

**INFLUENCE OF AN INTERNAL HEAT SOURCE OR SINK
ON THE MAGNETOCONVECTION OF A MICROPOLAR
FLUID IN A VERTICAL CHANNEL**

A. Borrelli^{1 §}, G. Giantesio², M.C. Patria³

^{1,3}Dipartimento di Matematica e Informatica

Università di Ferrara

via Machiavelli 30, 44121 Ferrara, ITALY

²Dipartimento di Matematica e Fisica

Università Cattolica del Sacro Cuore

via Musei 41, 25121 Brescia, ITALY

Abstract: This work examines the effects of an external uniform magnetic field and of an internal heat source or sink on the steady mixed convection in the fully developed flow of a micropolar fluid filling a vertical channel under the Oberbeck-Boussinesq approximation. The two boundaries are kept at different uniform temperatures. The velocity, the microrotation, the temperature and the induced magnetic field are analytically obtained. A selected set of pictures and Tables are reported in order to illustrate the influence of the internal heat parameter on the trend of the motion and in particular on the reverse flow.

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Key Words: micropolar fluids, MHD fully developed flow, mixed convection, Boussinesq approximation, vertical channel, heat source/sink

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[§]Correspondence author

Nomenclature

C	constant such that $P = -Cx_1 + p_0$
c_0, c_d, c_a	angular viscosity coefficients
$2d$	channel width
\mathbf{E}	electric field
$\mathbf{g} = -g\mathbf{e}_1$	gravity acceleration
Gr	Grashof number defined by (5) ₉
\mathbf{H}	total magnetic field
$h(y)$	dimensionless function describing the induced magnetic field defined by (5) ₁₅
$H_0\mathbf{e}_2$	external uniform magnetic field ($H_0 > 0$)
$H_1(x_2)$	induced magnetic field component in the x_1 -direction
I	microinertia coefficient
k	fluid thermal conductivity
$k_{1,2}$	heat transfer coefficients evaluated at $\Pi_{1,2}$
l	characteristic length defined by (5) ₂
L	dimensionless constant defined by (5) ₃
M^2	Hartmann number defined by (5) ₅
M_p^2	micropolar parameter defined by (5) ₄
N^2	coupling number defined by (5) ₁ ($0 < N^2 < 1$)
Nu	Nusselt number
p	pressure
$P = p + \mu_e \frac{H_1^2}{2} + \rho_0 g x_1$	difference between the hydromagnetic pressure and the hydrostatic pressure
p_0	arbitrary constant
Q	constant rate of volumetric internal heat generation ($Q > 0$)
r^2	internal heat parameter defined by (5) ₁₆
Re	Reynolds number defined by (5) ₈
$T = T(x_2)$	temperature

T_0	reference temperature
T_1, T_2	uniform temperatures ($T_2 > T_1$)
\mathbf{v}	velocity field
$v(y)$	dimensionless function describing the velocity defined by (5) ₁₂
V_0	characteristic velocity defined by (5) ₆
$v_1(x_2)$	velocity component in the x_1 -direction
\mathbf{w}	microrotation field
$w(y)$	dimensionless function describing the microrotation defined by (5) ₁₃
$w_3(x_2)$	microrotation component in the x_3 -direction
y	dimensionless transverse coordinate defined by (5) ₁₁

Greek symbols

α, β, γ	dimensionless constants defined by (10)
α_T	thermal expansion coefficient
η_e	magnetic diffusivity $\left(\eta_e = \frac{1}{\mu_e \sigma_e}\right)$
$\vartheta(y)$	dimensionless temperature defined by (5) ₁₄
λ	buoyancy coefficient defined by (5) ₁₀
μ	Newtonian viscosity coefficient ($\mu > 0$)
μ_e	magnetic permeability
μ_r	dynamic microrotation viscosity coefficient
ν_0	constant defined by (5) ₇
ρ_0	mass density at the temperature T_0
σ_e	electrical conductivity
$\tau_{1,2}$	skin friction at the plates $\Pi_{1,2}$
$\tau_{p1,2}$	skin couple friction at the plates $\Pi_{1,2}$

1. Introduction

The aim of this paper is to extend the results obtained in [1] when an internal heat source or sink is present. More precisely, we study the mixed magnetoconvection in the fully developed flow of a micropolar fluid filling a vertical channel under the Oberbeck-Boussinesq approximation ([2]). About the theory of micropolar fluids we refer to [3], [4] and [5].

The magnetoconvection in a channel has many relevant applications: geothermal reservoirs, cooling of nuclear reactors, electric transmission cables, thermal insulation, electronic chips, semiconductor wafers, etc.

In our study we solve the problem of the steady mixed convection of a Boussinesquian electrically conducting micropolar fluid in a vertical channel under the action of an external uniform magnetic field normal to the direction of the velocity and to the channel. The walls are kept at constant temperatures T_1 and T_2 ($T_1 < T_2$). The internal heat source or sink here considered depends only on the temperature difference $T - T_0$ and so it is not known a priori. This kind of source/sink occurs in exothermic/endothermic chemical and biochemical processes. There are many papers in the literature concerning such internal heat generation or absorption in several different physical situations ([6], [7], [8], etc.).

The first paper on the fully developed free convection of a micropolar fluid in a vertical channel is [9]; this work has been generalized in [10] in order to consider also the mass transfer. In [11], [12] mixed convection flow with symmetric and asymmetric heating is examined in the presence of heat source-sink.

As far as the influence of an external magnetic field on the convective flow of a micropolar fluid in a vertical channel is concerned, the most exhaustive paper is [1], where there is studied the case in the absence of an internal heat source/sink.

In the present paper we neglect the dissipation terms in the energy equation, as it is usual in the Oberbeck-Boussinesq approximation ([13]), so that we obtain the explicit solution of the problem which takes into account the induced magnetic field. We point out that the induced magnetic field is neglected in most of the works concerning the convective flow in a vertical channel, also in the simpler case of a Newtonian fluid.

The paper is organized in this way:

In Section 2 we formulate the problem from the physical point of view and Section 3 is devoted to solve it in the presence of an internal heat source. We first determine the temperature by solving a boundary value problem which is well-set only if the internal heat generation parameter r^2 does not take particular values. Then, in order to find the analytical expression of the velocity, microrotation and induced magnetic field, we have to examine three cases (as in [1]) related to the strength of the external uniform magnetic field. Finally, we integrate the boundary value problem which describes the motion in the three cases.

In Section 4 we solve the same problem when an internal heat sink is present. In order to find the analytical expression of the velocity, microrotation and induced magnetic field, we have to distinguish if the square root of the internal heat generation parameter is not a root, is a simple root or is a double root of the characteristic equation and we furnish the solutions in all cases.

In Section 5 we make some comments about the flow and we compute the Nusselt numbers, the skin and skin couple frictions. We also furnish the flow in the absence of external magnetic field and in the Newtonian case.

In Section 6 we give some pictures in order to show the influence of r^2 on the flow and we point out some differences about the trend of the flow between the case of heat generation and the case of heat absorption. We refer to [1] for the complete study of the influence of the other material parameters on the flow; as in [1] and [15], for suitable values of the buoyancy parameter λ , the reverse flow occurs. Precisely, there exists a critical value λ^* such that

- $\lambda > 0$: if $\lambda > \lambda^*$, then the reverse flow occurs near the coldest wall;
- $\lambda < 0$: if $\lambda < \lambda^*$, then the reverse flow occurs near the hottest wall.

The value of λ^* decreases as r increases if there is a source (the decreasing is very significant when $r = 3$), λ^* increases as r increases if there is a sink so that we can conclude that the presence of a source (sink) tends to favor (prevent) the occurrence of the reverse flow. As far as we know, the influence of r^2 on the reverse flow has not been studied before in the literature.

If the buoyancy parameter vanishes (symmetric heating), then the phenomenon of the reverse flow does not appear.

2. Basic Equations

In this Section we briefly formulate the problem. The Boussinesquian, electrically conducting micropolar fluid fills the region \mathcal{S} between two infinite rigid, fixed, non-electrically conducting vertical plates Π_1, Π_2 separated by a distance $2d$ (Figure 1). We assume the region outside the plane to be a vacuum (free space). The coordinate axes are fixed in order to have

$$\begin{aligned} \mathcal{S} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, x_2 \in (-d, d)\}, \\ \Pi_i &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, x_2 = (-1)^i d\}, \quad i = 1, 2 \end{aligned} \quad (1)$$

and x_1 -axis is vertical upward.

Our aim is to study the steady mixed convection in the fully developed flow of the fluid under the action of an external uniform magnetic field $H_0 \mathbf{e}_2$ normal to planes $\Pi_{1,2}$ ($H_0 > 0$) and in the presence of an internal heat source or sink neglecting the viscous and ohmic dissipation terms.

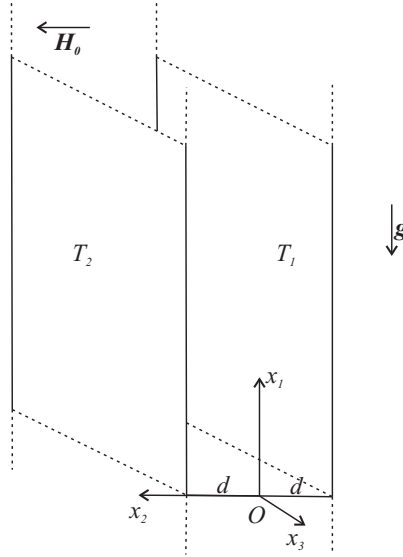


Figure 1: Physical configuration and coordinate system ($T_2 > T_1$).

This flow in the absence of external mechanical body forces, body couples and free electric charges under the Oberbeck-Boussinesq approximation is governed by the following equations ([5])

$$\begin{aligned}
 \rho_0 \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla P + (\mu + \mu_r) \Delta \mathbf{v} + 2\mu_r (\nabla \times \mathbf{w}) + \mu_e (\nabla \times \mathbf{H}) \times \mathbf{H} \\
 &\quad - \rho_0 \alpha_T (T - T_0) \mathbf{g}, \\
 \rho_0 I \mathbf{v} \cdot \nabla \mathbf{w} &= (c_a + c_d) \Delta \mathbf{w} + (c_0 + c_d - c_a) \nabla (\nabla \cdot \mathbf{w}) + 2\mu_r (\nabla \times \mathbf{v} - 2\mathbf{w}), \\
 \nabla \cdot \mathbf{v} &= 0, \\
 \eta_e \Delta \mathbf{H} &= \nabla \times (\mathbf{H} \times \mathbf{v}), \\
 \nabla \cdot \mathbf{H} &= 0, \\
 \rho_0 c_v \nabla T \cdot \mathbf{v} &= k \Delta T \pm Q (T - T_0) \quad \text{in } \mathcal{S}. \tag{2}
 \end{aligned}$$

The sign + in front of Q denotes heat generation whereas the sign - represents heat absorption.

All the material parameters are positive constants and μ_e is equal to the magnetic permeability of free space. As it is usual in the Boussinesq approximation ([13]), in equation (2)₆ the dissipative terms have been neglected.

We search $\mathbf{v}, \mathbf{w}, \mathbf{H}, T$ in the following form:

$$\mathbf{v} = v_1(x_2)\mathbf{e}_1, \quad \mathbf{w} = w_3(x_2)\mathbf{e}_3, \quad \mathbf{H} = H_0\mathbf{e}_2 + H_1(x_2)\mathbf{e}_1, \quad T = T(x_2), \quad (3)$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical base of \mathbb{R}^3 . Thank to (3), $\mathbf{v}, \mathbf{w}, \mathbf{H}$ are divergence free.

The previous unknown functions satisfy the boundary conditions

$$v_1(\pm d) = 0, \quad w_3(\pm d) = 0, \quad H_1(\pm d) = 0, \quad T(-d) = T_1, \quad T(d) = T_2. \quad (4)$$

We choice the reference temperature $T_0 = \frac{T_1 + T_2}{2}$ and we assume $T_2 > T_1$.

By virtue of (3) and (2)₁, we deduce $P = P(x_1) = -Cx_1 + p_0$ (C, p_0 some constants). By putting

$$\begin{aligned} N^2 &= \frac{\mu_r}{\mu + \mu_r}, \quad l^2 = \frac{c_d + c_a}{4\mu}, \quad L = \frac{d}{l}, \quad M_p^2 = N^2 L^2, \quad M^2 = \frac{\sigma_e}{\mu} \mu_e^2 H_0^2 d^2, \\ V_0 &= \frac{Cd^2}{\mu}, \quad \nu_0 = \frac{\mu + \mu_r}{\rho_0}, \quad Re = \frac{V_0 d}{\nu_0}, \\ Gr &= \frac{\alpha_T g (T_2 - T_1) d^3}{\nu_0^2}, \quad \lambda = \frac{Gr}{Re}, \quad y = \frac{x_2}{d}, \\ v(y) &= \frac{v_1(dy)}{V_0}, \quad w(y) = \frac{dw_3(dy)}{V_0}, \quad \vartheta(y) = \frac{T(dy) - T_0}{T_2 - T_1}, \quad h(y) = \frac{H_1(dy)}{V_0 \sqrt{\sigma_e \mu}}, \\ r^2 &= \frac{Qd^2}{k}, \end{aligned} \quad (5)$$

equations (2) written in dimensionless form reduce to:

$$\begin{aligned} v'' + 2N^2 w' + M(1 - N^2)h' + \lambda \vartheta + 1 - N^2 &= 0, \\ w'' - \frac{M_p^2}{2(1 - N^2)}v' - \frac{M_p^2}{1 - N^2}w &= 0, \\ h'' + Mv' &= 0, \\ \vartheta'' \pm r^2 \vartheta &= 0 \quad \text{in } [-1, 1]. \end{aligned} \quad (6)$$

We recall that the coupling number N^2 belongs to $(0, 1)$.

Boundary conditions (4) in dimensionless form become

$$v(\pm 1) = 0, \quad w(\pm 1) = 0, \quad h(\pm 1) = 0, \quad \vartheta(\pm 1) = \pm \frac{1}{2}. \quad (7)$$

3. Flow in the Case of Internal Heat Source

We begin by considering the case of an internal heat source, which means to have the sign + in (6)₄. The integration of this equation taking into account (7)₄ shows that the solution does not exist if r takes the values $n\pi$ where $n = 1, 2, \dots$ and that the solution is not unique if r takes the values $(2n + 1)\frac{\pi}{2}$ where $n = 0, 1, \dots$. Otherwise the unique solution is given by:

$$\vartheta(y) = \frac{\sin(ry)}{2 \sin r} \quad \text{in } [-1, 1]. \tag{8}$$

On taking into account the values of r which make not well-set the problem (6)₄, (7)₄, we will only consider $r \in (0, \pi) \setminus \{\pi/2\}$.

In order to determine the other unknown functions, we begin by differentiating (6)_{1,2}. By virtue of (6)₃ and (8), w has to satisfy the following linear ordinary differential equation:

$$w^{IV} - \alpha w'' + \beta w = \frac{\gamma r}{\sin r} \cos(ry), \tag{9}$$

where

$$\alpha = M_p^2 + (1 - N^2)M^2, \quad \beta = M_p^2 M^2, \quad \gamma = -\frac{\lambda M_p^2}{4(1 - N^2)}. \tag{10}$$

The general solution of equation (9) depends on the sign of the discriminant $\Delta = \alpha^2 - 4\beta$ of the algebraic equation

$$\xi^4 - \alpha \xi^2 + \beta = 0. \tag{11}$$

Therefore we solve problem (6), (7) in the three cases: $\Delta > 0$, $\Delta = 0$, $\Delta < 0$.

These cases are all possible from the physical point of view because they represent the following situations:

- if $\Delta > 0$, then $0 < H_0 < \sqrt{\frac{\mu}{\sigma_e \mu_e d(1 + N)}} \frac{M_p}{\sigma_e \mu_e d(1 + N)}$ or $H_0 > \sqrt{\frac{\mu}{\sigma_e \mu_e d(1 - N)}} \frac{M_p}{\sigma_e \mu_e d(1 - N)}$
weak or strong external uniform magnetic field;
- if $\Delta = 0$, then $H_0 = \sqrt{\frac{\mu}{\sigma_e \mu_e d(1 + N)}} \frac{M_p}{\sigma_e \mu_e d(1 + N)}$ or $H_0 = \sqrt{\frac{\mu}{\sigma_e \mu_e d(1 - N)}} \frac{M_p}{\sigma_e \mu_e d(1 - N)}$
critical external uniform magnetic field;
- if $\Delta < 0$, then $\sqrt{\frac{\mu}{\sigma_e \mu_e d(1 + N)}} \frac{M_p}{\sigma_e \mu_e d(1 + N)} < H_0 < \sqrt{\frac{\mu}{\sigma_e \mu_e d(1 - N)}} \frac{M_p}{\sigma_e \mu_e d(1 - N)}$
bounded external uniform magnetic field.

Independently on the strength of the external magnetic field, a particular solution w^* of (9) is

$$w^*(y) = \frac{\gamma r}{R \sin r} \cos (ry), \quad R = r^4 + \alpha r^2 + \beta \tag{12}$$

so that the general solution of (9) is given by

$$w(y) = w_0(y) + \frac{\gamma r}{R \sin r} \cos (ry), \tag{13}$$

where $w_0(y)$ is the general solution of the homogeneous equation associate to (9) whose expression depends on the strength of the external magnetic field.

3.1. $\Delta > 0$: Weak or Strong External Uniform Magnetic Field

In this situation equation (11) admits the following real routes

$$\begin{aligned} \xi_1 &= \sqrt{\frac{\alpha - \sqrt{\Delta}}{2}} = \frac{1}{2} \left[\sqrt{(M_p + M)^2 - N^2 M^2} - \sqrt{(M_p - M)^2 - N^2 M^2} \right], \\ \xi_2 &= \sqrt{\frac{\alpha + \sqrt{\Delta}}{2}} = \frac{1}{2} \left[\sqrt{(M_p + M)^2 - N^2 M^2} + \sqrt{(M_p - M)^2 - N^2 M^2} \right], \\ \xi_3 &= -\xi_1, \quad \xi_4 = -\xi_2, \end{aligned} \tag{14}$$

After some calculations we find that the solution of problem (6)_{1,2,3}, (7)_{1,2,3} is given by

$$\begin{aligned} v(y) &= \frac{M_p^2}{\xi_2^2 - \xi_1^2} \frac{A_1^+ \sinh \xi_2 [\cosh(\xi_1 y) - \cosh \xi_1] - A_2^+ \sinh \xi_1 [\cosh(\xi_2 y) - \cosh \xi_2]}{\sinh \xi_1 \sinh \xi_2} \\ &\quad + \frac{2\gamma}{R} \frac{A_1^+ D_2^+ \sinh(\xi_1 y) - A_2^+ D_1^+ \sinh(\xi_2 y)}{A_1^+ \sinh \xi_1 \cosh \xi_2 - A_2^+ \sinh \xi_2 \cosh \xi_1} - \frac{2\gamma S}{R \sin r} \sin(ry), \\ w(y) &= \frac{M_p^2}{2(\xi_2^2 - \xi_1^2)} \frac{\sinh \xi_2 \sinh(\xi_1 y) - \sinh \xi_1 \sinh(\xi_2 y)}{\sinh \xi_1 \sinh \xi_2} \\ &\quad + \frac{\gamma}{R} \frac{D_2^+ \cosh(\xi_1 y) - D_1^+ \cosh(\xi_2 y)}{A_1^+ \sinh \xi_1 \cosh \xi_2 - A_2^+ \sinh \xi_2 \cosh \xi_1} + \frac{\gamma r}{R \sin r} \cos(ry), \\ h(y) &= \frac{M_p^2}{M(\xi_2^2 - \xi_1^2)} \frac{B_1^+ \sinh \xi_2 \sinh(\xi_1 y) - B_2^+ \sinh \xi_1 \sinh(\xi_2 y)}{\sinh \xi_1 \sinh \xi_2} - \frac{y}{M} \end{aligned}$$

$$\begin{aligned}
 & + \frac{2}{M R} \frac{\gamma B_1^+ D_2^+ [\cosh(\xi_1 y) - \cosh \xi_1] - B_2^+ D_1^+ [\cosh(\xi_2 y) - \cosh \xi_2]}{A_1^+ \sinh \xi_1 \cosh \xi_2 - A_2^+ \sinh \xi_2 \cosh \xi_1} \\
 & + \frac{2\gamma M S}{R r \sin r} [\cos r - \cos(ry)]. \tag{15}
 \end{aligned}$$

where

$$\begin{aligned}
 S &= \frac{(1 - N^2)r^2 + M_p^2}{M_p^2}, \\
 A_1^+ &= \frac{1 - N^2}{M_p^2} \xi_1 - \frac{1}{\xi_1}, \quad A_2^+ = \frac{1 - N^2}{M_p^2} \xi_2 - \frac{1}{\xi_2}, \quad B_1^+ = 1 - \frac{\xi_1^2}{M_p^2}, \quad B_2^+ = 1 - \frac{\xi_2^2}{M_p^2}, \\
 D_1^+ &= A_1^+ r \cot r \sinh \xi_1 + S \cosh \xi_1, \quad D_2^+ = A_2^+ r \cot r \sinh \xi_2 + S \cosh \xi_2. \tag{16}
 \end{aligned}$$

3.2. $\Delta = 0$: Critical External Uniform Magnetic Field

In this case, equation (11) admits the following real routes

$$\xi_1 = \xi_2 = \sqrt{\frac{\alpha}{2}} = \sqrt{M_p M} =: \xi, \quad \xi_3 = \xi_4 = -\xi_1 \tag{17}$$

so that problem (6)_{1,2,3}, (7)_{1,2,3} has the following solution

$$\begin{aligned}
 v(y) &= \frac{M_p^2}{2\xi} \frac{A_1^0 [\cosh \xi \cosh(\xi y) - y \sinh \xi \sinh(\xi y) - 1] + A_2^0 \sinh \xi [\cosh \xi - \cosh(\xi y)]}{\sinh^2 \xi} \\
 & + \frac{2\gamma}{R} \frac{(SA_2^0 \cosh \xi - A_1^0 D_2^0) \sinh(\xi y) + A_1^0 D_1^0 y \cosh(\xi y)}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} - \frac{2\gamma S}{R \sin r} \sin(ry), \\
 w(y) &= \frac{M_p^2}{4\xi} \frac{\cosh \xi \sinh(\xi y) - y \sinh \xi \cosh(\xi y)}{\sinh^2 \xi} \\
 & + \frac{\gamma}{R} \frac{D_1^0 y \sinh(\xi y) - (A_2^0 r \cot r \sinh \xi + D_2^0) \cosh(\xi y)}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} + \frac{\gamma r}{R \sin r} \cos(ry), \\
 h(y) &= \frac{M_p^2}{2\xi M} \frac{(B_1^0 \cosh \xi + 2B_2^0 \sinh \xi) \sinh(\xi y) - B_1^0 y \sinh \xi \cosh(\xi y)}{\sinh^2 \xi} - \frac{y}{M} \\
 & + \frac{2\gamma}{MR} \left\{ B_1^0 \left[r \cot r + \frac{D_1^0 y \sinh(\xi y) - (A_2^0 r \cot r \sinh \xi + D_2^0) \cosh(\xi y)}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} \right] \right\}
 \end{aligned}$$

$$+ 2B_2^0 \frac{D_1^0 [\cosh \xi - \cosh(\xi y)]}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} \left\} + \frac{2\gamma MS}{Rr \sin r} [\cos r - \cos(ry)], \tag{18}$$

where

$$A_1^0 = \frac{1 - N^2}{M_p^2} \xi - \frac{1}{\xi}, \quad A_2^0 = \frac{1 - N^2}{M_p^2} + \frac{1}{\xi^2}, \quad B_1^0 = 1 - \frac{\xi^2}{M_p^2}, \quad B_2^0 = \frac{\xi}{M_p^2},$$

$$D_1^0 = A_1^0 r \cot r \sinh \xi + S \cosh \xi, \quad D_2^0 = A_1^0 r \cot r \cosh \xi + S \sinh \xi. \tag{19}$$

3.3. $\Delta < 0$: Bounded External Uniform Magnetic Field

Equation (11) admits the following complex routes

$$\xi_1 = \frac{1}{2} \left[\sqrt{(M_p + M)^2 - N^2 M^2} - i \sqrt{N^2 M^2 - (M_p - M)^2} \right],$$

$$\xi_2 = \frac{1}{2} \left[\sqrt{(M_p + M)^2 - N^2 M^2} + i \sqrt{N^2 M^2 - (M_p - M)^2} \right],$$

$$\xi_3 = -\xi_1, \quad \xi_4 = -\xi_2. \tag{20}$$

Putting

$$\delta = \frac{1}{2} \sqrt{(M_p + M)^2 - N^2 M^2}, \quad \sigma = \frac{1}{2} \sqrt{N^2 M^2 - (M_p - M)^2},$$

$$A_1^- = \left(\frac{1 - N^2}{M_p^2} - \frac{1}{M_p M} \right) \delta, \quad A_2^- = \left(\frac{1 - N^2}{M_p^2} + \frac{1}{M_p M} \right) \sigma,$$

$$B_1^- = \frac{M_p^2 - M^2(1 - N^2)}{2M_p^2}, \quad B_2^- = -\frac{2}{M_p^2} \sigma \delta,$$

$$D_1^- = r \cot r (A_1^- \cos \sigma \sinh \delta - A_2^- \sin \sigma \cosh \delta) + S \cos \sigma \cosh \delta,$$

$$D_2^- = r \cot r (A_1^- \sin \sigma \cosh \delta + A_2^- \cos \sigma \sinh \delta) + S \sin \sigma \sinh \delta, \tag{21}$$

the solution of the problem is given by:

$$v(y) = \frac{M_p^2}{2\delta\sigma(\sin^2 \sigma + \sinh^2 \delta)} \left[(A_1^- \sin \sigma \cosh \delta - A_2^- \cos \sigma \sinh \delta) \cos(\sigma y) \cosh(\delta y) \right.$$

$$\left. - (A_1^- \cos \sigma \sinh \delta + A_2^- \sin \sigma \cosh \delta) \sin(\sigma y) \sinh(\delta y) + A_2^- \sinh \delta \cosh \delta \right]$$

$$\begin{aligned}
& - A_1^- \cos \sigma \sin \sigma \Big] + \frac{2\gamma}{R} \\
& \times \frac{(A_2^- D_1^- - A_1^- D_2^-) \cos(\sigma y) \sinh(\delta y) + (A_1^- D_1^- + A_2^- D_2^-) \sin(\sigma y) \cosh(\delta y)}{A_1^- \sin \sigma \cos \sigma + A_2^- \sinh \delta \cosh \delta} \\
& - \frac{2\gamma S}{R \sin r} \sin(ry), \\
w(y) = & \frac{M_p^2 \sin \sigma \cosh \delta \cos(\sigma y) \sinh(\delta y) - \cos \sigma \sinh \delta \sin(\sigma y) \cosh(\delta y)}{4\delta \sigma \sin^2 \sigma + \sinh^2 \delta} \\
& + \frac{\gamma D_1^- \sin(\sigma y) \sinh(\delta y) - D_2^- \cos(\sigma y) \cosh(\delta y)}{R A_1^- \sin \sigma \cos \sigma + A_2^- \sinh \delta \cosh \delta} + \frac{\gamma r}{R \sin r} \cos(ry), \\
h(y) = & \frac{M_p^2}{2M\delta \sigma (\sin^2 \sigma + \sinh^2 \delta)} \left[(B_1^- \sin \sigma \cosh \delta - B_2^- \cos \sigma \sinh \delta) \right. \\
& \times \cos(\sigma y) \sinh(\delta y) \\
& \left. - (B_1^- \cos \sigma \sinh \delta + B_2^- \sin \sigma \cosh \delta) \sin(\sigma y) \cosh(\delta y) \right] - \frac{y}{M} \\
& + \frac{2\gamma}{MR(A_1^- \sin \sigma \cos \sigma + A_2^- \sinh \delta \cosh \delta)} \left\{ \right. \\
& (B_2^- D_1^- - B_1^- D_2^-) [\cos(\sigma y) \cosh(\delta y) - \cos \sigma \cosh \delta] \\
& \left. + (B_1^- D_1^- + B_2^- D_2^-) [\sin(\sigma y) \sinh(\delta y) - \sin \sigma \sinh \delta] \right\} \\
& + \frac{2\gamma MS}{Rr \sin r} [\cos r - \cos(ry)]. \tag{22}
\end{aligned}$$

4. Flow in the Case of Internal Heat Sink

Now we suppose that there is an internal heat sink in the channel so that we have a sign $-$ in (6)₄. The integration of this equation taking into account (7)₄ furnishes:

$$\vartheta(y) = \frac{\sinh(ry)}{2 \sinh r} \quad \text{in } [-1, 1]. \tag{23}$$

In this case equation (9) becomes:

$$w^{IV} - \alpha w'' + \beta w = \frac{\gamma r}{\sinh r} \cosh(ry). \tag{24}$$

The general solution of equation (24) depends on the sign of Δ ; it is given by

$$w(y) = w_0(y) + w^*(y), \tag{25}$$

where $w_0(y)$ is the general solution of the homogeneous equation associate to (24) and the particular solution $w^*(y)$ is given by

$$\begin{aligned} w^*(y) &= \frac{\gamma r}{R \sinh r} \cosh(ry), & \text{if } r \text{ is not root of (11) } (\Delta \gtrless 0), \\ w^*(y) &= \frac{\gamma}{2(2r^2 - \alpha) \sinh r} y \sinh(ry), & \text{if } r \text{ is a simple root of (11) } (\Delta > 0), \\ w^*(y) &= \frac{\gamma r}{2(6r^2 - \alpha) \sinh r} y^2 \cosh(ry), & \text{if } r \text{ is double root of (11) } (\Delta = 0), \end{aligned} \tag{26}$$

where in this Section

$$R = r^4 - \alpha r^2 + \beta.$$

We now furnish the solution of the problem by using the expressions in (15), (18), (22) with suitable replacements. We recall that now we have to take into account two more cases. For the sake of brevity, we omit the calculations.

4.1. $\Delta > 0$: Weak or Strong External Uniform Magnetic Field

The roots of (11) are given by (14), but we have to distinguish two subcases.

4.1.1. r is not a Route of (11):

The solution is obtained from (15) replacing the last term in the expressions of v, w, h with

$$+ \frac{2\gamma S}{R \sinh r} \sinh(ry), + \frac{\gamma r}{R \sinh r} \cosh(ry), + \frac{2\gamma MS}{Rr \sinh r} [\cosh r - \cosh(ry)],$$

respectively, where now

$$S = \frac{(1 - N^2)r^2 - M_p^2}{M_p^2},$$

$$D_1^+ = A_1^+ r \coth r \sinh \xi_1 - S \cosh \xi_1, \quad D_2^+ = A_2^+ r \coth r \sinh \xi_2 - S \cosh \xi_2.$$

4.1.2. r is a Simple Route of (11):

In this subcase, we hence have

$$r = \xi_i, \quad i = 1, 2, \quad 2r^2 - \alpha = (-1)^i(\xi_2^2 - \xi_1^2).$$

After some calculations we get three expressions analogous to (15) where now

$$D_1^+ = A_1^+ \xi_i^2 \sinh \xi_1 - [A_i^+ \xi_i + 2 + A_i^+ \xi_i^2 \coth \xi_i] \cosh \xi_1,$$

$$D_2^+ = A_2^+ \xi_i^2 \sinh \xi_2 - [A_i^+ \xi_i + 2 + A_i^+ \xi_i^2 \coth \xi_i] \cosh \xi_2,$$

the constant R need to be replaced with $2(-1)^i(\xi_2^2 - \xi_1^2)\xi_i^2$, and the last term in v, w, h with

$$\begin{aligned} & + \frac{\gamma}{(-1)^i(\xi_2^2 - \xi_1^2)\xi_i \sinh \xi_i} \left[\left(A_i^+ + \frac{2}{\xi_i} \right) \sinh(\xi_i y) + A_i^+ \xi_i y \cosh(\xi_i y) \right], \\ & + \frac{\gamma}{2(-1)^i(\xi_2^2 - \xi_1^2) \sinh \xi_i} y \sinh(\xi_i y), \\ & + \frac{\gamma}{M(-1)^i(\xi_2^2 - \xi_1^2)\xi_i \sinh \xi_i} \left\{ -2 \frac{(-1)^i(\xi_2^2 - \xi_1^2) - \xi_i^2}{M_p^2} [\cosh \xi_i - \cosh(\xi_i y)] \right. \\ & \quad \left. + B_i^+ \xi_i [y \sinh(\xi_i y) - \sinh \xi_i] \right\}, \end{aligned}$$

respectively.

4.2. $\Delta = 0$: Critical External Uniform Magnetic Field

In this situation, equation (11) admits the real routes given by (17).

We have to distinguish two subcases.

4.2.1. r is not a Route of (11):

The solution is obtained from (18) replacing the last term in the expressions of v, w, h with

$$-\frac{2\gamma S}{R \sinh r} \sinh(ry), \quad +\frac{\gamma r}{R \sinh r} \cosh(ry), \quad -\frac{2\gamma MS}{Rr \sinh r} [\cosh r - \cosh(ry)],$$

respectively, $\cot r$ with $\coth r$ in h and in D_1^0, D_2^0 given by (19), while now

$$S = \frac{M_p^2 - (1 - N^2)r^2}{M_p^2}. \tag{27}$$

4.2.2. r is a Double Route of (11):

$$r^2 = \xi^2 = \alpha/2.$$

Putting

$$\begin{aligned} D_1^0 &= A_1^0 \coth \xi \sinh \xi - \left(2A_2^0 \coth \xi + A_1^0 - \frac{2}{\xi^3} \right) \cosh \xi, \\ D_2^0 &= A_1^0 \coth \xi \cosh \xi - \left(2A_2^0 \coth \xi + A_1^0 - \frac{2}{\xi^3} \right) \sinh \xi, \end{aligned} \tag{28}$$

the solution is obtained from (18) replacing in the expressions of v, w, h the constant R with 8ξ , $\cot r$ with $\coth r$ and the last term with

$$\begin{aligned} &+ \frac{\gamma}{2\xi \sinh \xi} A_2^0 y \cosh(\xi y) + \frac{\gamma}{4\xi \sinh \xi} \left(A_1^0 y^2 - \frac{2}{\xi^3} \right) \sinh(\xi y), \\ &+ \frac{\gamma}{8\xi \sinh \xi} y^2 \cosh(\xi y), \\ &+ \frac{\gamma}{4MM_p^2 \xi \sinh \xi} \left\{ 4\xi [\sinh \xi - y \sinh(\xi y)] + (\xi^2 - M_p^2) [\cosh \xi - y^2 \cosh(\xi y)] \right. \\ &\left. - 6[\cosh \xi - \cosh(\xi y)] \right\}, \end{aligned}$$

respectively, where

$$S = -2A_2^0 \coth \xi - A_1^0 + \frac{2}{\xi^3}.$$

4.3. $\Delta < 0$: Bounded External Uniform Magnetic Field

In this situation, equation (11) admits the complex routes given by (20). After some calculations we deduce that v, w, h have the expressions given by (22) substituting $\cot r$ with $\coth r$ and the last term with

$$-\frac{2\gamma S}{R \sinh r} \sinh(ry), + \frac{\gamma r}{R \sinh r} \cosh(ry), - \frac{2\gamma MS}{Rr \sinh r} [\cosh r - \cosh(ry)]$$

respectively where S is given by (27).

5. Comments on the Flow

We make some comments on the flow both in the case of internal heat generation both in the case of internal heat absorption.

- If $r \ll 1$, then the solution tends to that one of the same problem in the absence of heat source or sink (see[1]).
- It is easy to verify that the electric field \mathbf{E} is given by $E_0 \mathbf{e}_3$ where E_0 is a constant independent of λ and r , whose expression is given in [1]. Moreover outside the planes where there is a vacuum we have $\mathbf{E} = E_0 \mathbf{e}_3$, $\mathbf{H} = H_0 \mathbf{e}_2$, by virtue of the usual transmission conditions for the electromagnetic field across $\Pi_{1,2}$.
- The Nusselt number at $\Pi_{1,2}$ defined by

$$\text{Nu}_{1,2} = \frac{k_{1,2} d}{k} = \frac{d}{T_2 - T_1} \frac{dT}{dx_2} \Big|_{x_2=\pm d} = \vartheta'(\pm 1),$$

has the following form

$$\text{Nu}_{1,2} = \frac{r}{2} \cot r, \quad \text{if there is a source,}$$

$$\text{Nu}_{1,2} = \frac{r}{2} \coth r, \quad \text{if there is a sink.}$$

If r tends to 0^+ then $\text{Nu}_{1,2}$ tends to $1/2$ according to [1].

The heat flux vector at the walls in the case of heat generation is given by

$$\mathbf{q} = -\frac{(T_2 - T_1)kr}{2d} \cot r \mathbf{e}_2,$$

so that the sign of $\text{Nu}_{1,2}$ and the direction of \mathbf{q} depend on the values of r . In the case of heat absorption it is sufficient to replace $\cot r$ with $\coth r$: the sign of $\text{Nu}_{1,2}$ is positive and the direction of \mathbf{q} is opposite of \mathbf{e}_2 as in the absence of source or sink ([1]).

- The skin friction ($\boldsymbol{\tau}_{1,2}$) and the skin couple friction ($\boldsymbol{\tau}_{p1,2}$) at both plates are given by

$$\boldsymbol{\tau}_{1,2} = (\mu + \mu_r) \frac{V_0}{d} v'(\pm 1) \mathbf{e}_1; \quad \boldsymbol{\tau}_{p1,2} = (c_d + c_a) \frac{V_0}{d^2} w'(\pm 1) \mathbf{e}_3.$$

The expression of $\boldsymbol{\tau}_{1,2}$ is related to the occurrence of the reverse flow, as we will see in the next section.

- If $C = 0$, then the solution is obtained from the previous expressions writing only the terms having coefficient γ . Of course in the dimensionless variables, the reference velocity V_0 cannot be expressed in terms of C .
- If $H_0 = 0$ and there is a source, then the motion is given by

$$\begin{aligned}
 v(y) = & \frac{1}{2}(1 - y^2) + \frac{N^2[\cosh(M_p y) - \cosh M_p]}{M_p \sinh M_p} \\
 & - \frac{\lambda N^2 M_p^2 \cot r [\sinh(M_p y) - y \sinh M_p]}{2r(1 - N^2)(r^2 + M_p^2)(M_p \cosh M_p - N^2 \sinh M_p)} \\
 & + \frac{\lambda[M_p^2 + (1 - N^2)r^2]}{2r^2(r^2 + M_p^2)(1 - N^2)} \\
 & \times \left\{ \frac{\sin(ry)}{\sin r} - \frac{y M_p \cosh M_p - N^2 \sinh(M_p y)}{M_p \cosh M_p - N^2 \sinh M_p} \right\}, \\
 w(y) = & \frac{1}{2} \left[y - \frac{\sinh(M_p y)}{\sinh M_p} \right] \\
 & - \frac{\lambda M_p [M_p^2 + (1 - N^2)r^2] [\cosh(M_p y) - \cosh M_p]}{4r^2(1 - N^2)(r^2 + M_p^2)(M_p \cosh M_p - N^2 \sinh M_p)} \\
 & - \frac{\lambda M_p^2}{4r^2(r^2 + M_p^2)(1 - N^2)} \\
 & \times \left\{ r \frac{\cos(ry)}{\sin r} - \frac{r \cot r [M_p \cosh(M_p y) - N^2 \sinh M_p]}{M_p \cosh M_p - N^2 \sinh M_p} \right\}. \quad (29)
 \end{aligned}$$

If there is a sink, then we have to consider two cases according to the value of r .

If $r \neq M_p$, then the motion is obtained from (29) by replacing the trigonometric functions \sin, \cos, \cot with the hyperbolic functions \sinh, \cosh, \coth , the constant $r^2 + M_p^2$ with $r^2 - M_p^2$ and $M_p^2 + (1 - N^2)r^2$ with $M_p^2 - (1 - N^2)r^2$.

If $r = M_p$, then

$$\begin{aligned}
 v(y) = & \frac{1}{2}(1 - y^2) + \frac{N^2[\cosh(M_p y) - \cosh M_p]}{M_p \sinh M_p} \\
 & + \frac{\lambda N^2}{4M_p(1 - N^2) \sinh M_p} \left[y \cosh(M_p y) \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \left. \frac{(M_p \sinh M_p - N^2 \cosh M_p) \sinh(M_p y) - M_p y}{M_p \cosh M_p - N^2 \sinh M_p} \right], \\
 & - \frac{\lambda(2 - N^2)}{4M_p^2(1 - N^2)} \left[\frac{\sinh(M_p y)}{\sinh M_p} - \frac{M_p y \cosh M_p - N^2 \sinh(M_p y)}{M_p \cosh M_p - N^2 \sinh M_p} \right], \\
 w(y) = & \frac{1}{2} \left[y - \frac{\sinh(M_p y)}{\sinh M_p} \right] \\
 & + \frac{\lambda[2 - N^2 - M_p N^2 \coth M_p][\cosh(M_p y) - \cosh M_p]}{8M_p(1 - N^2)(M_p \cosh M_p - N^2 \sinh M_p)} \\
 & - \frac{\lambda}{8(1 - N^2)} \left[y \frac{\sinh(M_p y)}{\sinh M_p} - \frac{M_p \cosh(M_p y) - N^2 \sinh M_p}{M_p \cosh M_p - N^2 \sinh M_p} \right]. \tag{30}
 \end{aligned}$$

- If \mathcal{S} is occupied by a Boussinesquian, electrically conducting Newtonian fluid, then its MHD mixed convective flow is governed by (6)₁ with $w = 0$, $N = 0$, (6)_{3,4} and the boundary conditions (7)_{1,3,4}. The dimensionless temperature ϑ is given by (8) and

$$\begin{aligned}
 v(y) = & \frac{\cosh M - \cosh(My)}{M \sinh M} + \frac{\lambda}{2(M^2 + r^2)} \left[\frac{\sin(ry)}{\sin r} - \frac{\sinh(My)}{\sinh M} \right], \\
 h(y) = & \frac{1}{M} \left[\frac{\sinh(My)}{\sinh M} - y \right] + \frac{\lambda[\cosh(My) - \cosh M]}{2(M^2 + r^2) \sinh M} \\
 & + \frac{\lambda M[\cos(ry) - \cos r]}{2r(M^2 + r^2) \sin r},
 \end{aligned}$$

if we assume the presence of an internal heat source.

If there is an internal heat sink, then the dimensionless temperature ϑ is given by (23) and we have to distinguish two cases.

If $r \neq M$, then in the previous expressions of v, h we have to replace the functions \sin, \cos with \sinh, \cosh , the constant $M^2 + r^2$ with $M^2 - r^2$ and we have to change the sign in the last term in h .

If $r = M$, then

$$\begin{aligned}
 v(y) = & \frac{\cosh M - \cosh(My)}{M \sinh M} \\
 & + \frac{\lambda}{4M \sinh^2 M} [\cosh M \sinh(My) - y \sinh M \cosh(My)],
 \end{aligned}$$

$$h(y) = \frac{1}{M} \left[\frac{\sinh(My)}{\sinh M} - y \right] + \frac{\lambda(1 + M \coth M)}{4M^2 \sinh M} [\cosh M - \cosh(My)] + \frac{\lambda}{4M} \left[y \frac{\sinh(My)}{\sinh M} - 1 \right].$$

- In our study we have assumed $T_2 > T_1$; if $T_2 = T_1$ then it is easy to prove that $\vartheta = 0$. Hence we obtain that the flow is not influenced by the internal heat source or sink.

Due to the geometry of the problem, all the previous relations hold for all $y \in [-1, 1]$.

6. Numerical Simulations and Discussions

The aim of this Section is to present a selected set of graphical and numerical results illustrating the effects on the flow of the presence of an internal heat source or sink. This target can be obtained by studying the behavior of the flow when r varies. Actually, the internal heat parameter r^2 ($r \in (0, \pi) \setminus \{\pi/2\}$ in the case of internal source) is related to the constant rate of volumetric internal heat generation.

As it is proved in the previous Sections, the solution of the problem depends also on the values of some other relevant physical dimensionless parameters which describe the micropolar properties of the fluid (N and M_p), the electromagnetic properties of the flow (M^2) and the buoyancy forces due to the gravity (λ). Nevertheless, when the value of r is fixed, the influence of the other parameters on the flow is not modified in a relevant way by the presence of the source or sink, so that we refer to [1] for a complete study of the behavior of the motion when the parameters N^2 , M_p , M^2 and λ change. Briefly, the behavior of the micropolar flow differs highly from the Newtonian one as N^2 increases and M_p decreases, the velocity and microrotation decrease as M^2 increases and for suitable vales of the buoyancy parameter λ the reverse flow occurs as we will see below.

We provide Figures 2, 3 in order to show the influence of $r \in (0, 3]$ on the temperature, velocity, microrotation and induced magnetic field. The graphics given for $r = 0$ represent the flow in the absence of internal heat source or sink studied in [1].

As far as the temperature is concerned, the trend depends only on the parameter r : if $0 < r \leq 1$, it is approximatively linear. If there is a source then in the other cases ($1 < r \leq 3$) the temperature ϑ takes minimum and maximum

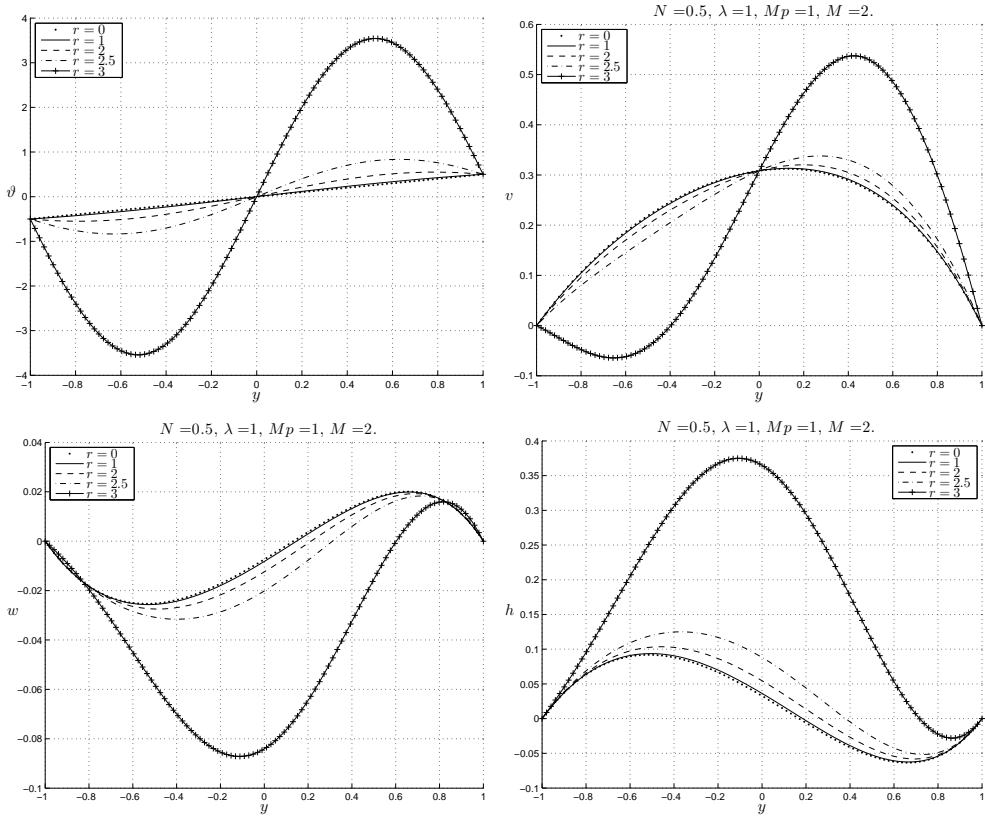


Figure 2: **Heat source:** the effect of r on the temperature (which depends only on r), on the velocity, on the microrotation and on the induced magnetic field.

which are very accentuated for r close to π ; if there is a sink then ϑ is always an increasing function. For the other functions, in the case of source, if $r = 3$ then the behavior is very different with respect to the other values and the reverse flow appears for the velocity. When an internal heat sink is present, the behavior of the velocity, the microrotation and the magnetic field is very slightly influenced by r .

The reverse flow appears when the dimensional velocity and the gradient of P have the same direction. In our problem, we have that the gradient of P ($\nabla P = -C\mathbf{e}_1$) depends only on the constant C , which appears in the expression of the buoyancy parameter λ . Hence, in order to study the occurrence of the reverse flow, it is useful to provide Figure 4, where we can see the behavior

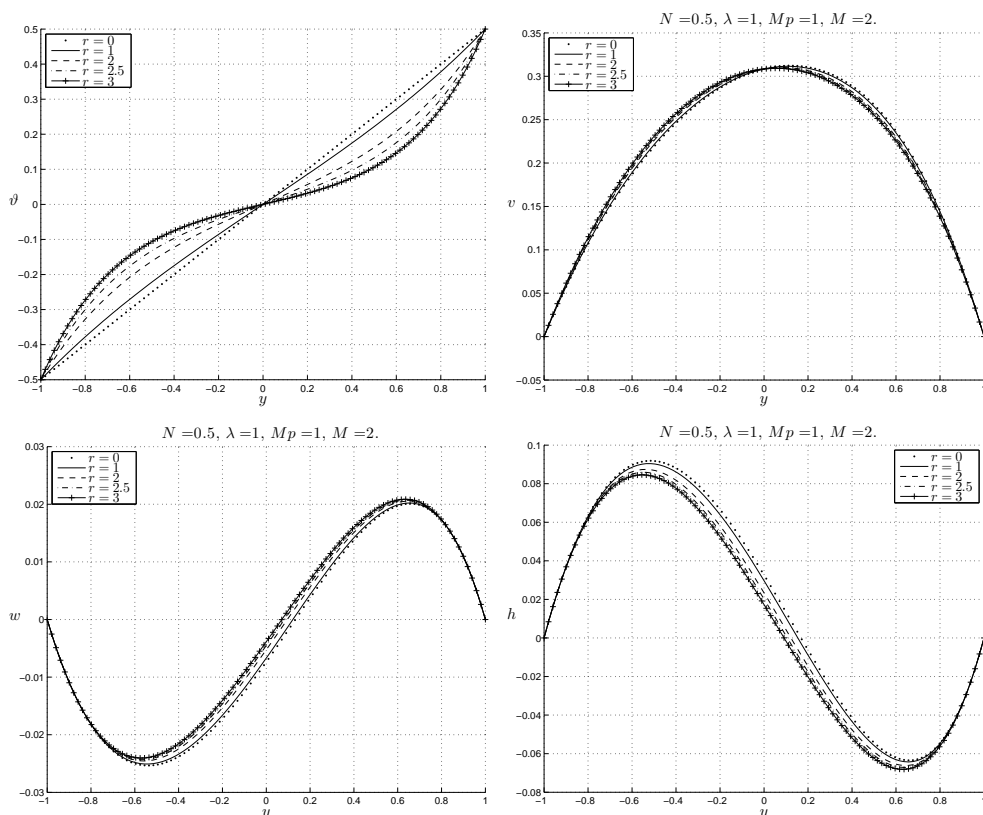


Figure 3: **Heat sink**: the effect of r on the temperature (which depends only on r), on the velocity, on the microrotation and on the induced magnetic field.

of the velocity when λ changes in the presence of a source or a sink.

If $\lambda = 0$ (symmetric heating, i.e. $T_1 = T_2$), then the pictures show that v is always positive so that the dimensional velocity ($\mathbf{v} = v_1\mathbf{e}_1$) and the gradient of P ($\nabla P = -C\mathbf{e}_1$) have opposite direction, provided $C \neq 0$ (different choices of the values of the other parameters do not modify the profile of the velocity). Hence, in the case $\lambda = 0$ the reverse flow does not occur. Therefore, the occurrence of the reverse flow is a feature of the mixed convection in the case of asymmetric heating. Actually, there exists a critical value λ^* of λ such that

- $C > 0$ ($\iff \lambda > 0$): if $\lambda \leq \lambda^*$, then the reverse flow does not appear; if $\lambda > \lambda^*$, then the reverse flow occurs near the coldest wall;

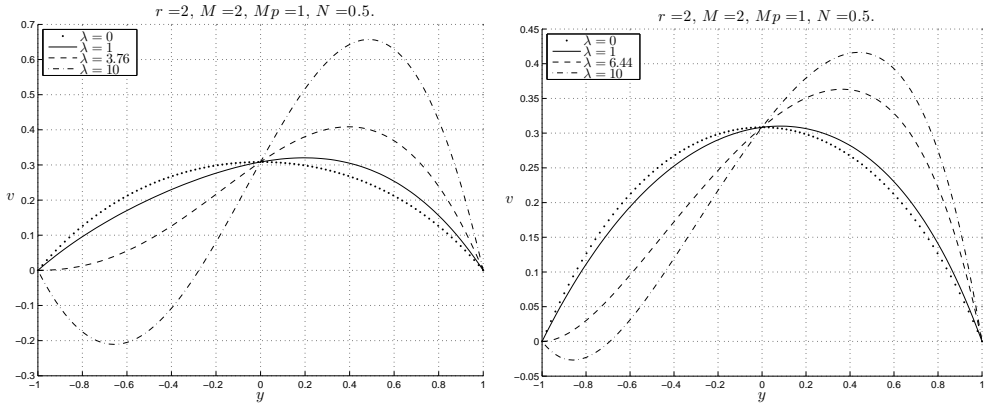


Figure 4: The effect of λ on the velocity when an internal heat source (first picture) or sink (second picture) is present ($\Delta = 0$).

- $C < 0$ ($\iff \lambda < 0$): if $\lambda \geq \lambda^*$, then the reverse flow does not appear; if $\lambda < \lambda^*$, then the reverse flow occurs near the hottest wall.

The value of λ^* depends on the other parameters and it is computed by putting $\tau_{1,2}$ equal to zero. In the fifth and sixth columns of Table 1 the values of λ_{source}^* and λ_{sink}^* are reported when r , M_p , N , and M change (first four columns) for $C > 0$. From this Table we can easily obtain the corresponding critical values of λ^* when $C < 0$ because the profiles of v for negative values of λ can be found by symmetry from the corresponding graphics of v when $\lambda > 0$.

From the examination of Table 1 it appears that λ^* is an increasing function in M , a decreasing function in N and it is not influenced in a relevant way by M_p as in [1]. Moreover λ^* decreases as r increases if there is a source (the decreasing is very significant when $r = 3$), while λ^* increases as r increases if there is a sink. So the presence of a source tends to favor the occurrence of the reverse flow, while the presence of a sink tends to prevent the occurrence of the reverse flow. As far as we know, this phenomenon has not been pointed out in the literature.

As in other physical situations ([14], [1]), the influence of M on λ^* shows that the presence of the external magnetic field tends to prevent the occurrence of the reverse flow. For fixed values of r and M , we have that λ^* is smaller than in Newtonian case and when $N \rightarrow 1^-$ λ^* differs highly from the corresponding value in the Newtonian case (see Tables 1 and 2).

Finally we consider the behavior of the induced magnetic field, which is usually neglected in the literature. Figure 5 shows the influence of the strength

r	M_p	N	M	λ^*_{source}	λ^*_{sink}	λ	M^*_{source}	M^*_{sink}	
1	1	0.20	1	5.6993	6.4941	1	3.2482	3.2883	
			2	6.6439	7.5021	5	3.1882	3.2082	
			10	20.3388	21.4946	10	3.2362	3.2442	
		0.90	1	0.9801	1.1253	1	6.3033	6.3033	
			2	1.0262	1.1751	5	6.5636	6.5836	
			10	2.0204	2.2175	10	6.5796	6.6036	
		5	0.20	1	5.6911	6.4848	1	3.2082	3.2482
				2	6.6451	7.5025	5	3.1562	3.1722
				10	20.3653	21.5173	10	3.2082	3.2122
	0.90		1	0.9956	1.1402	1	3.8889	3.8529	
			2	1.0868	1.2372	5	4.2172	4.2252	
			10	2.4024	2.5832	10	4.2372	4.2533	
	2	1	0.20	1	4.2881	7.5354	1	3.1842	3.3283
				2	5.0879	8.6107	5	3.1922	3.2402
				10	17.9197	22.8685	10	3.2322	3.2603
0.90			1	0.7269	1.3178	1	6.3554	6.3353	
			2	0.7647	1.3719	5	6.5315	6.6076	
			10	1.6396	2.4624	10	6.5395	6.6396	
5			0.20	1	4.2819	7.5247	1	3.1481	3.2883
				2	5.0902	8.6099	5	3.1602	3.2082
				10	17.9536	22.8867	10	3.2002	3.2322
		0.90	1	0.7419	1.3315	1	3.9930	3.8248	
			2	0.8194	1.4342	5	4.2012	4.2292	
			10	2.0402	2.8040	10	4.2092	4.2773	
3		1	0.20	1	0.8536	8.9956	1	3.1682	3.3283
				2	1.0659	10.1415	5	3.2082	3.2963
				10	6.5355	24.6235	10	3.2122	3.3043
	0.90		1	0.1391	1.5917	1	6.4635	6.3594	
			2	0.1483	1.6505	5	6.4675	6.6476	
			10	0.4169	2.7877	10	6.4675	6.6957	
	5		0.20	1	0.8524	8.9830	1	3.1401	3.2883
				2	1.0673	10.1388	5	3.1802	3.2603
				10	6.5701	24.6362	10	3.1842	3.2723
		0.90	1	0.1439	1.6025	1	4.1491	3.8208	
			2	0.1643	1.7105	5	4.1572	4.2292	
			10	0.6293	3.0944	10	4.1572	4.3093	

Table 1: Micropolar: Critical values λ^* , M^* of λ and M .

of the external magnetic field and the electromagnetic properties of the fluid on h in the case of a source or a sink. From these pictures, it appears that the absolute value of h is an increasing function in M until M reaches a critical

r	M	λ^*_{source}	λ^*_{sink}	λ	M^*_{source}	M^*_{sink}
1	1	5.9618	6.7915	1	3.1855	3.2231
	2	6.9806	7.8782	5	3.1303	3.1404
	10	21.5860	22.7928	10	3.1729	3.1779
2	1	4.4876	7.8782	1	3.1529	3.2657
	2	5.3513	9.0368	5	3.1253	3.1729
	10	19.0558	24.2260	10	3.1679	3.1955
3	1	0.8945	9.4014	1	3.1078	3.2607
	2	1.1245	10.6351	5	3.1479	3.2281
	10	7.0219	26.0555	10	3.1504	3.2356

Table 2: Newtonian: Critical values of λ and M .

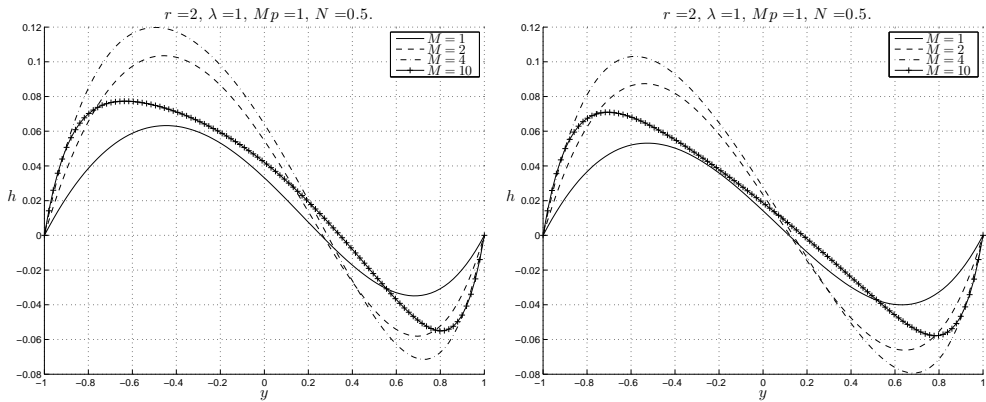


Figure 5: The effect of M on the induced magnetic field in the case of a source or a sink, respectively. If $M = 1, 2, 4, 10$, then $\Delta < 0, \Delta = 0, \Delta > 0, \Delta > 0$, respectively.

value M^* . If $M > M^*$, then the absolute value of h becomes a decreasing function in M . This behavior of h has been first emphasized in [1].

In the eighth and ninth columns of Table 1 we can read the values of M^*_{source} and M^*_{sink} when r, M_p, N , and λ increase (first, second, third and seventh columns). We note that M^* does not depend on r in a relevant way.

Finally, it is interesting to compare these values with the corresponding value of M^* in the Newtonian case (Table 2). In the micropolar fluid, M^* is always greater than in the Newtonian fluid and its value increases as N increases. This behavior can be expected because when $N \rightarrow 1^-$ the fluid differs highly from the Newtonian one.

Acknowledgments

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