(\(K, E\))-SOFT UNIFORMITIES AND
L-FUZZY (\(K, E\))-SOFT NEIGHBORHOOD SYSTEMS

Yong Chan Kim\(^1\)\(^\S\), A.A. Ramadan\(^2\)

\(^1\)Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo 210-702, KOREA

\(^2\)Mathematics Department
Faculty of Science
Beni-Suef University
Beni-Suef, EGYPT

Abstract: In this paper, we define (\(K, E\))-soft uniformities in complete residuated lattices. We investigate the relations among (\(K, E\))-soft uniformities, (\(K, E\))-soft topologies and L-fuzzy (\(K, E\))-soft neighborhood systems. We give their examples.

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1. Introduction

Molodtsov [15,16] introduced the theory of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. Maji et al. [12,13] gave the first practical application of soft sets in decision making problems. Many researchers have contributed towards the algebraic structure of soft set theory [1-5,7]. Shabir and Naz [23] introduced the study of soft...

In this paper, we define $(K,E)$-soft uniformities in complete residuated lattices. We investigate the relations among $(K,E)$-soft uniformities, $(K,E)$-soft topologies and $L$-fuzzy $(K,E)$-soft neighborhood systems. We give their examples.

2. Preliminaries

**Definition 1.** [8] A structure $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, 0, 1)$ is a complete lattice where 0 is the bottom element and 1 is the top element;
(L2) $(L, \odot, 1)$ is a commutative monoid;
(L3) adjointness properties,i.e.

$$x \leq y \rightarrow z \text{ iff } x \odot y \leq z.$$ 

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, ^*, 0, 1)$ be a complete residuated lattice.

**Lemma 2.** [8,9] Let $(L, \vee, \wedge, \odot, \rightarrow, ^*, 0, 1)$ be a complete residuated lattice. For each $x, y, z, x_i, y_i \in L$, the following properties hold.

1. If $y \leq z$, then $x \odot y \leq x \odot z$.
2. If $y \leq z$, then $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.
3. $x \rightarrow y = 1$ iff $x \leq y$.
4. $x \rightarrow 1 = 1$ and $1 \rightarrow x = x$.
5. $x \odot y \leq x \wedge y$.
6. $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma} (x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma} (x_i \odot y)$. 
by,

\[ \forall \phi \text{ where } E \]

for each \( f \) and each \( \psi \).

Throughout this paper, \( X \) refers to an initial universe, \( E \) and \( K \) are the sets of all parameters for \( X \), and \( L^X \) is the set of all \( L \)-fuzzy sets on \( X \).

**Definition 3.** [4] A map \( f \) is called an \( L \)-fuzzy soft set on \( X \), where \( f \) is a mapping from \( E \) into \( L^X \), i.e., \( f_e := f(e) \) is an \( L \)-fuzzy set on \( X \), for each \( e \in E \). The family of all \( L \)-fuzzy soft sets on \( X \) is denoted by \((L^X)^E\). Let \( f \) and \( g \) be two \( L \)-fuzzy soft sets on \( X \).

1. \( f \) is an \( L \)-fuzzy soft subset of \( g \) and we write \( f \subseteq g \) if \( f_e \leq g_e \), for each \( e \in E \). \( f \) and \( g \) are equal if \( f \subseteq g \) and \( g \subseteq f \).
2. The intersection of \( f \) and \( g \) is an \( L \)-fuzzy soft set \( h = f \cap g \), where \( h_e = f_e \wedge g_e \), for each \( e \in E \).
3. The union of \( f \) and \( g \) is an \( L \)-fuzzy soft set \( h = f \cup g \), where \( h_e = f_e \vee g_e \), for each \( e \in E \).
4. An \( L \)-fuzzy soft set \( h = f \circ g \) is defined as \( h_e = f_e \circ g_e \), for each \( e \in E \).
5. The complement of an \( L \)-fuzzy soft sets on \( X \) is denoted by \( f^* \), where \( f^* : E \rightarrow L^X \) is a mapping given by \( f_e^* = (f_e)^* \), for each \( e \in E \).
6. \( f \) is called a null \( L \)-fuzzy soft set and is denoted by \( 0_X \), if \( f_e(x) = 0 \), for each \( e \in E \), \( x \in X \).
7. \( f \) is called an absolute \( L \)-fuzzy soft set and is denoted by \( 1_X \), if \( f_e(x) = 1 \), for each \( e \in E \), \( x \in X \) and \((1_X)_e(x) = 1 \).

**Definition 4.** [4] Let \( \varphi : X \rightarrow Y \) and \( \psi : E_1 \rightarrow E_2 \) be two mappings, where \( E_1 \) and \( E_2 \) are parameters sets for the crisp sets \( X \) and \( Y \), respectively. Then \( \varphi_\psi : (L^X)^E_1 \rightarrow (L^Y)^E_2 \) is called a fuzzy soft mapping.

1. For \( f \in (L^X)^E_1 \), the image of \( f \) under the fuzzy soft mapping \( \varphi_\psi \) defined by, \( \forall k \in K, \forall y \in Y \),

\[
\varphi_\psi(f)_{e_2}(y) = \begin{cases} 
\bigvee_{x \in \varphi^{-1}\{y\}}(\bigvee_{(e_1)_{e_1}}(f_{e_1}(x))), & \text{if } x \in \varphi^{-1}\{y\}, e_1 \in \psi^{-1}\{e_2\} \\
0, & \text{otherwise}
\end{cases}
\]

2. For \( g \in (L^X)^E_2 \), the pre-image of \( g \) defined by

\[
\varphi_\psi^{-1}(g)_{e}(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E_1, \forall x \in X.
\]
(3) The soft mapping $\varphi_{\psi} : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ is called injective (resp. surjective, bijective) if $f$ and $\phi$ are both injective (resp. surjective, bijective).

**Lemma 5.** [10] Let $\varphi_{\psi} : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ be a soft mapping. Then we have the following properties. For $f, f_i \in (L^X)^{E_1}$ and $g, g_i \in (L^Y)^{E_2},$

1. $g \sqsupseteq \varphi_{\psi}(\varphi_{\psi}^{-1}(g))$ with equality if $\varphi_{\psi}$ is surjective,
2. $f \sqsubseteq \varphi_{\psi}^{-1}(\varphi_{\psi}(f))$ with equality if $\varphi_{\psi}$ is injective,
3. if $\varphi_{\psi}$ is injective,

$$\varphi_{\psi}(f)_{e_2}(y) = \begin{cases} f_{e_1}(x), & \text{if } x \in \varphi_{\psi}^{-1}(y), e_1 \in \psi^{-1}(e_2) \\ 0, & \text{otherwise} \end{cases}$$

4. $\varphi_{\psi}^{-1}(g^*) = (\varphi_{\psi}^{-1}(g))^*,$
5. $\varphi_{\psi}^{-1}(\bigvee_{i \in I} g_i) = \bigvee_{i \in I} \varphi_{\psi}^{-1}(g_i),$ 
6. $\varphi_{\psi}^{-1}(\bigwedge_{i \in I} g_i) = \bigwedge_{i \in I} \varphi_{\psi}^{-1}(g_i),$ 
7. $\varphi_{\psi}(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} \varphi_{\psi}(f_i),$ 
8. $\varphi_{\psi}(\bigwedge_{i \in I} f_i) \sqsubseteq \bigwedge_{i \in I} \varphi_{\psi}(f_i)$ with equality if $\varphi_{\psi}$ is injective,
9. $\varphi_{\psi}^{-1}(g_1 \circ g_2) = \varphi_{\psi}^{-1}(g_1) \circ \varphi_{\psi}^{-1}(g_2),$ 
10. $\varphi_{\psi}(f_1 \circ f_2) \sqsubseteq \varphi_{\psi}(f_1) \circ \varphi_{\psi}(f_2)$ with equality if $\varphi_{\psi}$ is injective.

**Lemma 6.** [20] Define a binary mapping $S : (L^X)^{E} \times (L^X)^{E} \rightarrow L$ by

$$S(f, g) = \bigwedge_{x \in X} \bigwedge_{e \in E} (f_e(x) \rightarrow g_e(x)) \ \forall \ f, g \in (L^X)^{E}, \ \forall \ e \in E.$$ 

Then $\forall f, g, h, m, n \in (L^X)^{E}$ the following statements hold.

1. $f \sqsubseteq g$ iff $S(f, g) = 1.$
2. If $f \sqsubseteq g,$ then $S(h, f) \leq S(h, g)$ and $S(f, h) \geq S(g, h).$
3. $S(f, h) \circ S(h, g) \leq S(f, g).$ Moreover, $\bigvee_{h \in (L^X)^{E}} (S(f, h) \circ S(h, g)) = S(f, g)$
4. $S(f, g) \circ S(m, n) \leq S(f \circ m, g \circ n).$
5. If $\varphi_{\psi} : (X, E) \rightarrow (Y, F)$ is a fuzzy soft mapping, then $S(p, q) \leq S(\varphi_{\psi}^{-1}(p), \varphi_{\psi}^{-1}(q)),$ for each $p, q \in (L^Y)^{F}.$

**Definition 7.** [11] A set $\tau = \{\tau_k \subset P((L^X)^{E}) \mid k \in K\}$ for each $k \in K$ is called a $(K, E)$-soft topology on $X$ if it satisfies the following conditions for each $k \in K.$

1. $(SO1) \quad 0_X, 1_X \in \tau_k,$
2. $(SO2) \quad \text{If } f, g \in \tau_k, \text{ then } (f \circ g) \in \tau_k.$
3. $(SO3) \quad \text{If } f_i \in \tau_k, \sqcup_{i \in I} f_i \in \tau_k.$
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The pair \((X, \tau)\) is called a \((K, E)\)-soft topological space. Let \((X, \tau^1)\) be a \((K_1, E_1)\)-soft topological space and \((Y, \tau^2)\) be a \((K_2, E_2)\)-soft topological space. Let \(\varphi : X \to Y\), \(\psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) be mappings. Then \(\varphi_\psi,\eta\) from \((X, \tau^1)\) into \((Y, \tau^2)\) is called soft continuous if \(\varphi^{-1}_\psi(f) \in (\tau^2)_k \ \forall f \in (\tau^2)_{\eta(k)}, k \in K_1\).

**Definition 8.** A set \(U = \{U_k \subset P((L^X \times X)^E) \mid k \in K\}\) is called a \((K, E)\)-soft quasi-uniformity on \(X\) iff the following conditions are fulfilled

1. **(QU1)** \(1_{X \times X} \in U_k\),
2. **(QU2)** If \(\alpha \leq u\) and \(v \in U_k\), then \(u \in U_k\),
3. **(QU3)** For every \(u, v \in U_k\), \(u \circ v \in U_k\),
4. **(QU4)** If \(u \in U_k\) then \(1_\Delta \leq u\) where

\[
(1_\Delta)_e(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y, \end{cases}
\]

5. **(QU5)** For each \(u \in U_k\), there exists \(v \in U_k\) such that \(v \circ v \leq u\) where

\[
(v \circ v)_e(x, y) = \bigvee_{z \in X} v_e(x, z) \odot v_e(z, y), \quad \forall x, y \in X, e \in E.
\]

The pair \((X, U)\) is called a \((K, E)\)-soft quasi-uniform space.

A \((K, E)\)-soft quasi-uniformity \(U\) on \(X\) is said to be stratified if

1. **(S)** if \(u \in U_k\), then \(\alpha \circ u \in U_k\).

A \((K, E)\)-soft quasi-uniformity \(U\) on \(X\) is said to be \((K, E)\)-soft uniformity if

1. **(U)** if \(u \in U_k\), then \(u^{-1} \in U_k\) where \((u^{-1})_e(x, y) = u_e(y, x)\).

Let \((X, U^1)\) be a \((K_1, E_1)\)-soft quasi-uniform space and \((Y, U^2)\) be a \((K_2, E_2)\)-soft quasi-uniform space. Let \(\varphi : X \to Y\), \(\psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) be mappings. Then \(\varphi_\psi,\eta\) from \((X, U^1)\) into \((Y, U^2)\) is called soft uniformly continuous if \((\varphi \times \varphi)_\psi^{-1}(v) \in (U^1)_k \ \forall v \in (U^2)_{\eta(k)}, k \in K_1\).

**Definition 9.** [20] An \(L\)-fuzzy \((K, E)\)-soft neighborhood system on \(X\) is a set \(N = \{N^x \mid x \in X\}\) of mappings \(N^x : K \to L^{(L^X)^E}\) such that for each \(k \in K\):

1. **(SN1)** \(N^x_k(1_X) = 1\) and \(N^x_k(0_X) = 0\),
2. **(SN2)** \(N^x_k(f \odot g) \geq N^x_k(f) \odot N^x_k(g)\) for each \(f, g \in (L^X)^E\),
3. **(SN3)** If \(f \geq g\), then \(N^x_k(f) \leq N^x_k(g)\),
4. **(SN4)** \(N^x_k(f) \leq f_e(x)\) for all \(f \in (L^X)^E\) and \(e \in E\).
5. **(SN5)** \(N^x_k(f) \leq \bigvee \{N^y_k(g) \mid g_e(y) \subseteq N^y_k(f), \ \forall y \in X, e \in E\}\).

The previous axiom can be reformulated in the following way
(SN5) \( \forall f \in (L^X)^E \) and \( x \in X \), \( N_k^x(f) \leq N_k^x(N_k^{-}(f)) \), where \( N_k^{-}(f) \in (L^X)^E \) is defined by

\[
(N_k^{-}(f))_e(y) = N_k^y(f) \quad \forall y \in Y, e \in E.
\]

An \( L \)-fuzzy \((K, E)\)-soft neighborhood system is called stratified if

\[
(N_k^{-}(\alpha \circ f)) \geq \alpha \circ N_k^{-}(f) \quad \text{for all} \quad f \in (L^X)^E \quad \text{and} \quad \alpha \in L.
\]

The pair \((X, N)\) is called an \( L \)-fuzzy \((K, E)\)-soft neighborhood space.

Let \((X, N)\) be an \( L \)-fuzzy \((K_1, E_1)\)-soft neighborhood space and \((Y, M)\) be an \( L \)-fuzzy \((K_2, E_2)\)-soft neighborhood space. Let \( \varphi : X \to Y \), \( \psi : E_1 \to E_2 \) and \( \eta : K_1 \to K_2 \) be mappings. Then \( \varphi \psi, \eta \) from \((X, N)\) into \((Y, M)\) is called soft \( N \)-continuous at every \( x \in X \) if \( M_{\eta(k)}(\varphi^{-1}(f)) \leq N_k^x(\varphi^{-1}(f)) \) \( \forall f \in (L^Y)^{E_2}, k \in K_1 \).

**Theorem 10.** Let \((X, \tau)\) be a \((K, E)\)-soft topological space. Define a map \( N_k^\tau : X \to L(L^X)^E \) by

\[
(N_k^\tau(f)) = \bigvee \{ \bigwedge_{e \in E} g_e(x) \mid g \sqsubseteq f, g \in \tau_k \}.
\]

Then the following properties hold.

1. \((X, N^\tau)\) is a \( L \)-fuzzy \((K, E)\)-soft neighborhood space.
2. If \( \tau \) is enriched, then \( N^\tau \) is stratified and

\[
(N_k^\tau(f)) = \bigvee \big( \bigwedge_{g \in \tau} \big( \bigwedge_{e \in E} g_e(x) \circ S(g, f) \big) \big).
\]


**Theorem 11.** Let \((X, U)\) be an \((K, E)\)-soft quasi uniform space. Define two maps \( (rN^U)_k^{-}, (rN^U)_k^{+} : X \to L(L^X)^E \) by

\[
(rN^U)_k^{-}(f) = \bigvee_{u \in U_k} S(u[x], f), \quad \forall f \in (L^X)^E, \ x \in X,
\]

\[
(lN^U)_k^{+}(f) = \bigvee_{u \in U_k} S(u[[x]], f), \quad \forall f \in (L^X)^E, \ x \in X,
\]

where \((u[x])e(y) = u_e(y, x)\) and \((u[[x]])e(y) = u_e(x, y)\).

Then
(1) \((X, rN^U)\) is a stratified \(L\)-fuzzy \((K, E)\)-soft neighborhood space.

(2) \((X, lN^U)\) is a stratified \(L\)-fuzzy \((K, E)\)-soft neighborhood space.

(3) \((rN^U)^{(x)_k}(f) = \bigvee \{ \land_{e \in X} g_e(x) \mid u[g] \subseteq f, u \in U_k \} = \bigvee \{ \land_{e \in X} g_e(x) \circ S(u[g], f) \mid u \in U \} \) where

\[(u[g])_e(x) = u_e[g_e](x) = \bigvee_{y \in X} u_e(x, y) \circ g_e(y),\]

(4) \((lN^U)^{(x)_k}(f) = \bigvee \{ g(x) \mid u[[g]] \subseteq f \mid u \in U \} = \bigvee \{ \land_{e \in X} g_e(x) \circ S(u[[g]], f) \mid u \in U \} \) where

\[(u[[g]])_e(x) = u_e[[g_e]](x) = \bigvee_{y \in X} u_e(y, x) \circ g_e(y),\]

**Proof.** (1) (SN1) For \(u \in U\), by (QU4), \(1_\Delta \subseteq u\). Then

\[(rN^U)^{(x)_k}(0_X) = \bigvee_{u \in U_k} S(u[x], 0_X) \leq \bigvee_{u \in U_k} (u_e(x, x) \to 0) = 0.\]

Hence \((rN^U)^{(x)_k}(0_X) = 0\). Also, \((rN^U)^{(x)_k}(1_X) = 1\), because

\[(rN^U)^{(x)_k}(1_X) \geq \bigwedge_{y \in X} ((1_\Delta)_e(x, y) \to (1_X)_e(y)) = 1.\]

(SN2) By Lemma 6(4), we have

\[(rN^U)^{(x)_k}(f) \circ (rN^U)^{(x)_k}(g) = \bigvee_{u \in U_k} S(u[x], f) \circ \bigvee_{v \in U_k} S(v[x], g) \leq \bigvee_{u \in U_k} S((u \circ v)[x], f \circ g) \leq \bigvee_{w \in U_k} S(w[x], f \circ g) = (rN^U)^{(x)_k}(f \circ g).\]

(SN3) By Lemma 6(3), we have

\[(rN^U)^{(x)_k}(f) = \bigvee_{u \in U_k} S(u[x], f) \leq \bigvee_{u \in U_k} S(u[x], g) = (rN^U)^{(x)_k}(g).\]

(SN4) For \(u \in U\), by (QU4), \(1_\Delta \subseteq u\). We have

\[(rN^{U_k})^{(x)_k}(f) = \bigvee_{u \in U_k} \land_{y \in X} \land_{c \in E} (u_e(y, x) \to f_c(y)) \leq \bigvee_{u \in U_k} (u_e(x, x) \to f_c(x)) \leq f_c(x).\]
\[(SN5)\]
\[
\begin{align*}
(rN^U)_k^x(f) &= \bigvee_{u \in U_k} S(u[x], f) \\
&= \bigvee_{u \in U_k} \bigwedge_{y \in X} \bigwedge_{e \in E} (u_e(y, x) \rightarrow f_e(y)) \\
&\leq \bigvee_{v \in U_k} \bigwedge_{y \in X} \bigwedge_{e \in E} ((v_e \circ v_e)(y, x) \rightarrow f_e(y)) \\
&= \bigvee_{v \in U_k} \bigwedge_{y \in X} \bigwedge_{e \in E} \left( (v_e(z, x) \circ v_e(y, z)) \rightarrow f_e(y) \right) \\
&= \bigvee_{v \in U_k} \bigwedge_{y \in X} \bigwedge_{z \in X} (v_e(z, x) \rightarrow (v_e(y, z) \rightarrow f_e(y)) \\
&= \bigvee_{v \in U_k} \bigwedge_{z \in X} (v_e(z, x) \rightarrow \bigwedge_{y \in X} (v_e(y, z) \rightarrow f_e(y)).
\end{align*}
\]

Put \( g_e(z) = \bigwedge_{y \in X} (v_e(y, z) \rightarrow f_e(y)) \). For all \( g_e(z) \leq (rN^U)_k^x(f) \) for each \( z \in X, e \in E, \bigwedge_{e \in E} g_e(x) \leq (rN^U)_k^x(f) \). Thus,
\[
\begin{align*}
(rN^U)_k^x(f) \\
&\leq \bigvee_{v \in U_k} \{ \bigwedge_{z \in X} \bigwedge_{e \in E} (v_e(z, x) \rightarrow g_e(z)) \mid g_e(z) \leq (rN^U)_k^x(f) \} \\
&\leq \bigvee_{v} \{ (rN^U)_k^x(g) \mid g_e(z) \leq (rN^U)_k^x(f) \}.
\end{align*}
\]

Thus, \((X, rN^U)\) is an \( L\)-fuzzy \((K, E)\)-soft neighborhood space.

Since \( \alpha \circ (u[x])_e(y) \circ S(u[x], f) \leq \alpha \circ (u[x])_e(y) \circ ((u[x])_e(y) \rightarrow f_e(y)) \leq \alpha \circ f_e(y) \), we have
\[
\alpha \circ S(u[x], f) \leq S(u[x], \alpha \circ f).
\]

Thus, \( rN^U \) is stratified from:
\[
\begin{align*}
\alpha \circ (rN^U)_k^x(f) &= \alpha \circ \bigvee_{u \in U_k} S(u[x], f) = \bigvee_{u \in U_k} (\alpha \circ S(u[x], f)) \\
&\leq \bigvee_{u \in U_k} (S(u[x], \alpha \circ f)) = (rN^U)_k^x(\alpha \circ f).
\end{align*}
\]

(2) It is similarly proved as (1).

(3) Put \( \gamma = \bigvee \{ \bigwedge_{e \in X} g_e(x) \mid u[g] \subseteq f, u \in U_k \} \). We show that \( (rN^U)_k^x(f) = \gamma \) from the following statements.

Let \( g_e(y) = \bigwedge_{x \in X} (u_e(x, y) \rightarrow f_e(x)) \). Then
\[
\begin{align*}
u_e[g_e](z) &= \bigvee_{y \in X} (u_e(z, y) \circ g_e(y)) \\
&= \bigvee_{y \in X} (u_e(z, y) \circ (\bigwedge_{x \in X} (u_e(x, y) \rightarrow f_e(x)))) \\
&\leq \bigvee_{y \in X} (u_e(z, y) \circ (u_e(z, y) \rightarrow f_e(z))) \leq f_e(z).
\end{align*}
\]

Hence \( (rN^U)_k^x(f) \leq \gamma \).

Let \( u_e[g_e](z) = \bigvee_{y \in X} (u_e(z, y) \circ g_e(y)) \leq f_e(z) \). Then
\[
g_e(y) \leq \bigwedge_{z \in X} (u_e(z, y) \rightarrow f_e(z)).
\]
Hence \((rN^U)^x_k(f) \geq \gamma\).

Put \(\delta = \sqrt{\{(\bigwedge_{e \in E} g_e(x)) \odot S(u[g], f) \mid u \in U_k\}}\). We show that \(\delta = \gamma\) from the following statements.

Let \(g \in (L^X)^E\) with \(u[g] \leq f\) and \(u \in U_k\). Then \(S(u[g], f) = 1\). Hence \((\bigwedge_{e \in E} g_e(x)) \odot S(u[g], f) = (\bigwedge_{e \in E} g_e(x)) \leq \delta(x)\). So, \(\gamma \leq \delta\).

Let \((\bigwedge_{e \in E} g_e) \odot S(u[g], f)\) with \(u \in U\). Since

\[
\begin{align*}
\mu_e[(\bigwedge_{e \in E} g_e) \odot S(u[g], f)](x) &= \bigvee_{y \in X}((\mu_e(x, y) \odot (\bigwedge_{e \in E} g_e(y))) \odot S(u[g], f)) \\
&\leq \mu_e[g_e](x) \odot S(u[g], f) \leq f_e(x)
\end{align*}
\]

we have \(\mu_e[(\bigwedge_{e \in E} g_e) \odot S(u[g], f)] \leq f_e\). Then \(\bigwedge_{e \in E} g_e(x) \odot S(u[g], f) \leq \gamma\). Thus, \(\delta = \gamma\).

**Theorem 12.** Let \((X, U)\) be a \((K, E)\)-soft quasi-uniform space, \((X, rN^U)\) and \((X, lN^U)\) \(L\)-fuzzy \((K, E)\)-soft neighborhood spaces. Define \((\tau^r_U)_k, (\tau^l_U)_k \subset (L^X)^E\) as follows

\[
(\tau^r_U)_k = \{f \in (L^X)^E \mid f_e(x) = (rN^U)^x_k(f), \forall x \in X, e \in E\},
\]

\[
(\tau^l_U)_k = \{f \in (L^X)^E \mid f_e(x) = (lN^U)^x_k(f), \forall x \in X, e \in E\}.
\]

Then,

1. \((\tau^r_U)_k = \{(\tau^r_U)_k \mid k \in K\}\) is an enriched \((K, E)\)-soft topology on \(X\).
2. \((\tau^l_U)_k = \{(\tau^l_U)_k \mid k \in K\}\) is an enriched \((K, E)\)-soft topology on \(X\).
3. \(rN^U = N^r_U\).
4. \(lN^U = N^l_U\).

**Proof.** (1) (SO1) Since \((rN^U)^x_k(1_X) = 1\) and \((rN^U)^x_k(0_X) = 0\), we have \(1_X, 0_X \in (\tau^r_U)_k\).

(SO2) Let \(f, g \in (\tau^r_U)_k\) with \((rN^U)^x_k(f) = f_e(x)\) and \((rN^U)^x_k(g) = g_e(x)\). Since \((rN^U)^x_k(f \odot g) \geq (rN^U)^x_k(f) \odot (rN^U)^x_k(g) = (f \odot g)_e(x)\) and (SN4), then \(f \odot g \in (\tau^r_U)_k\).

(SO3) Let \(i_1 \in (\tau^r_U)_k\) for all \(i \in \Gamma\). Since \((rN^U)^x_k(\bigcup_{i \in \Gamma} f_i) = \bigcup_{i \in \Gamma} (rN^U)^x_k(f_i) = (\tau^r_U)_k\), then \(\bigcup_{i \in \Gamma} f_i \in (\tau^r_U)_k\).

(R) Let \(f \in (\tau^r_U)_k\). Since \((rN^U)^x_k(\alpha \odot f) \geq \alpha \odot (rN^U)^x_k(f) = \alpha \odot f_e(x)\) and (SN4), then \(\alpha \odot f \in (\tau^r_U)_k\).

(2) It is similarly proved as (1).

(3) Since \((rN^U)^x_k(f) \leq (rN^U)^x_k((rN^U)^x_k(f)) \leq (rN^U)^x_k(f)\) from (SN3) and (SN5), \((rN^U)^x_k(f) = (rN^U)^x_k((rN^U)^x_k(f))\) for all \(x \in X\). Since \((rN^U)^x_k(f) \in \tau^r_U\), by the definition of \(N\tau^r_U\), \((rN^U)^x_k(f) \leq (\tau^r_U)^x_k(f)\).
Since \((N^\tau U)^c_k(f) = \bigvee \{ \wedge_{e \in E} (g_i)_e(x) \mid g_i \subseteq f, g_i \in (\tau^c_{U})_k \}\) and \(\wedge_{e \in E} (g_i)_e(x) = (rN^U)^c_k(g_i)\), then

\[
\bigvee_i (g_i)_e(x) = \bigvee_i (rN^U)^c_k(g_i) \leq (rN^U)^c_k((N^\tau U)^c_k(f)) = (rN^U)^c_k(\bigcup_i g_i) \leq \bigvee_i (g_i)_e(x).
\]

Hence \((rN^U)^c_k((N^\tau U)^c_k(f)) = (N^\tau U)^c_k(f)\). Since \((N^\tau U)^c_k(f) \leq f(x)\) for all \(e \in E\), by (SN3), \((N^\tau U)^c_k(f) = (rN^U)^c_k(N^\tau U_k(f)) \leq (rN^U)^c_k(f)\). So, \(rN^U = N^\tau U\).

(4) It is similarly proved as (3).

**Theorem 13.** Let \((X, U)\) be a \((K_1, E_1)\)-soft quasi-uniform space and \((Y, U)\) be a \((K_2, E_2)\)-soft quasi-uniform space. Let \(\varphi : X \to Y\), \(\psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) be mappings. If \(\varphi;\eta : (X, U) \to (Y, V)\) is soft uniformly continuous, then

1. \(\varphi;\eta : (X, rN^U) \to (Y, rN^V)\) is soft \(N\)-continuous.
2. \(\varphi;\eta : (X, lN^U) \to (Y, lN^V)\) is soft \(N\)-continuous.
3. \(\varphi;\eta : (X, \tau^\tau U) \to (Y, \tau^\tau V)\) is soft continuous.
4. \(\varphi;\eta : (X, \tau^\tau U) \to (Y, \tau^\tau V)\) is soft continuous.

**Proof.** (1) First we show that \(\varphi;\eta^{-1}((v[\varphi(x)]_\psi(e)) = ((\varphi \times \varphi)^{-1}_\psi(v)[x])_e\) from

\[
\varphi^{-1}_\psi((v[\varphi(x)]_\psi(e))(z) = (v[\varphi(x)]_\psi(e)(\varphi(z)) = v_\psi(e)(\varphi(z), \varphi(x))
= (\varphi \times \varphi)^{-1}_\psi(v_\psi(e))(z, x) = ((\varphi \times \varphi)^{-1}_\psi(v)[x])_e(z).
\]

Thus we have

\[
S(v[\varphi(x)], f) = \wedge_{y \in Y} \wedge_{e_2 \in E_2} (v[\varphi(x)]_e_2(y) \to f_{e_2}(y)) \leq \wedge_{z \in X} \wedge_{e_1 \in E_1} (v[\varphi(x)]_e_1(z) \to f_{e_1}(\varphi(z)))
= \wedge_{z \in X} \wedge_{e_1 \in E_1} (\varphi^{-1}_\psi((v[\varphi(x)]_e_1)(z) \to \varphi^{-1}_\psi(f_{e_1}(z)))
= \wedge_{z \in X} \wedge_{e_1 \in E_1} ((\varphi \times \varphi)^{-1}_\psi(v)[x]_e_1(z) \to (\varphi \times \varphi)^{-1}_\psi(f)[x]_e_1(z))
= S(((\varphi \times \varphi)^{-1}_\psi(v)[x], \varphi^{-1}_\psi(f)).
\]

\[
(rN^V)^\varphi(x)_k(f) = \bigvee_{v \in V_{\eta(k)}} S(v[\varphi(x)], f) \leq \bigvee_{v \in V_{\eta(k)}} S(((\varphi \times \varphi)^{-1}_\psi(v)[x], \varphi^{-1}_\psi(f)) \leq \bigvee_{((\varphi \times \varphi)^{-1}_\psi(v)[x] \varphi^{-1}_\psi(f)) \leq (rN^U)^c_k(\varphi^{-1}_\psi(f)).
\]

(2) It is similarly proved as (1).
(3) Let $f \in (\tau^\top V)_{\eta(k)}(f)$. Then $f_{\psi(e)}(\varphi(x)) = (rN^V)^{\varphi(x)}_{\eta(k)}(f)$. Then $\varphi\psi^{-1}(f)_{\tau(e)}(x) = \varphi\psi^{-1}((rN^V)^{\varphi(x)}_{\eta(k)}(f))_{\tau(e)}(x)$. Since $(rN^V)^{\varphi(x)}_{\eta(k)}(f) \leq (rN^U)^{\varphi\psi^{-1}(f)}_{\eta(k)}(f)$,

$$\varphi\psi^{-1}(f)_{\tau(e)}(x) = \varphi\psi^{-1}((rN^V)^{\varphi(x)}_{\eta(k)}(f))_{\tau(e)}(x) = (rN^V)^{\varphi(x)}_{\eta(k)}(f) \leq (rN^U)^{\varphi\psi^{-1}(f)}_{\eta(k)}(f).$$

By (SN3), $\varphi\psi^{-1}(f) = (rN^U)^{\varphi\psi^{-1}(f)}_{k}$. Hence $\varphi\psi^{-1}(f) \in (\tau^\top U)_k$.

(4) It is similarly proved as (3).

**Example 14.** Let $X = \{h_i \mid i \in \{1, 2, 3\}\}$ with $h_i =$ house and $E = \{e, b\}$ with $e =$ expensive, $b =$ beautiful. Let $(L = [0, 1], \odot = \land, \rightarrow, 0, 1)$ be a complete residuated lattice defined by

$$x \odot y = x \land y, \quad x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Put $f \in (L^X)^E$ such that

$$f_e(h_1) = 0.3, f_e(h_2) = 0.5, f_e(h_3) = 0.3$$
$$f_b(h_1) = 0.7, f_b(h_2) = 0.9, f_b(h_3) = 0.4$$

Put $K = \{k_1, K_2\}$ and $w, v \in (L^{X \times X})^E$ such that

$$w_e = \begin{pmatrix} 1 & 0.3 & 0.5 \\ 0.6 & 1 & 0.7 \\ 0.5 & 0.3 & 1 \end{pmatrix}, \quad w_b = \begin{pmatrix} 1 & 0.5 & 0.4 \\ 0.5 & 1 & 0.4 \\ 0.4 & 0.4 & 1 \end{pmatrix},$$

$$v_e = \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.5 & 1 \end{pmatrix}, \quad v_b = \begin{pmatrix} 1 & 0.5 & 0.4 \\ 0.7 & 1 & 0.4 \\ 0.6 & 0.5 & 1 \end{pmatrix}.$$  

Define $U_{k_1} = \{u \in (L^{X \times X})^E \mid u \geq w\}$ and $U_{k_2} = \{u \in (L^{X \times X})^E \mid u \geq v\}$.

(1) Since $w_e \circ w_e = w_e$ and $v_e \circ v_e = v_e$ for all $e \in E$, $U = \{U_{k_1}, U_{k_2}\}$ is a $(K, E)$-soft quasi-uniformity on $X$.

(2) Since $(rN^U)^x_{k}(f) = \bigvee_{u \in U_k} S(u[x], f)$, we have

$$(rN^U)^{h_1}_{k_1}(f) = \bigvee_{u \in U_{k_1}} S(u[h_1], f) = f_e(h_1) \land (0.6 \rightarrow f_e(h_2))$$
$$(0.5 \rightarrow f_e(h_3)) \land f_b(h_1) \land (0.5 \rightarrow f_b(h_2)) \land (0.4 \rightarrow f_b(h_3)),$$

$$(rN^U)^{h_2}_{k_1}(f) = \bigvee_{u \in U_{k_1}} S(u[h_2], f) = (0.3 \rightarrow f_e(h_1)) \land f_e(h_2)$$
$$(0.3 \rightarrow f_e(h_3)) \land (0.5 \rightarrow f_b(h_1)) \land f_b(h_2) \land (0.4 \rightarrow f_b(h_3)),$$

$$(rN^U)^{h_3}_{k_1}(f) = \bigvee_{u \in U_{k_1}} S(u[h_3], f) = (0.5 \rightarrow f_e(h_1)) \land (0.7 \rightarrow f_e(h_2))$$
$$\land f_e(h_3) \land (0.4 \rightarrow f_b(h_1)) \land (0.4 \rightarrow f_b(h_2)) \land f_b(h_3).$$
Then \((r^U_{N})_{k_1}^1(f) = 0.3, (r^U_{N})_{k_1}^2(f) = 0.5, (r^U_{N})_{k_1}^3(f) = 0.3\).

\((r^U_{N})_{k_2}^1(f) = \bigvee_{u \in U_{k_2}} S(u[h_1], f) = f_e(h_1) \land (0.4 \rightarrow f_e(h_2))
\land (0.5 \rightarrow f_e(h_3)) \land f_b(h_1) \land (0.7 \rightarrow f_b(h_2)) \land (0.6 \rightarrow f_b(h_3)),
\( (r^U_{N})_{k_2}^2(f) = \bigvee_{u \in U_{k_2}} S(u[h_2], f) = (0.6 \rightarrow f_e(h_1)) \land f_e(h_2)
\land (0.5 \rightarrow f_e(h_3)) \land (0.5 \rightarrow f_b(h_1)) \land f_b(h_2) \land (0.5 \rightarrow f_b(h_3)),
\( (r^U_{N})_{k_2}^3(f) = \bigvee_{u \in U_{k_2}} S(u[h_3], f) = (0.8 \rightarrow f_e(h_1)) \land (0.4 \rightarrow f_e(h_2))
\land f_e(h_3) \land (0.4 \rightarrow f_b(h_1)) \land (0.4 \rightarrow f_b(h_2)) \land f_b(h_3).

Then \((r^U_{N})_{k_2}^1(f) = 0.3, (r^U_{N})_{k_2}^2(f) = 0.3, (r^U_{N})_{k_2}^3(f) = 0.3\).

(3) Since \((L^{U}_{N})_{k}^x(f) = \bigvee_{u \in U} S(u[[x]], f)\), we have

\[(L^{U}_{N})_{k_1}^1(f) = \bigvee_{u \in U_{k_1}} S(u[[h_1]], f) = f_e(h_1) \land (0.3 \rightarrow f_e(h_2))
\land (0.5 \rightarrow f_e(h_3)) \land f_b(h_1) \land (0.5 \rightarrow f_b(h_2)) \land (0.4 \rightarrow f_b(h_3)),
\( (L^{U}_{N})_{k_1}^2(f) = \bigvee_{u \in U_{k_1}} S(u[h_2], f) = (0.6 \rightarrow f_e(h_1)) \land f_e(h_2)
\land (0.7 \rightarrow f_e(h_3)) \land (0.5 \rightarrow f_b(h_1)) \land f_b(h_2) \land (0.4 \rightarrow f_b(h_3)),
\( (L^{U}_{N})_{k_1}^3(f) = \bigvee_{u \in U_{k_1}} S(u[h_3], f) = (0.5 \rightarrow f_e(h_1)) \land (0.3 \rightarrow f_e(h_2))
\land f_e(h_3) \land (0.4 \rightarrow f_b(h_1)) \land (0.4 \rightarrow f_b(h_2)) \land f_b(h_3).

Then \((L^{U}_{N})_{k_1}^1(f) = 0.3, (L^{U}_{N})_{k_1}^2(f) = 0.3, (L^{U}_{N})_{k_1}^3(f) = 0.3\).

(4) Since \((\tau_{U}^r)_{k} = \{ f \in (L^{X})^E \mid f_e(x) = (r^U_{N})_{k}^x(f), \forall x \in X, e \in E\}\) from

Theorem 12, \(f_e = f_b\), we have

\[ f \in (\tau_{U}^r)_{k_1} \text{ iff } \begin{cases} f = \alpha_X, \\ f_e(h_1) \leq 0.6 \rightarrow f_e(h_2), f_e(h_1) \leq 0.5 \rightarrow f_e(h_3), \\ f_e(h_2) \leq 0.5 \rightarrow f_e(h_1), f_e(h_2) \leq 0.4 \rightarrow f_e(h_3), \\ f_e(h_3) \leq 0.5 \rightarrow f_e(h_1), f_e(h_3) \leq 0.7 \rightarrow f_e(h_3), \end{cases} \]
(5) Since \((τ^U_k)_k = \{ f \in (L^X)^E \mid f_e(x) = (\text{In}U)_k^e(f), \forall x \in X, e \in E \}\) from Theorem 12, \(f_e = f_b\), we have

\[
\begin{align*}
  f \in (τ^U_k)_k & \quad \text{iff} \quad f = \alpha_X, \\
  f_e(h_1) & \leq 0.5 \rightarrow f_e(h_2), f_e(h_1) \leq 0.5 \rightarrow f_e(h_3), \\
  f_e(h_2) & \leq 0.7 \rightarrow f_e(h_1), f_e(h_2) \leq 0.7 \rightarrow f_e(h_3), \\
  f_e(h_3) & \leq 0.8 \rightarrow f_e(h_1), f_e(h_3) \leq 0.5 \rightarrow f_e(h_3),
\end{align*}
\]

Put \((g_e = g_b)(h_1) = 0.5, (g_e = g_b)(h_2) = 0.8, (g_e = g_b)(h_3) = 0.6\). Then \(g \in (τ^U_k)_k\) but \(g \notin (τ^U_k)_k\) because

\[
f_e(h_3) = 0.6 \leq 0.8 \rightarrow f_e(h_3) = 0.5.
\]

Put \((h_e = h_b)(h_1) = 0.8, (h_e = h_b)(h_2) = 0.5, (h_e = h_b)(h_3) = 0.9\). Then \(h \in (τ^U_k)_k\) but \(h \notin (τ^U_k)_k\) because

\[
f_e(h_1) = 0.8 \leq 0.6 \rightarrow f_e(h_2) = 0.5.
\]

References


