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ON PLANARITY OF 3-JUMP GRAPHS

Varanoot Khemmani[1], Chira Lumduanhom[2 §],
Sriwan Muangloy[3], Massiri Muanphet[4], Kittisak Tipnuch[5]
[1,2,3,4,5]Department of Mathematics
Srinakharinwirot University
Sukhumvit 23, Bangkok, 10110, THAILAND

Abstract: For a graph G of size $m \geq 1$ and edge-induced subgraphs F and H of size k where $1 \leq k \leq m$, the subgraph H is said to be obtained from the subgraph F by an edge jump if there exist four distinct vertices u, v, w and x such that $uv \in E(F)$, $wx \in E(G) - E(F)$, and $H = F - uv + wx$. The k-jump graph $J_k(G)$ is that graph whose vertices correspond to the edge-induced subgraphs of size k of G where two vertices F and H of $J_k(G)$ are adjacent if and only if H can be obtained from F by an edge jump.

All connected graphs G for whose $J_3(G)$ is planar are determined.

AMS Subject Classification: 05C10, 05C12

Key Words: k-jump distance, k-jump graph, 3-jump graph, planar graph

1. Introduction

The concept of the k-jump graph of a nonempty graph G of size m where $1 \leq k \leq m$ was introduced by Chartrand, Hevia, Jarrett and Schultz [1]. Let G be a graph of size $m \geq 1$ and F and H be edge-induced subgraphs of size k of G where $1 \leq k \leq m$. The subgraph H is said to be obtained from the subgraph F by an edge jump if there exist four distinct vertices u, v, w and x such that $uv \in E(F)$, $wx \in E(G) - E(F)$, and $H = F - uv + wx$. It is obvious that if H is obtained from F by an edge jump then F is obtained from

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§Correspondence author

H by an edge jump. If there is a sequence $F = H_0, H_1, \ldots, H_\ell = H$ where $\ell \geq 1$ of edge-induced subgraphs of size k such that H_{i+1} is obtained from H_i by an edge jump for $0 \leq i \leq \ell - 1$, then we say that F can be *j-transformed* into H. The minimum number of edge jumps required to j-transform F into H is the *k-jump distance* from F to H. For a graph G of size $m \geq 1$ and an integer k with $1 \leq k \leq m$, the *k-jump graph* $J_k(G)$ is that graph whose vertices correspond to the edge-induced subgraphs of size k of G where two vertices F and H of $J_k(G)$ are adjacent if and only if the k-jump distance between edge-induced subgraphs F and H is 1 that is, H is obtained from F by an edge jump. We can label each vertex of $J_k(G)$ by listing all the edges of the respective subgraph. The concept of the k-jump graph is illustrated in Figure 1. In particular, if $k = 1$ then the graph $J_1(G) = J(G)$ is called the *jump graph* of G. Moreover, $J(G) = \overline{L(G)}$, the complement of the line graph of G. In [3] all connected graphs G for whose $J_2(G)$ is planar are determined in terms of a finite set S of graphs, namely a connected graph G has a planar 2-jump graph if and only if G is a subgraph of some element of S. The goal of this paper is to characterize all connected graphs having a planar 3-jump graph along the same lines as the characterization of connected graphs having a planar 2-jump graph.

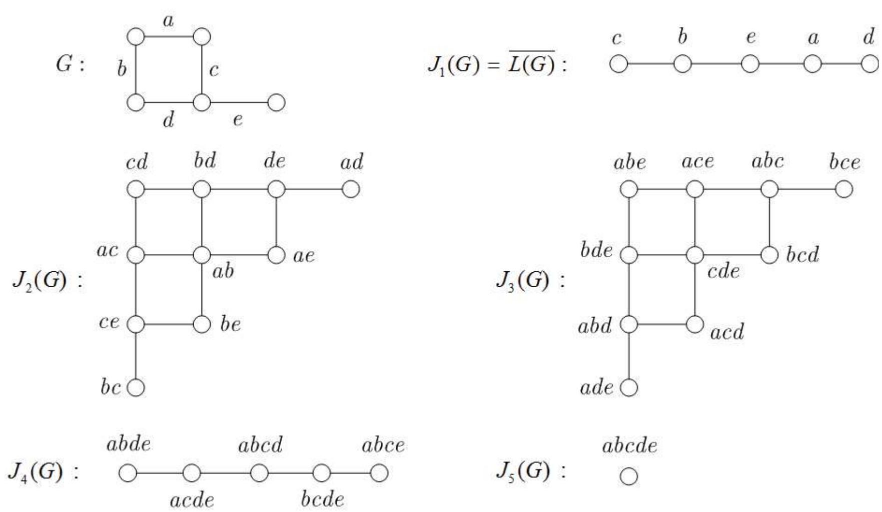

Figure 1: The k-jump graphs of a graph

The following results appeared in [3] and [4] will be useful for us later.

Theorem 1.1. ([4]) *A graph is planar if and only if it contains no* *subgraph isomorphic to* K_5 *or* $K_{3,3}$ *or a subdivision of one of these graphs.*

Theorem 1.2. ([3]) *If* G *is a graph of size* $m \geq 1$ *and* k *is an integer with* $1 \leq k < m$, *then* $J_k(G) = J_{m-k}(G)$.

Theorem 1.3. ([3]) *If a graph* G *is a subdivision of a graph* H *and* $J_k(H)$ *is nonplanar for some positive integer* k, *then* $J_k(G)$ *is nonplanar.*

The reader is referred to the book [2] by Chartrand, Lesniak and Zhang for basic definitions and terminology not described here.

2. Connected Graphs with Planar 3-Jump Graphs

In this section we will focus our attention on the planarity of the 3-jump graph $J_3(G)$ for a connected graph G of size at least 3. As we mentioned earlier, our aim is to determine a finite set S of graphs with the property that a connected graph G has a planar 3-jump graph if and only if G is a subgraph of some element of S.

In [3] it is shown that for the path P_n and cycle C_n of order n, $J_2(P_n)$ is nonplanar if and only if $n \geq 6$ and $J_2(C_n)$ is nonplanar if and only if $n \geq 5$. Similar results can be obtained for $J_3(P_n)$ and $J_3(C_n)$. The 3-jump graphs $J_3(P_5)$, $J_3(P_6)$, $J_3(C_4)$, and $J_3(C_5)$ are shown in Figure 2 which we see that $J_3(P_5)$ and $J_3(C_4)$ are planar while $J_3(P_6)$ and $J_3(C_5)$ are nonplanar. Therefore, by Theorems 1.1 and 1.3, the following results are immediate.

Corollary 2.1. *For* $n \geq 4$, $J_3(P_n)$ *is nonplanar if and only if* $n \geq 6$.

Corollary 2.2. *For* $n \geq 3$, $J_3(C_n)$ *is nonplanar if and only if* $n \geq 5$.

We now present a simple but useful lemma.

Lemma 2.3. *If* H *is a subgraph of a connected graph* G *then, for each* k, $J_k(H)$ *is a subgraph of* $J_k(G)$.

Proof. Let k be an integer such that $1 \leq k \leq m$ where m is the size of G. If v_{H_1} is a vertex of $J_k(H)$ that corresponds with an edge-induced subgraph H_1 of size k of H then since H is a subgraph of G, H_1 is certainly an edge-induced subgraph of size k of G. Thus v_{H_1} is a vertex of $J_k(G)$ and so $V(J_k(H)) \subseteq V(J_k(G))$. On the other hand, if $e = v_{H_1}v_{H_2} \in E(J_k(H))$ then in a graph H, H_1 is obtained from H_2 by an edge jump. Now, since H is a subgraph of G,

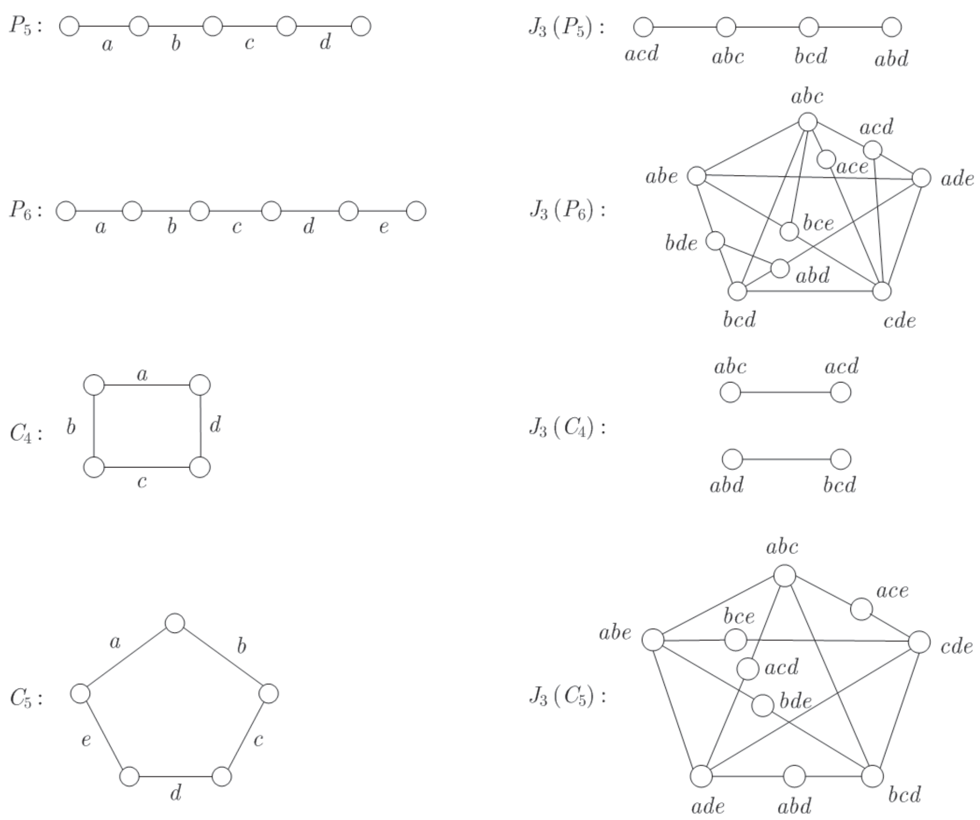

Figure 2: The 3-jump graphs of P_5, P_6, C_4 and C_5

it follows that in G, H_1 is also obtained from H_2 by an edge jump. Therefore $e = v_{H_1}v_{H_2} \in E(J_k(G))$ and so $E(J_k(H)) \subseteq E(J_k(G))$. □

Theorem 1.1 and lemma 2.3 give us the following results.

Lemma 2.4. *If G is a graph containing a subgraph H where H is the union of edge-disjoint subgraphs H_1 of size at least 2 and H_2 of size at least 4 such that (1) H_1 contains edges b and e, (2) H_2 contains edges a, c, d and f where a is not adjacent to c, and (3) edges b and e are not adjacent to both d and f in H, then $J_3(G)$ is nonplanar.*

Proof. Since there exists a subgraph of the 3-jump graph $J_3(H)$ that is isomorphic to a subdivision of $K_{3,3}$ as shown in Figure 3, it follows that $J_3(H)$ and so $J_3(G)$ are nonplanar. □

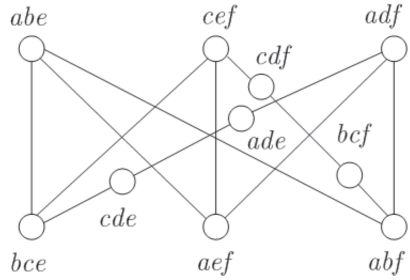

Figure 3: A subgraph of $J_3(H)$ of Lemma 2.4

Lemma 2.5. *If G is a graph containing a subgraph H where H is the union of edge-disjoint subgraphs H_1 of size at least 2 and H_2 of size at least 4 such that (1) H_1 contains nonadjacent edges a and f, (2) H_2 contains edges b, c, d and e where b and d are not adjacent, (3) a is not adjacent to c in H, and (4) f is not adjacent to two edges b and e in H, then $J_3(G)$ is nonplanar.*

Proof. Since the 3-jump graph $J_3(H)$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$ as shown in Figure 4, it follows that $J_3(H)$ and thus $J_3(G)$ are nonplanar. □

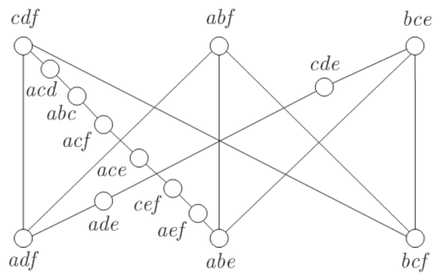

Figure 4: A subgraph of $J_3(H)$ of Lemma 2.5

Lemma 2.6. *If G is a graph containing three subgraphs G_1, G_2, and G_3 where G_1 is isomorphic to $K_{1,3}$ or K_3, and G_2 and G_3 are both isomorphic to P_2 such that for each $i \in \{2,3\}$, G_1 and G_i are disjoint subgraphs, and G_2 and G_3 are edge-disjoint subgraphs, then $J_3(G)$ is nonplanar.*

Proof. If the edge sets of three subgraphs G_1, G_2, and G_3 of G are $E(G_1) = \{e, f, g\}$, $E(G_2) = \{a\}$, and $E(G_3) = \{d\}$ respectively, then G contains either

H_1, H_2, H_3 or H_4, shown in Figure 5, as a subgraph. Consequently, $J_3(G)$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$ as shown in Figure 6. Thus $J_3(G)$ is nonplanar. □

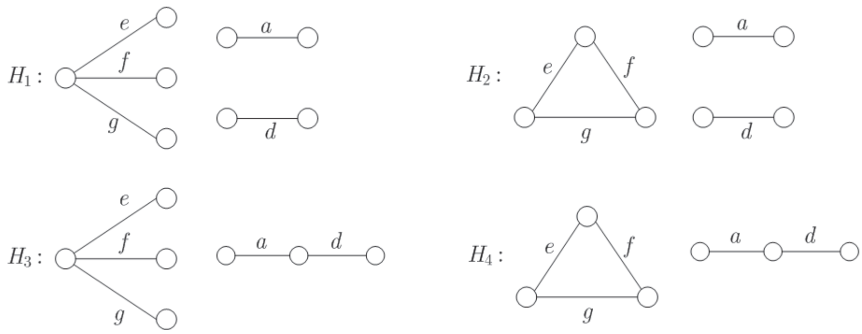

Figure 5: The four possible subgraphs for a graph G of Lemma 2.6

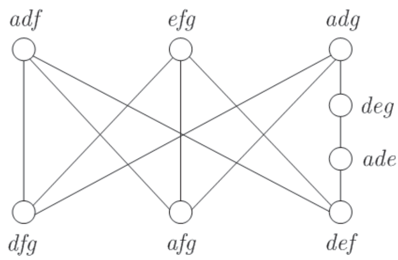

Figure 6: A subgraph of $J_3(G)$ of Lemma 2.6

We are next interested in showing that each of the graphs N_i for $1 \leq i \leq 20$ of Figure 7 and each of the graphs M_i for $1 \leq i \leq 11$ of Figure 10 have nonplanar and planar 3-jump graphs, respectively.

Theorem 2.7. *For each graph N_i, where $1 \leq i \leq 20$, of Figure 7, $J_3(N_i)$ is nonplanar.*

Proof. We have seen in Corollaries 2.1 and 2.2 that $J_3(P_6)$ and $J_3(C_5)$ are nonplanar. Thus it remains to show the nonplanarity for $J_3(N_i)$ where $1 \leq i \leq 18$.

For $i \in \{1, 5, 11, 16, 18\}$, some subgraph of $J_3(N_i)$ is shown in Figure 8(a), (b), (c), (d) and (e), respectively. Since, for each i, $J_3(N_i)$ contains a subgraph that is isomorphic to either a subdivision of K_5 or a subdivision of $K_{3,3}$, $J_3(N_i)$ is nonplanar.

For $i \in \{2, 3, 4, 7, 8, 12, 13, 15, 17\}$, N_i contains subgraphs G_1, G_2 and G_3 as mentioned in Lemma 2.6. (all edges of N_i are labeled to be corresponding with the edges of G_1, G_2 and G_3 in Lemma 2.6.) Thus, by Lemma 2.6, it follows that $J_3(N_i)$ is nonplanar.

For $i \in \{6, 9\}$, N_i contains a subgraph H as mentioned in Lemma 2.4. (all edges of N_i are labeled to be corresponding with the edges of H_1 and H_2 in Lemma 2.4.) Thus, by Lemma 2.4, $J_3(N_i)$ is nonplanar.

For $i \in \{10, 14\}$, N_i contains a subgraph H as mentioned in Lemma 2.5. (all edges of N_i are labeled to be corresponding with the edges of H_1 and H_2 in Lemma 2.5.) Thus, by Lemma 2.5, $J_3(N_i)$ is nonplanar. □

An alternative way to show that $J_3(N_i)$ where $i \in \{1, 2, \ldots, 20\} - \{10, 13, 14\}$ is nonplanar can be obtained from the following theorem.

Theorem 2.8. *If G is a connected graph of size m and H is a subgraph of G of size m' such that $J_k(H)$ is nonplanar then $J_{k+m-m'}(G)$ is nonplanar.*

Proof. If $H = G$ then the result is trivial. Assume that H is a proper subgraph of G and so $m - m' \geq 1$. We show that $J_{k+m-m'}(G)$ contains a subgraph F isomorphic to $J_k(H)$ which is nonplanar. Let $e_1, e_2, \ldots, e_{m-m'} \in E(G) - E(H)$. Now, for each vertex X of $J_k(H)$, let $e_1 e_2 \ldots e_{m-m'} X$ be a vertex of F. Thus $V(F) \subseteq V(J_{k+m-m'}(G))$. Since $e_1 e_2 \ldots e_{m-m'} X$ and $e_1 e_2 \ldots e_{m-m'} Y$ are adjacent in F if and only if X and Y are adjacent in $J_k(H)$, it follows that F is isomorphic to $J_k(H)$. □

In [3] it has been shown that for each graph H_i where $1 \leq i \leq 17$ in Figure 9, $J_2(H_i)$ is nonplanar. Observe that $H_{15} = N_1$, $H_{14} = N_2$, $H_9 = N_3$, $H_{13} \subsetneq N_4$, $H_{10} = N_5$, $H_{11} = N_6$, $H_{12} = N_7$, $H_1 \subsetneq N_8$, $H_2 = N_9$, $H_1 \subsetneq N_{11}$, $H_7 = N_{12}$, $H_4 = N_{15}$, $H_5 \subsetneq N_{16}$, $H_5 \subsetneq N_{17}$, $H_3 \subsetneq N_{18}$, $H_{17} = N_{19}$ and $H_{16} = N_{20}$. Thus by theorem 2.8, $J_3(N_i)$ is nonplanar.

We now turn our attention to graphs having a planar 3-jump graph. If $G = K_{1,m}$ is a star of size m then $J_k(G)$ is an empty graph of order $\binom{m}{k}$. Thus we have an immediate result.

Proposition 2.9. *If G is a star of size m then $J_k(G)$ is planar for every k where $1 \leq k \leq m$.*

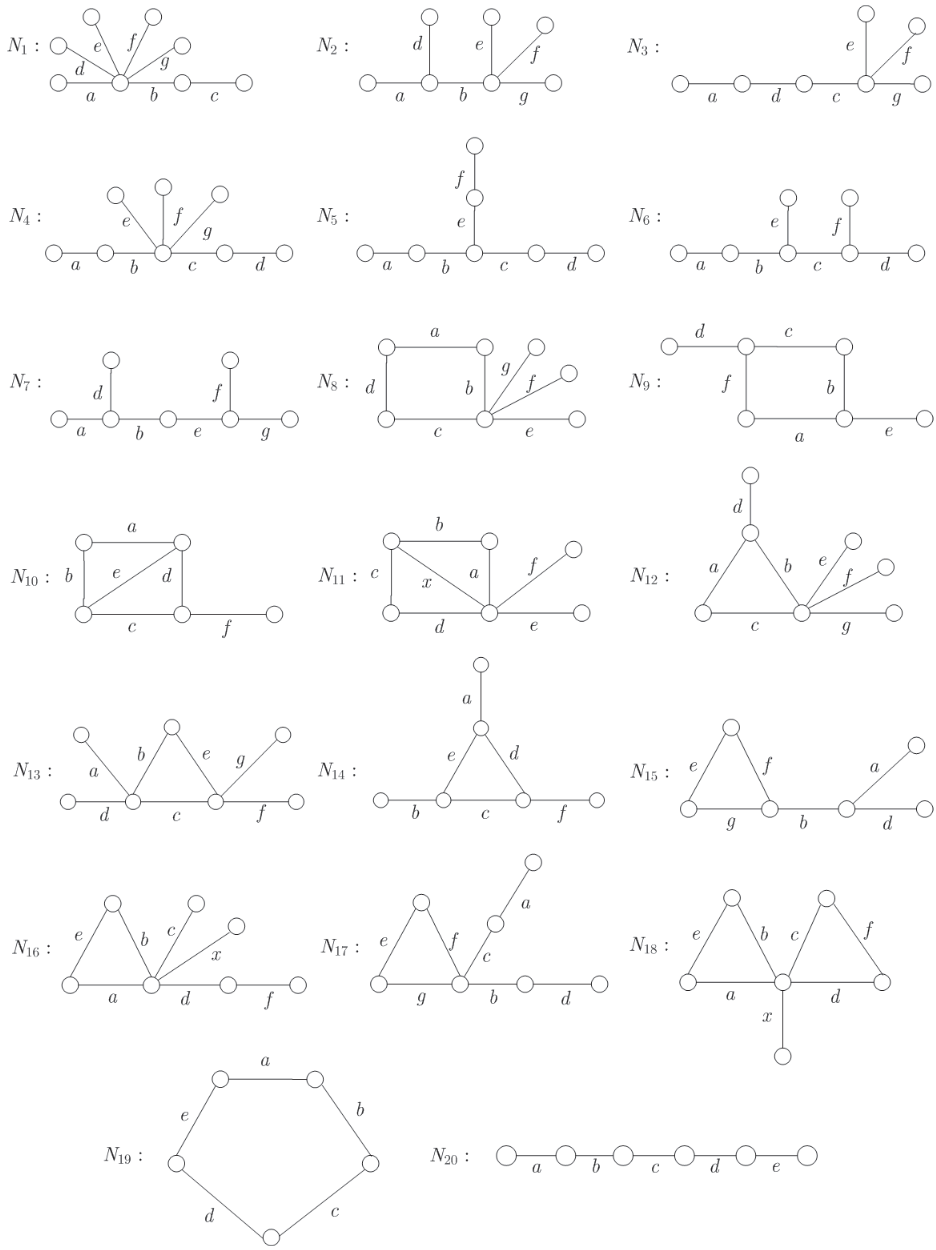

Figure 7: Graphs having a nonplanar 3-jump graph

Lemma 2.10. *For each graph M_i, where $1 \leq i \leq 11$, of Figure 10, $J_3(M_i)$*

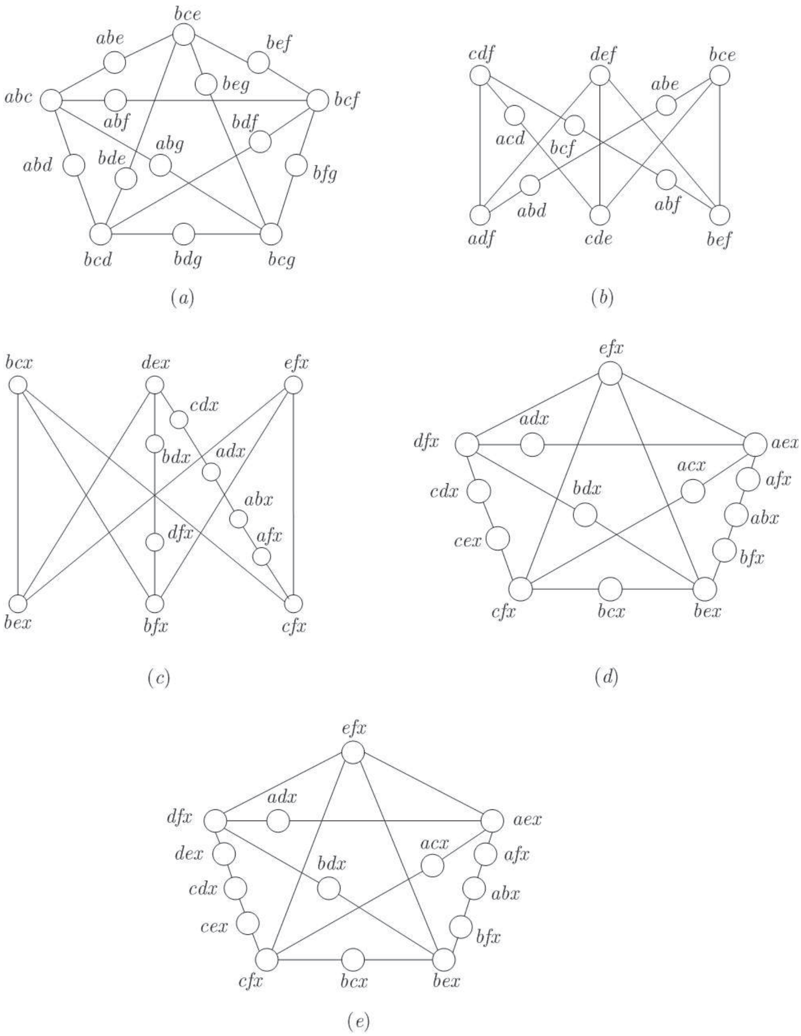

Figure 8: Some subgraph of N_i for $i = 1, 5, 11, 16$ and 18

is planar.

Proof. The 3-jump graph $J_3(M_i)$ of a graph M_i is shown in Figure 11 for $1 \leq i \leq 5$, and in Figure 12 for $6 \leq i \leq 11$. Thus, for each i, $J_3(M_i)$ is planar. $\square$

Next, we investigate that these graphs M_i, where $1 \leq i \leq 11$, of Figure 10 are maximal in the sense that its 3-jump graph is planar.

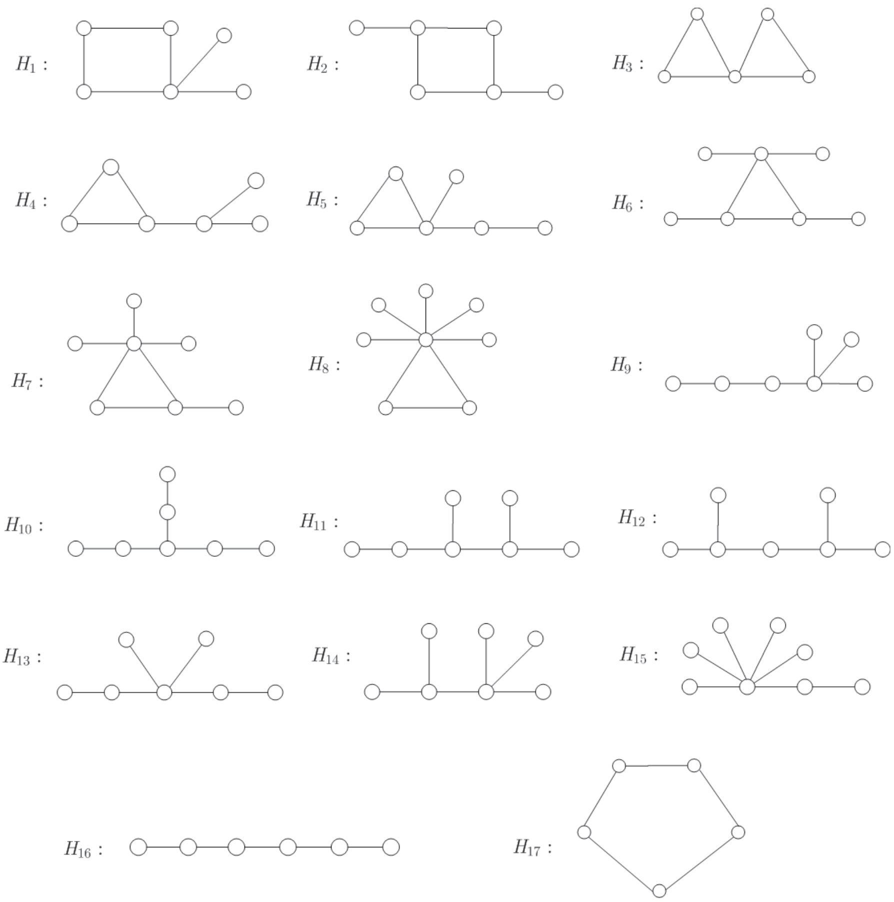

Figure 9: Graphs having a nonplanar 2-jump graph

Theorem 2.11. *For a connected graph G that is not a star, the 3-jump graph $J_3(G)$ is planar if and only if G is a subgraph of M_i for some i where $1 \leq i \leq 11$ of Figure 10.*

Proof. It has been shown in Lemma 2.10 that, for $1 \leq i \leq 11$, $J_3(M_i)$ is planar. Thus if G is a subgraph of M_i for some i where $1 \leq i \leq 11$ then $J_3(G)$ is planar by Lemma 2.3.

For the converse, let G be a connected graph that is not a star for whose $J_3(G)$ is planar. Then the connected graph G may or may not contain cycles.

Case 1. *G is a tree.* Since G cannot contain $N_{20} = P_6$, it follows that $\text{diam}(G) \leq 4$ and since G is not a star, we have that $\text{diam}(G) \geq 3$. Thus either $\text{diam}(G) = 3$ or $\text{diam}(G) = 4$. If $\text{diam}(G) = 3$ then G is a double star. Observe

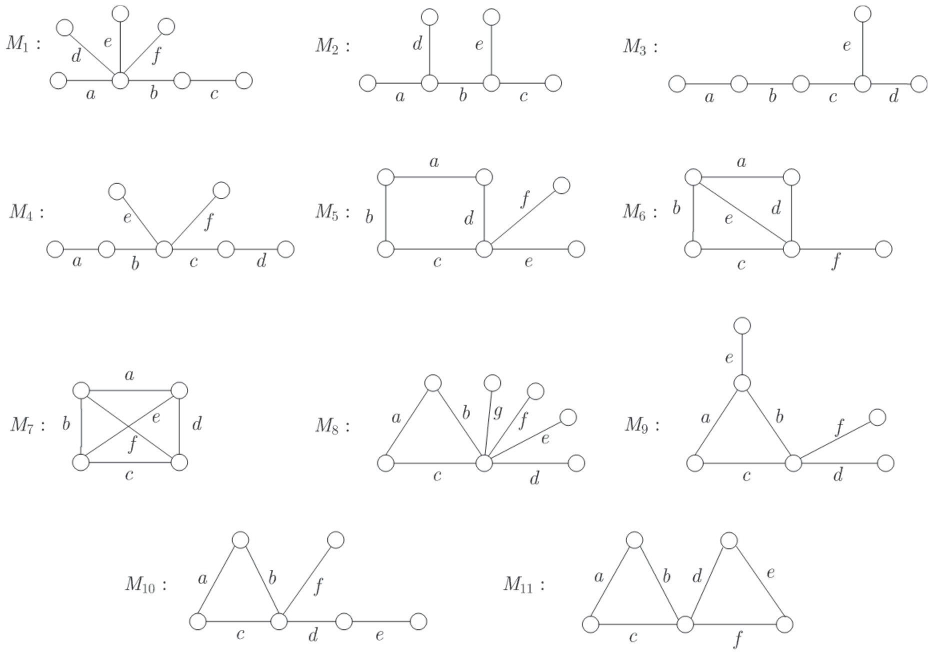

Figure 10: Graphs having a planar 3-jump graph

in this case that G cannot contain N_1 and N_2, for otherwise $J_3(G)$ is nonplanar which is a contradiction. Thus G is a subgraph of M_1 or M_2. (See Figure 13 for all graphs G with $\text{diam}(G) = 3$ and $J_3(G)$ is planar.) Now, we assume that $\text{diam}(G) = 4$. Then G cannot contain N_i, for $3 \leq i \leq 7$ as a subgraph. Therefore, G is a subgraph of M_3 or M_4. (See Figure 14 for all graphs G with $\text{diam}(G) = 4$ for whose $J_3(G)$ is planar.)

Case 2. *G contains cycles.* Since G cannot contain $N_{19} = C_5$, it follows that G contains C_3 or C_4. We consider the following three subcases.

Subcase 2.1. *G contains C_4 but not C_3.* Observe that G cannot contain N_i for each $i \in \{8, 9, 20\}$. Thus G is a subgraph of M_5. (See Figure 15 for all graphs G containing C_4 but not C_3 for whose $J_3(G)$ is planar.)

Subcase 2.2. *G contains both C_3 and C_4.* Then, for $i \in \{10, 11, 19, 20\}$, N_i cannot be a subgraph of G. Therefore G is a subgraph of M_6 or M_7. (See Figure 16 for all graphs G containing C_4 and C_3 for whose $J_3(G)$ is planar.)

Subcase 2.3. *G contains C_3 but not C_4.* Since the graph G cannot contain N_i where $12 \leq i \leq 18$ and N_{20}, G is a subgraph of M_8, M_9, M_{10} or M_{11}. (See Figure 17 for all graphs G containing C_3 but not C_4 for whose $J_3(G)$ is planar.) □

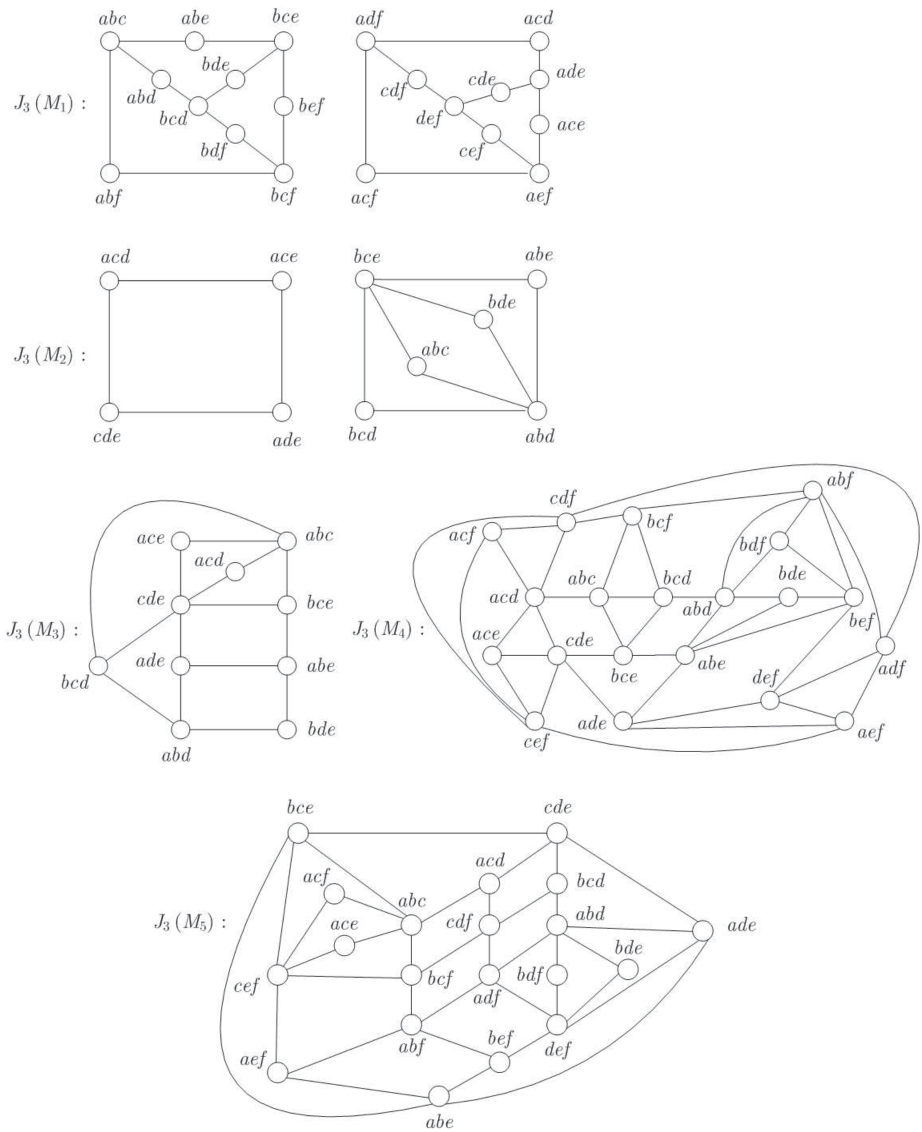

Figure 11: The 3-jump graphs of M_i for $1 \leq i \leq 5$

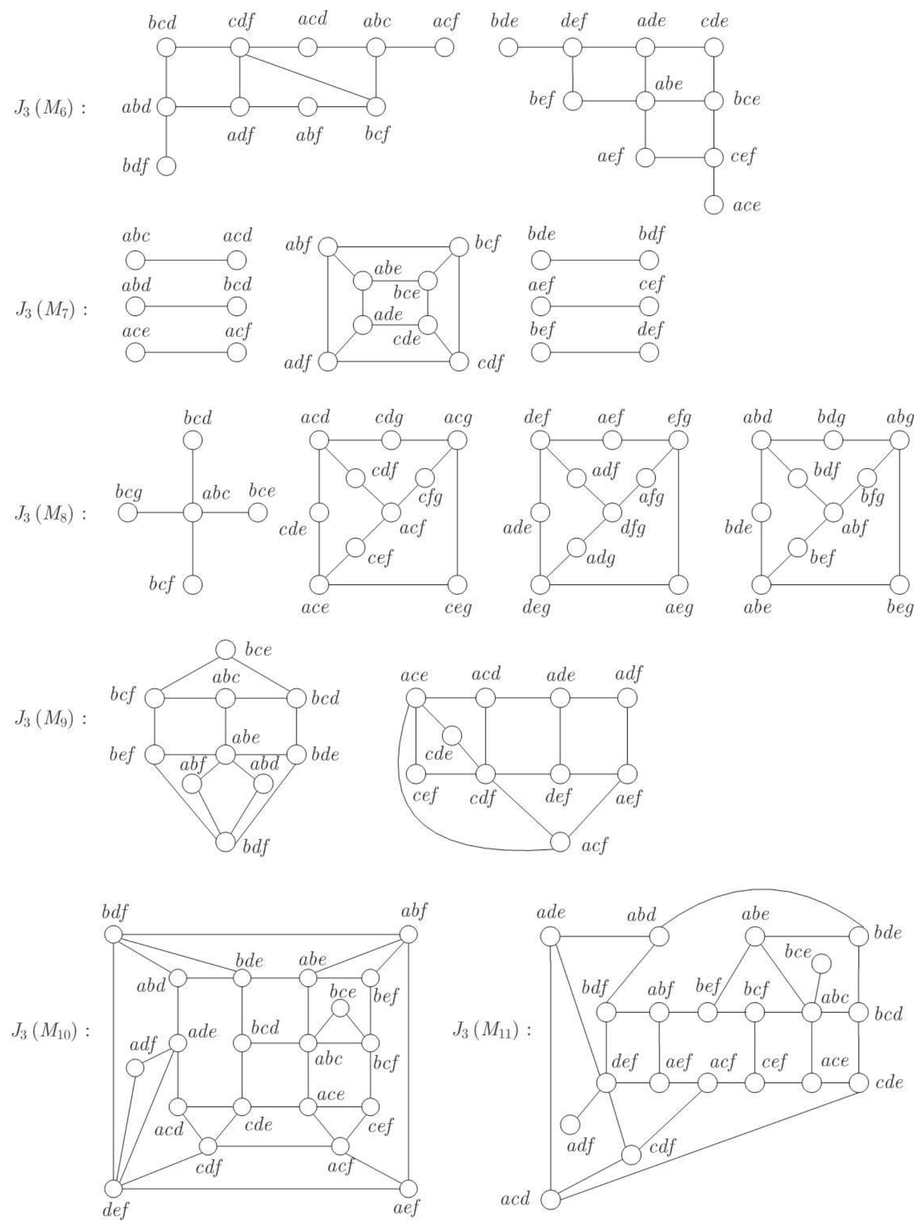

Figure 12: The 3-jump graphs of M_i for $6 \leq i \leq 11$

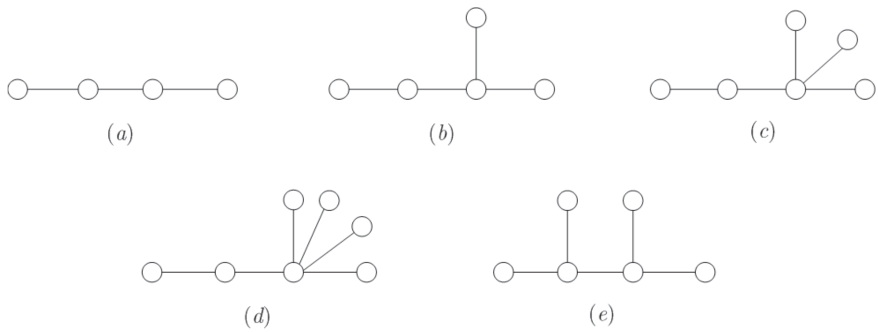

Figure 13: All trees with diameter 3 for whose the 3-jump graph is planar

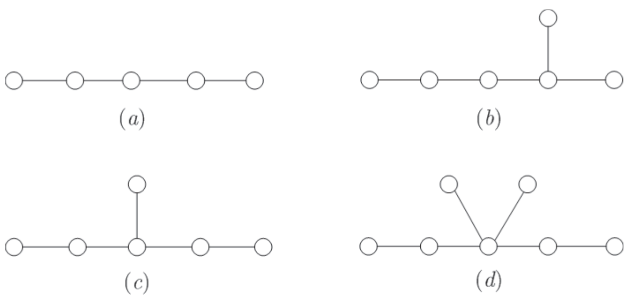

Figure 14: All trees with diameter 4 for whose the 3-jump graph is planar

We next present another characterization of graphs for whose the 3-jump graph is planar and also show that these graphs N_i, for $1 \leq i \leq 20$ are minimal according to its 3-jump graph being nonplanar.

Corollary 2.12. *For a connected graph G that is not a star, the 3-jump graph $J_3(G)$ is planar if and only if G does not contain any of N_i for $1 \leq i \leq 20$ of Figure 7 as a subgraph.*

Proof. If G contains N_i for some i where $1 \leq i \leq 20$, then $J_3(G)$ contains $J_3(N_i)$, by Lemma 2.3, and so $J_3(G)$ is nonplanar since $J_3(N_i)$ is nonplanar.

For the converse, we assume that G does not contain any of N_i for $1 \leq i \leq 20$. We consider two cases.

Case 1. *G is a tree.* Since G does not contain $N_{20} = P_6$, it follows that $\text{diam}(G) \leq 4$ and since G is not a star, we have that $\text{diam}(G) \geq 3$. Thus either

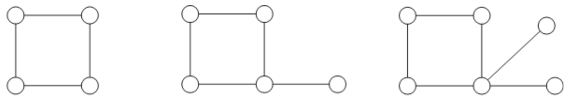

Figure 15: All graphs that contain C_4 but not C_3 for whose the 3-jump graph is planar

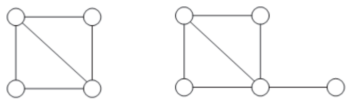

Figure 16: All graphs that contain C_3 and C_4 for whose the 3-jump graph is planar

$\text{diam}(G) = 3$ or $\text{diam}(G) = 4$. If $\text{diam}(G) = 3$ then since G does not contain N_1 and N_2, we have that G is a subgraph of M_1 or M_2. Thus $J_3(G)$ is planar. Now, if $\text{diam}(G) = 4$, then again since G does not contain N_i, where $3 \leq i \leq 7$ as a subgraph, G is a subgraph of M_3 or M_4. Therefore $J_3(G)$ is planar.

Case 2. *G contains cycles.* Since G does not contain $N_{19} = C_5$, it follows that G contains C_3 or C_4. If G contains C_4 but not C_3 then since G does not contain N_i for $i \in \{8, 9, 20\}$, G is a subgraph of M_5 and so $J_3(G)$ is planar. Next, if G contains both C_3 and C_4 then since none of N_i for $i \in \{10, 11, 19, 20\}$ is contained in G, G is a subgraph of M_6 or M_7 and thus $J_3(G)$ is planar. Finally, if G contains C_3 but not C_4 then since G does not contain N_i where $12 \leq i \leq 18$ and N_{20}, G is a subgraph of M_i for some $8 \leq i \leq 11$ and thus $J_3(G)$ is planar. □

3. Final Remarks

In this paper, we have characterized connected graphs whose the 3-jump graph is planar. A natural question arises what the characterization of a connected graph whose the k-jump graph where $4 \leq k \leq m - 4$ is planar.

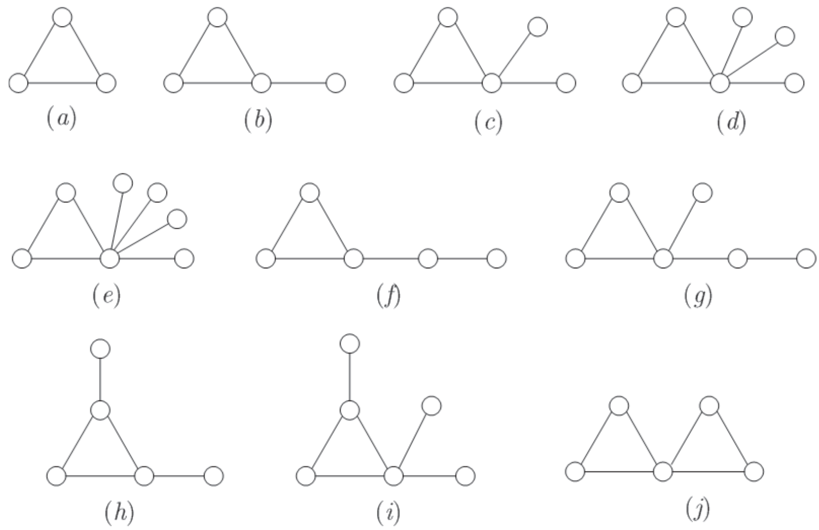

Figure 17: All graphs that contain C_3 but not C_4 for whose the 3-jump graph is planar

Acknowledgments

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