ON PLANARITY OF 3-JUMP GRAPHS

Varanoot Khemmani¹, Chira Lunduanhom²§, Sriwan Muangloy³, Massiri Muanphet⁴, Kittisak Tipnuch⁵
¹,²,³,⁴,⁵Department of Mathematics
Srinakharinwirot University
Sukhumvit 23, Bangkok, 10110, THAILAND

Abstract: For a graph $G$ of size $m \geq 1$ and edge-induced subgraphs $F$ and $H$ of size $k$ where $1 \leq k \leq m$, the subgraph $H$ is said to be obtained from the subgraph $F$ by an edge jump if there exist four distinct vertices $u, v, w$ and $x$ such that $uv \in E(F)$, $wx \in E(G) - E(F)$, and $H = F - uv + wx$. The $k$-jump graph $J_k(G)$ is that graph whose vertices correspond to the edge-induced subgraphs of size $k$ of $G$ where two vertices $F$ and $H$ of $J_k(G)$ are adjacent if and only if $H$ can be obtained from $F$ by an edge jump.

All connected graphs $G$ for whose $J_3(G)$ is planar are determined.

AMS Subject Classification: 05C10, 05C12
Key Words: $k$-jump distance, $k$-jump graph, 3-jump graph, planar graph

1. Introduction

The concept of the $k$-jump graph of a nonempty graph $G$ of size $m$ where $1 \leq k \leq m$ was introduced by Chartrand, Hevia, Jarrett and Schultz [1]. Let $G$ be a graph of size $m \geq 1$ and $F$ and $H$ be edge-induced subgraphs of size $k$ of $G$ where $1 \leq k \leq m$. The subgraph $H$ is said to be obtained from the subgraph $F$ by an edge jump if there exist four distinct vertices $u, v, w$ and $x$ such that $uv \in E(F)$, $wx \in E(G) - E(F)$, and $H = F - uv + wx$. It is obvious that if $H$ is obtained from $F$ by an edge jump then $F$ is obtained from...
by an edge jump. If there is a sequence $F = H_0, H_1, \ldots, H_\ell = H$ where $\ell \geq 1$ of edge-induced subgraphs of size $k$ such that $H_{i+1}$ is obtained from $H_i$ by an edge jump for $0 \leq i \leq \ell - 1$, then we say that $F$ can be $j$-transformed into $H$. The minimum number of edge jumps required to $j$-transform $F$ into $H$ is the $k$-jump distance from $F$ to $H$. For a graph $G$ of size $m \geq 1$ and an integer $k$ with $1 \leq k \leq m$, the $k$-jump graph $J_k(G)$ is that graph whose vertices correspond to the edge-induced subgraphs of size $k$ of $G$ where two vertices $F$ and $H$ of $J_k(G)$ are adjacent if and only if the $k$-jump distance between edge-induced subgraphs $F$ and $H$ is 1 that is, $H$ is obtained from $F$ by an edge jump. We can label each vertex of $J_k(G)$ by listing all the edges of the respective subgraph. The concept of the $k$-jump graph is illustrated in Figure 1. In particular, if $k = 1$ then the graph $J_1(G) = J(G)$ is called the jump graph of $G$. Moreover, $J(G) = \overline{L(G)}$, the complement of the line graph of $G$. In [3] all connected graphs $G$ for whose $J_2(G)$ is planar are determined in terms of a finite set $S$ of graphs, namely a connected graph $G$ has a planar 2-jump graph if and only if $G$ is a subgraph of some element of $S$. The goal of this paper is to characterize all connected graphs having a planar 3-jump graph along the same lines as the characterization of connected graphs having a planar 2-jump graph.

Figure 1: The $k$-jump graphs of a graph

The following results appeared in [3] and [4] will be useful for us later.

**Theorem 1.1.** ([4]) A graph is planar if and only if it contains no
subgraph isomorphic to $K_5$ or $K_{3,3}$ or a subdivision of one of these graphs.

**Theorem 1.2.** ([3]) If $G$ is a graph of size $m \geq 1$ and $k$ is an integer with $1 \leq k < m$, then $J_k(G) = J_{m-k}(G)$.

**Theorem 1.3.** ([3]) If a graph $G$ is a subdivision of a graph $H$ and $J_k(H)$ is nonplanar for some positive integer $k$, then $J_k(G)$ is nonplanar.

The reader is referred to the book [2] by Chartrand, Lesniak and Zhang for basic definitions and terminology not described here.

### 2. Connected Graphs with Planar 3-Jump Graphs

In this section we will focus our attention on the planarity of the 3-jump graph $J_3(G)$ for a connected graph $G$ of size at least 3. As we mentioned earlier, our aim is to determine a finite set $S$ of graphs with the property that a connected graph $G$ has a planar 3-jump graph if and only if $G$ is a subgraph of some element of $S$.

In [3] it is shown that for the path $P_n$ and cycle $C_n$ of order $n$, $J_2(P_n)$ is nonplanar if and only if $n \geq 6$ and $J_2(C_n)$ is nonplanar if and only if $n \geq 5$. Similar results can be obtained for $J_3(P_n)$ and $J_3(C_n)$. The 3-jump graphs $J_3(P_5)$, $J_3(P_6)$, $J_3(C_4)$, and $J_3(C_5)$ are shown in Figure 2 which we see that $J_3(P_5)$ and $J_3(C_4)$ are planar while $J_3(P_6)$ and $J_3(C_5)$ are nonplanar. Therefore, by Theorems 1.1 and 1.3, the following results are immediate.

**Corollary 2.1.** For $n \geq 4$, $J_3(P_n)$ is nonplanar if and only if $n \geq 6$.

**Corollary 2.2.** For $n \geq 3$, $J_3(C_n)$ is nonplanar if and only if $n \geq 5$.

We now present a simple but useful lemma.

**Lemma 2.3.** If $H$ is a subgraph of a connected graph $G$ then, for each $k$, $J_k(H)$ is a subgraph of $J_k(G)$.

**Proof.** Let $k$ be an integer such that $1 \leq k \leq m$ where $m$ is the size of $G$. If $v_{H_1}$ is a vertex of $J_k(H)$ that corresponds with an edge-induced subgraph $H_1$ of size $k$ of $H$ then since $H$ is a subgraph of $G$, $H_1$ is certainly an edge-induced subgraph of size $k$ of $G$. Thus $v_{H_1}$ is a vertex of $J_k(G)$ and so $V(J_k(H)) \subseteq V(J_k(G))$. On the other hand, if $e = v_{H_1}v_{H_2} \in E(J_k(H))$ then in a graph $H$, $H_1$ is obtained from $H_2$ by an edge jump. Now, since $H$ is a subgraph of $G$,
it follows that in $G$, $H_1$ is also obtained from $H_2$ by an edge jump. Therefore $e = v_{H_1}v_{H_2} \in E(J_k(G))$ and so $E(J_k(H)) \subseteq E(J_k(G))$. \hfill $\Box$

Theorem 1.1 and lemma 2.3 give us the following results.

**Lemma 2.4.** If $G$ is a graph containing a subgraph $H$ where $H$ is the union of edge-disjoint subgraphs $H_1$ of size at least 2 and $H_2$ of size at least 4 such that (1) $H_1$ contains edges $b$ and $e$, (2) $H_2$ contains edges $a$, $c$, $d$ and $f$ where $a$ is not adjacent to $c$, and (3) edges $b$ and $e$ are not adjacent to both $d$ and $f$ in $H$, then $J_3(G)$ is nonplanar.

**Proof.** Since there exists a subgraph of the 3-jump graph $J_3(H)$ that is isomorphic to a subdivision of $K_{3,3}$ as shown in Figure 3, it follows that $J_3(H)$ and so $J_3(G)$ are nonplanar. \hfill $\Box$
Lemma 2.5. If $G$ is a graph containing a subgraph $H$ where $H$ is the union of edge-disjoint subgraphs $H_1$ of size at least 2 and $H_2$ of size at least 4 such that (1) $H_1$ contains nonadjacent edges $a$ and $f$, (2) $H_2$ contains edges $b$, $c$, $d$ and $e$ where $b$ and $d$ are not adjacent, (3) $a$ is not adjacent to $c$ in $H$, and (4) $f$ is not adjacent to two edges $b$ and $e$ in $H$, then $J_3(G)$ is nonplanar.

Proof. Since the 3-jump graph $J_3(H)$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$ as shown in Figure 4, it follows that $J_3(H)$ and thus $J_3(G)$ are nonplanar.  

Lemma 2.6. If $G$ is a graph containing three subgraphs $G_1$, $G_2$, and $G_3$ where $G_1$ is isomorphic to $K_{1,3}$ or $K_3$, and $G_2$ and $G_3$ are both isomorphic to $P_2$ such that for each $i \in \{2, 3\}$, $G_1$ and $G_i$ are disjoint subgraphs, and $G_2$ and $G_3$ are edge-disjoint subgraphs, then $J_3(G)$ is nonplanar.

Proof. If the edge sets of three subgraphs $G_1$, $G_2$, and $G_3$ of $G$ are $E(G_1) = \{e, f, g\}$, $E(G_2) = \{a\}$, and $E(G_3) = \{d\}$ respectively, then $G$ contains either
$H_1$, $H_2$, $H_3$ or $H_4$, shown in Figure 5, as a subgraph. Consequently, $J_3(G)$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$ as shown in Figure 6. Thus $J_3(G)$ is nonplanar.

We are next interested in showing that each of the graphs $N_i$ for $1 \leq i \leq 20$ of Figure 7 and each of the graphs $M_i$ for $1 \leq i \leq 11$ of Figure 10 have nonplanar and planar 3-jump graphs, respectively.

**Theorem 2.7.** For each graph $N_i$, where $1 \leq i \leq 20$, of Figure 7, $J_3(N_i)$ is nonplanar.

*Proof.* We have seen in Corollaries 2.1 and 2.2 that $J_3(P_6)$ and $J_3(C_5)$ are nonplanar. Thus it remains to show the nonplanarity for $J_3(N_i)$ where $1 \leq i \leq 18$.

For $i \in \{1, 5, 11, 16, 18\}$, some subgraph of $J_3(N_i)$ is shown in Figure 8(a), (b), (c), (d) and (e), respectively. Since, for each $i$, $J_3(N_i)$ contains a subgraph
that is isomorphic to either a subdivision of $K_5$ or a subdivision of $K_{3,3}$, $J_3(N_i)$ is nonplanar.

For $i \in \{2, 3, 4, 7, 8, 12, 13, 15, 17\}$, $N_i$ contains subgraphs $G_1$, $G_2$ and $G_3$ as mentioned in Lemma 2.6. (all edges of $N_i$ are labeled to be corresponding with the edges of $G_1$, $G_2$ and $G_3$ in Lemma 2.6.) Thus, by Lemma 2.6, it follows that $J_3(N_i)$ is nonplanar.

For $i \in \{6, 9\}$, $N_i$ contains a subgraph $H$ as mentioned in Lemma 2.4. (all edges of $N_i$ are labeled to be corresponding with the edges of $H_1$ and $H_2$ in Lemma 2.4.) Thus, by Lemma 2.4, $J_3(N_i)$ is nonplanar.

For $i \in \{10, 14\}$, $N_i$ contains a subgraph $H$ as mentioned in Lemma 2.5. (all edges of $N_i$ are labeled to be corresponding with the edges of $H_1$ and $H_2$ in Lemma 2.5.) Thus, by Lemma 2.5, $J_3(N_i)$ is nonplanar.

An alternative way to show that $J_3(N_i)$ where $i \in \{1, 2, \ldots, 20\} - \{10, 13, 14\}$ is nonplanar can be obtained from the following theorem.

**Theorem 2.8.** If $G$ is a connected graph of size $m$ and $H$ is a subgraph of $G$ of size $m'$ such that $J_k(H)$ is nonplanar then $J_{k+m-m'}(G)$ is nonplanar.

**Proof.** If $H = G$ then the result is trivial. Assume that $H$ is a proper subgraph of $G$ and so $m - m' \geq 1$. We show that $J_{k+m-m'}(G)$ contains a subgraph $F$ isomorphic to $J_k(H)$ which is nonplanar. Let $e_1, e_2, \ldots, e_{m-m'} \in E(G) - E(H)$. Now, for each vertex $X$ of $J_k(H)$, let $e_1 e_2 \ldots e_{m-m'} X$ be a vertex of $F$. Thus $V(F) \subseteq V(J_{k+m-m'}(G))$. Since $e_1 e_2 \ldots e_{m-m'} X$ and $e_1 e_2 \ldots e_{m-m'} Y$ are adjacent in $F$ if and only if $X$ and $Y$ are adjacent in $J_k(H)$, it follows that $F$ is isomorphic to $J_k(H)$.

In [3] it has been shown that for each graph $H_i$ where $1 \leq i \leq 17$ in Figure 9, $J_2(H_i)$ is nonplanar. Observe that $H_{15} = N_1$, $H_{14} = N_2$, $H_9 = N_3$, $H_{13} \subsetneq N_4$, $H_{10} = N_5$, $H_{11} = N_6$, $H_{12} = N_7$, $H_1 \subsetneq N_8$, $H_2 = N_9$, $H_1 \subsetneq N_{11}$, $H_7 = N_{12}$, $H_4 = N_{15}$, $H_5 \subsetneq N_{16}$, $H_5 \subsetneq N_{17}$, $H_3 \subsetneq N_{18}$, $H_{17} = N_{19}$ and $H_{16} = N_{20}$. Thus by theorem 2.8, $J_3(N_i)$ is nonplanar.

We now turn our attention to graphs having a planar 3-jump graph. If $G = K_{1,m}$ is a star of size $m$ then $J_k(G)$ is an empty graph of order $\binom{m}{k}$. Thus we have an immediate result.

**Proposition 2.9.** If $G$ is a star of size $m$ then $J_k(G)$ is planar for every $k$ where $1 \leq k \leq m$. 

Figure 7: Graphs having a nonplanar 3-jump graph

Lemma 2.10. For each graph $M_i$, where $1 \leq i \leq 11$, of Figure 10, $J_3(M_i)$
is planar.

Proof. The 3-jump graph $J_3(M_i)$ of a graph $M_i$ is shown in Figure 11 for $1 \leq i \leq 5$, and in Figure 12 for $6 \leq i \leq 11$. Thus, for each $i$, $J_3(M_i)$ is planar.

Next, we investigate that these graphs $M_i$, where $1 \leq i \leq 11$, of Figure 10 are maximal in the sense that its 3-jump graph is planar.
Theorem 2.11. For a connected graph $G$ that is not a star, the 3-jump graph $J_3(G)$ is planar if and only if $G$ is a subgraph of $M_i$ for some $i$ where $1 \leq i \leq 11$ of Figure 10.

Proof. It has been shown in Lemma 2.10 that, for $1 \leq i \leq 11$, $J_3(M_i)$ is planar. Thus if $G$ is a subgraph of $M_i$ for some $i$ where $1 \leq i \leq 11$ then $J_3(G)$ is planar by Lemma 2.3.

For the converse, let $G$ be a connected graph that is not a star for whose $J_3(G)$ is planar. Then the connected graph $G$ may or may not contain cycles.

Case 1. $G$ is a tree. Since $G$ cannot contain $N_{20} = P_6$, it follows that $\text{diam}(G) \leq 4$ and since $G$ is not a star, we have that $\text{diam}(G) \geq 3$. Thus either $\text{diam}(G) = 3$ or $\text{diam}(G) = 4$. If $\text{diam}(G) = 3$ then $G$ is a double star. Observe
in this case that $G$ cannot contain $N_1$ and $N_2$, for otherwise $J_3(G)$ is nonplanar which is a contradiction. Thus $G$ is a subgraph of $M_1$ or $M_2$. (See Figure 13 for all graphs $G$ with diam$(G) = 3$ and $J_3(G)$ is planar.) Now, we assume that diam$(G) = 4$. Then $G$ cannot contain $N_i$, for $3 \leq i \leq 7$ as a subgraph. Therefore, $G$ is a subgraph of $M_3$ or $M_4$. (See Figure 14 for all graphs $G$ with diam$(G) = 4$ for whose $J_3(G)$ is planar.)

Case 2. $G$ contains cycles. Since $G$ cannot contain $N_{19} = C_5$, it follows that $G$ contains $C_3$ or $C_4$. We consider the following three subcases.

Subcase 2.1. $G$ contains $C_4$ but not $C_3$. Observe that $G$ cannot contain $N_i$ for each $i \in \{8, 9, 20\}$. Thus $G$ is a subgraph of $M_5$. (See Figure 15 for all graphs $G$ containing $C_4$ but not $C_3$ for whose $J_3(G)$ is planar.)

Subcase 2.2. $G$ contains both $C_3$ and $C_4$. Then, for $i \in \{10, 11, 19, 20\}$, $N_i$ cannot be a subgraph of $G$. Therefore $G$ is a subgraph of $M_6$ or $M_7$. (See Figure 16 for all graphs $G$ containing $C_4$ and $C_3$ for whose $J_3(G)$ is planar.)

Subcase 2.3. $G$ contains $C_3$ but not $C_4$. Since the graph $G$ cannot contain $N_i$ where $12 \leq i \leq 18$ and $N_{20}$, $G$ is a subgraph of $M_8$, $M_9$, $M_{10}$ or $M_{11}$. (See Figure 17 for all graphs $G$ containing $C_3$ but not $C_4$ for whose $J_3(G)$ is planar.)
Figure 11: The 3-jump graphs of $M_i$ for $1 \leq i \leq 5$
Figure 12: The 3-jump graphs of $M_i$ for $6 \leq i \leq 11$
We next present another characterization of graphs for whose the 3-jump graph is planar and also show that these graphs \( N_i \), for \( 1 \leq i \leq 20 \) are minimal according to its 3-jump graph being nonplanar.

**Corollary 2.12.** For a connected graph \( G \) that is not a star, the 3-jump graph \( J_3(G) \) is planar if and only if \( G \) does not contain any of \( N_i \) for \( 1 \leq i \leq 20 \) of Figure 7 as a subgraph.

**Proof.** If \( G \) contains \( N_i \) for some \( i \) where \( 1 \leq i \leq 20 \), then \( J_3(G) \) contains \( J_3(N_i) \), by Lemma 2.3, and so \( J_3(G) \) is nonplanar since \( J_3(N_i) \) is nonplanar.

For the converse, we assume that \( G \) does not contain any of \( N_i \) for \( 1 \leq i \leq 20 \). We consider two cases.

**Case 1.** \( G \) is a tree. Since \( G \) does not contain \( N_{20} = P_6 \), it follows that \( \text{diam}(G) \leq 4 \) and since \( G \) is not a star, we have that \( \text{diam}(G) \geq 3 \). Thus either
diam\( (G) \) = 3 or diam\( (G) \) = 4. If diam\( (G) \) = 3 then since \( G \) does not contain \( N_1 \) and \( N_2 \), we have that \( G \) is a subgraph of \( M_1 \) or \( M_2 \). Thus \( J_3(G) \) is planar. Now, if diam\( (G) \) = 4, then again since \( G \) does not contain \( N_i \), where \( 3 \leq i \leq 7 \) as a subgraph, \( G \) is a subgraph of \( M_3 \) or \( M_4 \). Therefore \( J_3(G) \) is planar.

\textbf{Case 2.} \( G \) contains cycles. Since \( G \) does not contain \( N_{19} = C_5 \), it follows that \( G \) contains \( C_3 \) or \( C_4 \). If \( G \) contains \( C_4 \) but not \( C_3 \) then since \( G \) does not contain \( N_i \) for \( i \in \{8,9,20\} \), \( G \) is a subgraph of \( M_5 \) and so \( J_3(G) \) is planar. Next, if \( G \) contains both \( C_3 \) and \( C_4 \) then since none of \( N_i \) for \( i \in \{10,11,19,20\} \) is contained in \( G \), \( G \) is a subgraph of \( M_6 \) or \( M_7 \) and thus \( J_3(G) \) is planar. Finally, if \( G \) contains \( C_3 \) but not \( C_4 \) then since \( G \) does not contain \( N_i \) where \( 12 \leq i \leq 18 \) and \( N_{20} \), \( G \) is a subgraph of \( M_i \) for some \( 8 \leq i \leq 11 \) and thus \( J_3(G) \) is planar.

\[ \square \]

\section{3. Final Remarks}

In this paper, we have characterized connected graphs whose the 3-jump graph is planar. A natural question arises what the characterization of a connected graph whose the \( k \)-jump graph where \( 4 \leq k \leq m - 4 \) is planar.
Figure 17: All graphs that contain $C_3$ but not $C_4$ for whose the 3-jump graph is planar.

Acknowledgments

Research is supported by Faculty of Science, Srinakharinwirot University, Year 2015.

References


