property

$$|u - u_I|_{H^s(K)} \leq Ch_K^{p+1-s}|u|_{H^{p+1}(K)} \quad \forall\ K \in \tau_h\ , s = 0, 1, 2$$

It will be useful to define it explicitly. It is defined in two steps. Let $\hat{K}$ be the

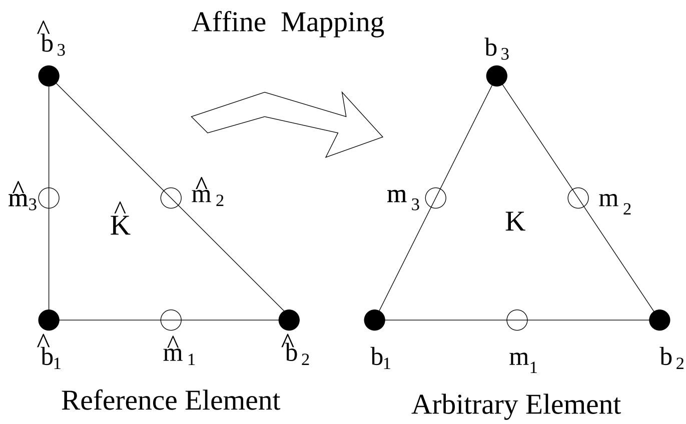

Figure 1

reference triangle with vertices $\hat{b_1}, \hat{b_2}, \hat{b_3}$ whose coordinates are $(0, 0), (1, 0), (0, 1)$ respectively and $\hat{m_i}$ be the midpoint of the side joining i and $i + 1$(modulo 3) vertices. We define

$$\hat{I}^1(\hat{u}) = \hat{u}(\hat{m}_1)\hat{\phi}_4 + \hat{u}(\hat{m}_2)\hat{\phi}_5 + \hat{u}(\hat{m}_3)\hat{\phi}_6$$

$\hat{I}^2(\hat{u}) = (\hat{u}(\hat{b}_1) - \hat{I}^1(\hat{u})(\hat{b}_1))\hat{\phi}_1 + (\hat{u}(\hat{b}_2) - \hat{I}^1(\hat{u})(\hat{b}_2))\hat{\phi}_2 + (\hat{u}(\hat{b}_3) - \hat{I}^1(\hat{u})(\hat{b}_3))\hat{\phi}_3$
Finally we define interpolation as

$$\begin{aligned}\hat{I}(\hat{u}) &= \hat{I}^1(\hat{u}) + \hat{I}^2(\hat{u}) \\ &= \sum_{j=1}^{6} \hat{L}_j(\hat{u})\hat{\phi}_j\end{aligned} \tag{6}$$

where $\hat{L}_i$ for $i = 1, 2, 3, 4, 5, 6$ continuous linear functional.

Now we will show that $P_2(\hat{K})$ is unisolvent with respect to these functionals, i.e. for an arbitrary polynomial $\hat{p} \in P_2(\hat{K})$, $\hat{L}_i(\hat{p}) = 0$ implies $\hat{p} = 0$.

Since $\hat{p} \in P_2(\hat{K})$ implies that $\hat{p}$ can be written as linear combination of basis of $P_2(\hat{K})$. Then

$$\begin{aligned}\hat{p} &= \sum_{i=1}^{6} C_i\phi_i. \\ C_4 &= \hat{p}(\hat{m}_1) = \hat{L}_4(\hat{p}) = 0 \\ C_5 &= \hat{p}(\hat{m}_2) = \hat{L}_5(\hat{p}) = 0 \\ C_6 &= \hat{p}(\hat{m}_3) = \hat{L}_6(\hat{p}) = 0 \\ \Rightarrow \hat{p} &= C_1\hat{\phi}_1 + C_2\hat{\phi}_2 + C_3\hat{\phi}_3\end{aligned}$$

again

$$\begin{aligned}C_1 &= \hat{p}(\hat{b}_1) = \hat{L}_1(\hat{p}) = 0 \\ C_2 &= \hat{p}(\hat{m}_2) = \hat{L}_2(\hat{p}) = 0 \\ C_3 &= \hat{p}(\hat{m}_3) = \hat{L}_3(\hat{p}) = 0 \\ \Rightarrow\ \hat{p} &= 0\end{aligned}$$

Similarly it can be shown that for arbitrary $\hat{p} \in P_2(\hat{K})$, $\hat{I}(\hat{p}) = \hat{p}$. Let $(K, P_2(K), \Sigma)$ be an affine finite element of $(\hat{K}, P_2(\hat{K}), \hat{\Sigma})$ where $\Sigma = \{L_1, L_2, L_3, L_4, L_5, L_6\}$ and $\hat{\Sigma} = \{\hat{L}_1, \hat{L}_2, \hat{L}_3, \hat{L}_4, \hat{L}_5, \hat{L}_6\}$.

Theorem 5. *Let $u \in H^{m+1}(K)$ be an arbitrary element. Then we have*

$$\|D^s(u - I_h u)\|_{L^2(K)} \leq C\ h_K^{m+1-s}\ \|D^{m+1}u\|_{L^2(K)} \tag{7}$$

where $s \leq m + 1$ and $m = 0, 1, 2$ and C is a positive constant does not depend on h_K.

Proof. Using Bramble-Hilbert lemma and properties of affine transformation. □

4. Conclusions

In this paper we have introduced a new finite element method with satisfies all the finite element properties. Also we have shown the interpolation estimates in the standard sobolev norm.

References

[1] P. G. Ciarlet, *The finite element method for elliptic problems*, SIAM, USA, 2002.

[2] S. C. Brenner and Ridgway Scott, *The mathematical theory of finite element methods*, Springer Science & Business Media, USA, 2008.

[3] Fortin, M and Soulie, M, A non-conforming piecewise quadratic finite element on triangles, *Inter. J. Numer. Engg.*,19,(1983),505-520.

[4] Irons, Bruce M and Razzaque, Abdur, Experience with the patch test for convergence of finite elements, *Mathematical Foundations of FEM with Appl to PDE*,557,(1972),587.

[5] Raphal Lger, Jonathan Viquerat, Clment Durochat, Claire Scheid, Stphane Lanteri, A parallel non conforming multi-element DGTD method for the simulation of electromagnetic wave interaction with metallic nanoparticles, *J. Comp. Appl. Math*,270(2014),330-342.

[6] Hassan Fahs, Stphane Lanteri, A high-order non-conforming discontinuous Galerkin method for time-domain electromagnetics, *J. Comp. Appl. Math.*,234,(2010),1088-1096.

[7] Hassan Fahs, High-order discontinuous Galerkin method for time-domain electromagnetics on non-conforming hybrid meshes, *Math Comp Simul.*,107,(2015),134-156.

[8] Antonietti P F, Mazzieri I, Quarteroni A, Rapetti F, Non-conforming high order approximations of the elastodynamics equation, *Comput. Meth. Appl. Mech. Engg*,209-212,(2012),212-238.

[9] Xianmin Xu, Zhiping Li, Non conforming finite element and artificial boundary in multi-atomic young measure approximation for micromagnetics, *Appl. Numer. Math.*,59:5,(2009),920-937.

[10] Erik Burman, Peter Hansbo, A stabilized non-conforming finite element method for incompressible flow, *Comput. Meth. Appl. Mech. Engg*,195,(2006),2882-2899.

[11] Julie Joie, Didier Graebling, Numerical simulation of polymer flows using non-conforming finite elements, *Comput. Fluids*,79,(2013),178-189.

[12] Ignacio Romero, Manfred Bischoff, Incompatible Bubbles: A non-conforming finite element formulation for linear elasticity, *Comput. Meth. Appl. Mech. Engg*,196,(2007),1662-1672.