International Journal of Pure and Applied Mathematics
Volume 108 No. 2 2016, 199-214
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v108i2.2

ijpam.eu

ρC($\mathcal{I}$)-COMPACT AND $\rho\mathcal{I}$-QHC SPACES

Néstor Raúl Pachón

Department of Mathematics
Colombian School of Engineering
AK 45 No 205-59, 111166, Bogotá, COLOMBIA

Abstract: In this paper we introduce and investigate two new ideal topological spaces, which are strong forms of Gupta-Noiri concepts. Interesting characterizations of this spaces are presented, as well as several useful properties of these. We compare this new spaces with C-compact and quasi-H-closed spaces.

AMS Subject Classification: 54D30, 54C10
Key Words: C-compact, quasi-H-closed, $\mathcal{I}$-compact, C($\mathcal{I}$)-compact, $\mathcal{I}$-QHC, $\rho\mathcal{I}$-compact

1. Introduction and Preliminaries

The ideal topological spaces has been introduced in Kuratowski [5] and Vaidyanathaswamy [12] books. The concept of compactness modulo an ideal was introduced by Newcomb [7], but popularized by Hamlett-Jancovic papers [3][4]. The C-compact spaces and QHC spaces were defined by Viglino [13] and Porter-Thomas [10], respectively, and are generalizations of compactness. In 2006 Gupta-Noiri [2] generalize Viglino and Porter through the notion of C($\mathcal{I}$)-compact and $\mathcal{I}$-QHC spaces.

In this paper we introduce and study the ρC($\mathcal{I}$)-compact and $\rho\mathcal{I}$-QHC spaces, which are strong forms of the Gupta-Noiri concepts. Interesting characterizations of this new spaces will also be presented, as well as their relationship with the $\rho\mathcal{I}$-compact spaces [9].

Received: March 18, 2016
Published: June 18, 2016

An ideal $\mathcal{I}$ in a set X is a subset of $\mathcal{P}(X)$, the power set of X, such that: (i) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some useful ideals in X are: (i) $\mathcal{P}(A)$, where $A \subseteq X$, (ii) $\mathcal{I}_f$, the ideal of all finite subsets of X, (iii) $\mathcal{I}_c$, the ideal of all countable subsets of X, (iv) $\mathcal{I}_n$, the ideal of all nowhere dense subsets in a topological space (X,τ).

If (X,τ) is a topological space and $\mathcal{I}$ is an ideal in X, then $(X,\tau,\mathcal{I})$ is called an *ideal space*.

If $(X,\tau,\mathcal{I})$ is an ideal space then the set $\mathcal{B} = \{U \backslash I : U \in \tau$ and $I \in \mathcal{I}\}$ is a base for a topology τ^*, finer than τ.

If (X,τ) is a topological space and $A \subseteq X$ then $\overline{A}$ (or $adh\,(A)$, or $adh_\tau\,(A)$) and $\overset{0}{A}$ (or $int(A)$, or $int_\tau(A)$) will, respectively, denote the closure and interior of A in (X,τ).

If (X,τ) is a topological space and $A \subseteq X$ then A is said to be *regular open* if $A = \overset{0}{\overline{A}}$, and A is defined to be *regular closed* if $A = \overline{\overset{0}{A}}$. If $A \subseteq \overset{0}{\overline{A}}$ then A is called *pre-open* [6]. The set of all pre-open subsets of X is denoted by $PO(X)$. If $A \subseteq \overset{0}{\overline{\overset{0}{A}}}$ then A is called *α-open* [8]. Clearly open $\Rightarrow$ α-open $\Rightarrow$ pre-open.

Moreover, if $\mathcal{I}$ is an ideal in X and $\mathcal{I} \cap \tau = \{\varnothing\}$, $\mathcal{I}$ is called *codense* [1]. If $\mathcal{I} \cap PO(X) = \{\varnothing\}$ then $\mathcal{I}$ is said to be *completely codense* [1].

2. $\rho\mathcal{I}$-QHC spaces

A topological space (X,τ) is said to be *quasi-H-closed*, or simply *QHC* [10], if for each open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

An ideal space $(X,\tau,\mathcal{I})$ is defined to be *$\mathcal{I}$-compact* [7] if for all open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

The space $(X,\tau,\mathcal{I})$ is said to be *$\mathcal{I}$-QHC* [2] if for all open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

It is noted that QHC $\Rightarrow$ $\mathcal{I}$-QHC.

An ideal space $(X,\tau,\mathcal{I})$ is defined to be *$\rho\mathcal{I}$-compact* [9] if for each family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $X \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

In this section we define the $\rho\mathcal{I}$-*QHC* spaces and study some of its properties and characterizations.

Definition 2.1 If $(X,\tau,\mathcal{I})$ is an ideal space and $A \subseteq X$, then A is said to be $\rho\mathcal{I}$-*QHC* if for all family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $A\backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $A\backslash \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$. The ideal space $(X,\tau,\mathcal{I})$ is said to be $\rho\mathcal{I}$-*QHC* if X is $\rho\mathcal{I}$-QHC.

Clearly $(X,\tau,\{\varnothing\})$ is $\rho\mathcal{I}$-QHC $\Leftrightarrow$ $(X,\tau,\{\varnothing\})$ is $\mathcal{I}$-QHC $\Leftrightarrow$ (X,τ) is QHC. It is also evident that $\rho\mathcal{I}$-QHC $\Rightarrow$ $\mathcal{I}$-QHC and $\rho\mathcal{I}$-compact $\Rightarrow$ $\rho\mathcal{I}$-QHC, but the converse, in general, are not true.

Example 2.1 We denote by $2\mathbb{Z}$ the set of even integers, and by $2\mathbb{Z}+1$ the set of odd integers.

Let τ be the topology on $\mathbb{Z}$ given by: If $V \subseteq \mathbb{Z}$ then $V \in \tau \Leftrightarrow$ [if $0 \in V$ then $2\mathbb{Z} \subseteq V$, and if $1 \in V$ then $2\mathbb{Z}+1 \subseteq V$].

Let $\mathcal{I} = \mathcal{P}\left[(2\mathbb{Z}+1)\cup\{0\}\right]$. We have that:

a) $(\mathbb{Z},\tau)$ is a QHC space, and then $(\mathbb{Z},\tau,\mathcal{I})$ is $\mathcal{I}$-QHC.

If $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of $\mathbb{Z}$ and $\mathbb{Z} = \bigcup\limits_{\alpha\in\Lambda} V_\alpha$, then there are $\alpha_0 \in \Lambda$ and $\alpha_1 \in \Lambda$ with $0 \in V_{\alpha_0}$ and $1 \in V_{\alpha_1}$. Then $2\mathbb{Z} \subseteq V_{\alpha_0}$ and $2\mathbb{Z}+1 \subseteq V_{\alpha_1}$, and so $\mathbb{Z} = \overline{V_{\alpha_0}} \cup \overline{V_{\alpha_1}}$.

b) $(\mathbb{Z},\tau,\mathcal{I})$ is not $\rho\mathcal{I}$-QHC.

$\mathbb{Z}\backslash \bigcup\limits_{n\neq 0}\{2n\} = (2\mathbb{Z}+1)\cup\{0\} \in \mathcal{I}$, but if $n \neq 0$ we have that $\overline{\{2n\}} = \{0,2n\}$, and if $\{n_1,n_2,...,n_r\} \subseteq \mathbb{Z}\backslash\{0\}$ then $\mathbb{Z}\backslash \bigcup\limits_{j=1}^{r} \overline{\{2n_j\}} \notin \mathcal{I}$.

In the Examples 3.1 and 3.2 we show $\rho\mathcal{I}$-QHC spaces.

It is easy to see that an open and closed subset of a $\rho\mathcal{I}$-QHC space is $\rho\mathcal{I}$-QHC.

In the next theorem we present interesting characterizations of $\rho\mathcal{I}$-QHC spaces. The proof is similar to that of Theorems 3.2 and 3.3, so we omit it.

If $\mathcal{I}$ is an ideal in a set X, a family $\mathcal{F}$ of subsets of X is said to have the *finite-intersection property modulo* $\mathcal{I}$, if for each $\mathcal{F}_0 \subseteq \mathcal{F}$, finite, we have that $\bigcap\limits_{V\in\mathcal{F}_0} V \notin \mathcal{I}$.

Theorem 2.1 *For an ideal space $(X,\tau,\mathcal{I})$, the following statements are equivalents:*

1) $(X,\tau,\mathcal{I})$ *is* $\rho\mathcal{I}$*-QHC.*

2) For each family $\{F_\alpha\}_{\alpha\in\Lambda}$ *of closed subsets of* X*, if* $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \in \mathcal{I}$*, there exists* $\Lambda_0 \subseteq \Lambda$*, finite, such that* $\bigcap\limits_{\alpha\in\Lambda_0} \overset{0}{F}_\alpha \in \mathcal{I}$*.*

3) For each family $\{F_\alpha\}_{\alpha\in\Lambda}$ *of closed subsets, if* $\left\{\overset{0}{F}_\alpha : \alpha\in\Lambda\right\}$ *has the finite-intersection property modulo* $\mathcal{I}$*, then* $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \notin \mathcal{I}$*.*

4) For each family $\{V_\alpha\}_{\alpha\in\Lambda}$ *of regular open subsets of* X*, if* $X\setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$*, there is* $\Lambda_0 \subseteq \Lambda$*, finite, such that* $X\setminus \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$*.*

5) For each family $\{F_\alpha\}_{\alpha\in\Lambda}$ *of regular closed subsets of* X*, if* $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \in \mathcal{I}$*, there is* $\Lambda_0 \subseteq \Lambda$*, finite, such that* $\bigcap\limits_{\alpha\in\Lambda_0} \overset{0}{F}_\alpha \in \mathcal{I}$*.*

6) For each family $\{F_\alpha\}_{\alpha\in\Lambda}$ *of regular closed subsets of* X*, if* $\left\{\overset{0}{F}_\alpha : \alpha\in\Lambda\right\}$ *has the finite-intersection property modulo* $\mathcal{I}$*, then* $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \notin \mathcal{I}$*.*

7) For each open filter base Ω *on* X *such that* $\Omega \subseteq \mathcal{P}(X)\setminus\mathcal{I}$*, one has* $\bigcap\limits_{V\in\Omega} \overline{V} \notin \mathcal{I}$*.*

It follows from a result in [11] that if (X,τ) is a topological space and $\mathcal{I}$ is a completely codense ideal in X, then (X,τ) and (X,τ^*) have the same regular open subsets, and $adh_\tau(V) = adh_{\tau^*}(V)$, for all $V \in \tau^*$. Then the following result is clear.

Theorem 2.2 *If* $\mathcal{I}$ *is a completely codense ideal in* X*, the space* $(X,\tau,\mathcal{I})$ *is* $\rho\mathcal{I}$*-QHC if and only if* $(X,\tau^*,\mathcal{I})$ *is* $\rho\mathcal{I}$*-QHC.*

Now we review the behavior of $\rho\mathcal{I}$-QHC spaces under continuous or open functions.

Theorem 2.3 *1) If* $(X,\tau,\mathcal{I})$ *is* $\rho\mathcal{I}$*-QHC and* $f:(X,\tau)\to(Y,\beta)$ *is a biyective continuous function, then* $(Y,\beta,f(\mathcal{I}))$ *is* $\rho f(\mathcal{I})$*-QHC, where* $f(\mathcal{I})$ *is the ideal* $\{f(I) : I\in\mathcal{I}\}$*.*

2) If $(X,\tau,\mathcal{I})$ *is* $\rho\mathcal{I}$*-QHC and* $f:(X,\tau)\to(Y,\beta)$ *is a continuous function, then* $(Y,\beta,\mathcal{J})$ *is* $\rho\mathcal{J}$*-QHC, where* $\mathcal{J}$ *is the ideal* $\left\{V\subseteq Y : f^{-1}(V)\in\mathcal{I}\right\}$*.*

3) If $(Y,\beta,\mathcal{J})$ is $\rho\mathcal{J}$-QHC and $f:(X,\tau)\to(Y,\beta)$ is a biyective and open function, then $\left(X,\tau,f^{-1}(\mathcal{J})\right)$ is $\rho f^{-1}(\mathcal{J})$-QHC, where $f^{-1}(\mathcal{J})$ is the ideal $\left\{f^{-1}(V):V\in\mathcal{J}\right\}$.

Proof. 1) Suppose that $\{W_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of Y with $Y\setminus\bigcup_{\alpha\in\Lambda}W_\alpha\in f(\mathcal{I})$. There exists $I\in\mathcal{I}$ such that $Y\setminus\bigcup_{\alpha\in\Lambda}W_\alpha=f(I)$. Since $X\setminus\bigcup_{\alpha\in\Lambda}f^{-1}(W_\alpha)=f^{-1}(f(I))=I\in\mathcal{I}$, there exists $\Lambda_0\subseteq\Lambda$, finite, with $X\setminus\bigcup_{\alpha\in\Lambda_0}\overline{f^{-1}(W_\alpha)}\in\mathcal{I}$. Given that f is continuous, $f^{-1}\left(Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{W_\alpha}\right)=X\setminus\bigcup_{\alpha\in\Lambda_0}f^{-1}(\overline{W_\alpha})\subseteq X\setminus\bigcup_{\alpha\in\Lambda_0}\overline{f^{-1}(W_\alpha)}$, and so $f^{-1}\left(Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{W_\alpha}\right)\in\mathcal{I}$. Then $Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{W_\alpha}=f\left(f^{-1}\left(Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{W_\alpha}\right)\right)\in f(\mathcal{I})$.

2) It is easy to see that $\mathcal{J}$ is an ideal in Y. Suppose that $\{W_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of Y with $Y\setminus\bigcup_{\alpha\in\Lambda}W_\alpha\in\mathcal{J}$. Since $X\setminus\bigcup_{\alpha\in\Lambda}f^{-1}(W_\alpha)=f^{-1}\left(Y\setminus\bigcup_{\alpha\in\Lambda}W_\alpha\right)\in\mathcal{I}$, there is $\Lambda_0\subseteq\Lambda$, finite, with $X\setminus\bigcup_{\alpha\in\Lambda_0}\overline{f^{-1}(W_\alpha)}\in\mathcal{I}$. Given that f is continuous, $X\setminus\bigcup_{\alpha\in\Lambda_0}f^{-1}(\overline{W_\alpha})\subseteq X\setminus\bigcup_{\alpha\in\Lambda_0}\overline{f^{-1}(W_\alpha)}$, and so $f^{-1}\left(Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{W_\alpha}\right)=X\setminus\bigcup_{\alpha\in\Lambda_0}f^{-1}(\overline{W_\alpha})\in\mathcal{I}$. Thus $Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{W_\alpha}\in\mathcal{J}$.

3) Suppose that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X, with $X\setminus\bigcup_{\alpha\in\Lambda}V_\alpha\in f^{-1}(\mathcal{J})$. There exists $J\in\mathcal{J}$ such that $X\setminus\bigcup_{\alpha\in\Lambda}V_\alpha=f^{-1}(J)$. Then $Y\setminus\bigcup_{\alpha\in\Lambda}f(V_\alpha)=J$ and so there is $\Lambda_0\subseteq\Lambda$, finite, with $Y\setminus\bigcup_{\alpha\in\Lambda_0}\overline{f(V_\alpha)}\in\mathcal{J}$. Given that f is open and biyective, f is closed, and so $\overline{f(V_\alpha)}\subseteq f\left(\overline{V_\alpha}\right)$, for each $\alpha\in\Lambda_0$. This implies that $Y\setminus\bigcup_{\alpha\in\Lambda_0}f(\overline{V_\alpha})\in\mathcal{J}$, and then $X\setminus\bigcup_{\alpha\in\Lambda_0}\overline{V_\alpha}\in f^{-1}(\mathcal{J})$.

We end this section by presenting a characterization of $\rho\mathcal{I}$-QHC spaces in terms of pre-open and α-open subsets. The proof is similar to that of Theorem 3.7.

Theorem 2.4 *If $(X,\tau,\mathcal{I})$ is an ideal space, the following statements are equivalents:*

1) $(X,\tau,\mathcal{I})$ is $\rho\mathcal{I}$-QHC.

2) For each family $\{V_\alpha\}_{\alpha\in\Lambda}$ *of pre-open subsets , if* $X\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ *then there exists* $\Lambda_0 \subseteq \Lambda$*, finite, with* $X\setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$*.*

3) For each family $\{V_\alpha\}_{\alpha\in\Lambda}$ *of* α*-open subsets , if* $X\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ *then there exists* $\Lambda_0 \subseteq \Lambda$*, finite, with* $X\setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$*.*

3. ρC($\mathcal{I}$)-compact spaces

A topological space (X,τ) is defined to be *C-compact* [13] if for each $F \subseteq X$, closed, and each τ-open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of F, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

An ideal space $(X,\tau,\mathcal{I})$ is said to be *C($\mathcal{I}$)-compact* [2] if for each $F \subseteq X$, closed, and each τ-open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of F, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F\setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

It is noted that C-compact $\Rightarrow$ QHC, C($\mathcal{I}$)-compact $\Rightarrow$ $\mathcal{I}$-QHC and that if (X,τ) is C-compact then $(X,\tau,\mathcal{I})$ is C($\mathcal{I}$)-compact.

In this section we introduce and study the $\rho C(\mathcal{I})$-compact spaces, which are stronger forms of C($\mathcal{I}$)-compactness and $\mathcal{I}$-QHC. We present some of its properties and characterizations.

Definition 3.1 The ideal space $(X,\tau,\mathcal{I})$ is said to be $\rho C(\mathcal{I})$*-compact* if for each closed subset F of X, and each family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X such that $F\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F\setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

Note that if $(X,\tau^*,\mathcal{I})$ is ρC($\mathcal{I}$)-compact then $(X,\tau,\mathcal{I})$ is ρC($\mathcal{I}$)-compact.

It is also clear that:

1) (X,τ) is C-compact $\Leftrightarrow$ $(X,\tau,\{\varnothing\})$ is $\rho C(\{\varnothing\})$-compact $\Leftrightarrow$ $(X,\tau,\{\varnothing\})$ is $C(\{\varnothing\})$-compact.

2) $\rho C(\mathcal{I})$-compact $\Rightarrow$ $\rho\mathcal{I}$-QHC.

3) $\rho C(\mathcal{I})$-compact $\Rightarrow$ $C(\mathcal{I})$-compact.

This implications are, in general, irreversible.

Example 3.1 1) We consider again the ideal space $(\mathbb{Z},\tau,\mathcal{I})$ of Example 2.1, which is not $\rho\mathcal{I}$-QHC. We will demonstrated that $(\mathbb{Z},\tau)$ is C-compact, and so $(\mathbb{Z},\tau,\mathcal{I})$ is C($\mathcal{I}$)-compact.

Let F be a closed subset of $\mathbb{Z}$ and $\{V_\alpha\}_{\alpha\in\Lambda}$ an open cover of F.

(i) If $F \cap \{0,1\} = \varnothing$ then $2\mathbb{Z} \cap F = \varnothing$ and $(2\mathbb{Z}+1) \cap F = \varnothing$, and so $F = \varnothing$. If $\alpha_0 \in \Lambda$ then $F \subseteq \overline{V_{\alpha_0}}$.

(ii) If $F \cap \{0,1\} = \{0,1\}$ then there are $\alpha_0 \in \Lambda$ and $\alpha_1 \in \Lambda$ such that $0 \in V_{\alpha_0}$ and $1 \in V_{\alpha_1}$. This implies that $\overline{V_{\alpha_0}} \cup \overline{V_{\alpha_1}} = X$ and $F \subseteq \overline{V_{\alpha_0}} \cup \overline{V_{\alpha_1}}$.

(iii) If $F \cap \{0,1\} = \{0\}$ then $(2\mathbb{Z}+1) \cap F = \varnothing$, and there exists $\alpha_0 \in \Lambda$ with $0 \in V_{\alpha_0}$. Thus $F \subseteq 2\mathbb{Z} \subseteq V_{\alpha_0} \subseteq \overline{V_{\alpha_0}}$.

(iv) If $F \cap \{0,1\} = \{1\}$ then $2\mathbb{Z} \cap F = \varnothing$, and there exists $\alpha_1 \in \Lambda$ with $1 \in V_{\alpha_0}$. Thus $F \subseteq 2\mathbb{Z}+1 \subseteq V_{\alpha_1} \subseteq \overline{V_{\alpha_1}}$.

Hence the space $(\mathbb{Z}, \tau, \mathcal{I})$ is $\mathrm{C}(\mathcal{I})$-compact. However this space is not $\rho\mathrm{C}(\mathcal{I})$-compact, because $(X, \tau, \mathcal{I})$ is not $\rho\mathcal{I}$-QHC.

2) Let $\mathcal{U}$ be the usual topology for $X = [0,1]$.

Let $F = \{1/n : n \in \mathbb{Z}^+\}$. We consider the topology $\mathcal{U}^*$ for X generated by $\mathcal{U} \cup \{X \backslash F\}$. A base for $\mathcal{U}^*$ is $\mathcal{B} = \mathcal{U} \cup \{V \backslash F : V \in \mathcal{U}\}$.

We have that:

a) F is closed and discrete in $(X, \mathcal{U}^*)$.

If $n \in \mathbb{Z}^+$, $r_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$, and if $W_n = \left(\frac{1}{n} - r_n, \frac{1}{n} + r_n\right) \cap X$, then $W_n \in \mathcal{U} \subseteq \mathcal{U}^*$ and $W_n \cap F = \left\{\frac{1}{n}\right\}$.

b) The family $\{W_n\}_{n \in \mathbb{Z}^+}$ is a $\mathcal{U}^*$-open cover of F.

c) F is nowhere dense in $(X, \mathcal{U}^*)$, because $int_{\mathcal{U}^*}\left(adh_{\mathcal{U}^*}(F)\right) = int_{\mathcal{U}^*}(F) = \varnothing$, since $\varnothing$ is the only element in $\mathcal{B}$ which is contained in F.

d) If $V \in \mathcal{U}^*$ then $adh_{\mathcal{U}^*}(V) = adh_{\mathcal{U}}(V)$.

It is clear that $adh_{\mathcal{U}^*}(V) \subseteq adh_{\mathcal{U}}(V)$. Suppose that $z \in adh_{\mathcal{U}}(V)$ and that $B \in \mathcal{B}$, with $z \in B$. If $B \in \mathcal{U}$ then $V \cap B \neq \varnothing$. If there exists $W \in \mathcal{U}$ such that $B = W \backslash F$ then $W \cap V \neq \varnothing$. Since F is nowhere dense in $(X, \mathcal{U}^*)$, we have that $(W \cap V) \cap (X \backslash F) \neq \varnothing$, and so $V \cap B \neq \varnothing$. Thus $z \in adh_{\mathcal{U}^*}(V)$.

e) The space $(X, \mathcal{U}^*)$ is not C-compact, and then $(X, \mathcal{U}^*, \{\varnothing\})$ is not $\rho\mathrm{C}(\{\varnothing\})$-compact.

If $n \in \mathbb{Z}^+$, $adh_{\mathcal{U}^*}(W_n) = adh_{\mathcal{U}}(W_n) = \left[\frac{1}{n} - r_n, \frac{1}{n} + r_n\right] \cap X$ and so $adh_{\mathcal{U}^*}(W_n) \cap F = \left\{\frac{1}{n}\right\}$. Hence $F \subseteq \bigcup\limits_{n \in \mathbb{Z}^+} W_n$, but if $n_1, n_2, \ldots, n_r \in \mathbb{Z}^+$, $F \nsubseteq \bigcup\limits_{j=1}^{r} adh_{\mathcal{U}^*}(W_{n_j})$.

f) The space $(X, \mathcal{U}^*)$ is QHC, and then $(X, \mathcal{U}^*, \{\varnothing\})$ is $\rho\{\varnothing\}$-QHC.

Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be a $\mathcal{U}^*$-open cover of X. There exists $\alpha_0 \in \Lambda$ such that $0 \in V_{\alpha_0}$. Let B be an element of $\mathcal{B}$ with $0 \in B \subseteq V_{\alpha_0}$.

(i) If $B \in \mathcal{U}$ then there exists $r \in \left(0, \frac{1}{2}\right) \backslash F$ with $[0, r) \subseteq B$. Then $[0, r] \subseteq adh_{\mathcal{U}^*}(B) \subseteq adh_{\mathcal{U}^*}(V_{\alpha_0})$.

The set $F\backslash[0,r]$ is finite. Suppose that $F\backslash[0,r]=\{f_1,f_2,...,f_n\}$, and that $f_1<f_2<\cdots<f_{n-1}=\frac{1}{2}<f_n=1$. For all $j\in\{1,2,...,n\}$ there exists $\alpha_j\in\Lambda$ with $f_j\in V_{\alpha_j}$, and there exists $\epsilon_j>0$ such that $(f_j-\epsilon_j,f_j+\epsilon_j)\subseteq V_{\alpha_j}$ if $j\in\{1,2,...,n-1\}$, $(f_n-\epsilon_n,f_n]\subseteq V_{\alpha_n}$ and $(f_j-\epsilon_j,f_j+\epsilon_j)\cap F=\{f_j\}$ for each $j\in\{1,2,...,n\}$.

Let $F_j=[f_j-\epsilon_j,f_j+\epsilon_j]$, $T_j=(f_j-\epsilon_j,f_j+\epsilon_j)$ if $j\in\{1,2,...,n-1\}$, $F_n=[f_n-\epsilon_n,f_n]$ and $T_n=(f_n-\epsilon_n,f_n]$.

Clearly $\{f_1,f_2,...,f_n\}\subseteq\bigcup\limits_{k=1}^{n}F_k\subseteq\bigcup\limits_{k=1}^{n}adh_{\mathcal{U}^*}(V_{\alpha_k})$.

Now, $[0,1]\backslash\left([0,r)\cup\bigcup\limits_{k=1}^{n}T_k\right)$ is a finite union of closed intervals, each of which disjoint of F. Suppose that $[0,1]\backslash\left([0,r)\cup\bigcup\limits_{k=1}^{n}T_k\right)=\bigcup\limits_{i=1}^{m}[a_i,b_i]$. It is easy to see that for every $i\in\{1,2,...,m\}$ there exists $\Lambda_i\subseteq\Lambda$, finite, such that $[a_i,b_i]\subseteq\bigcup\limits_{\alpha\in\Lambda_i}adh_{\mathcal{U}^*}(V_\alpha)$.

Therefore $X=\left(\bigcup\limits_{k=0}^{n}adh_{\mathcal{U}^*}(V_{\alpha_k})\right)\cup\left(\bigcup\limits_{i=1}^{m}\bigcup\limits_{\alpha\in\Lambda_i}adh_{\mathcal{U}^*}(V_\alpha)\right)$.

(ii) If there exists $V\in\mathcal{U}$ with $B=V\backslash F$, then there exists $r\in\left(0,\frac{1}{2}\right)\backslash F$ such that $[0,r)\subseteq V$, and so $[0,r]=adh_{\mathcal{U}^*}([0,r)\backslash F)\subseteq adh_{\mathcal{U}^*}(B)\subseteq adh_{\mathcal{U}^*}(V_{\alpha_0})$. Now we proceed as in case (i).

In conclusion, the space $(X,\mathcal{U}^*)$ is QHC.

Theorem 3.1 *If the space $(X,\tau,\mathcal{I})$ is $\rho\mathcal{I}$-compact then $(X,\tau,\mathcal{I})$ is $\rho C(\mathcal{I})$-compact.*

Proof. Suppose that K is a closed subset of X, and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X with $K\backslash\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$, this is, $X\backslash\left[(X\backslash K)\cup\bigcup\limits_{\alpha\in\Lambda}V_\alpha\right]\in\mathcal{I}$. By hypothesis, there exists $\Lambda_0\subseteq\Lambda$, finite, such that $X\backslash\left[(X\backslash K)\cup\bigcup\limits_{\alpha\in\Lambda_0}V_\alpha\right]\in\mathcal{I}$, this is, $K\backslash\bigcup\limits_{\alpha\in\Lambda_0}V_\alpha\in\mathcal{I}$. Hence $K\backslash\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$.

The converse of this theorem, in general, is not true.

Example 3.2 If $X=[0,\infty)$, $\tau=\{(r,\infty):r\geq 0\}\cup\{\varnothing,X\}$, and $\mathcal{I}=\mathcal{I}_f$ then we know that $(X,\tau,\mathcal{I})$ is not $\rho\mathcal{I}$-compact [9].

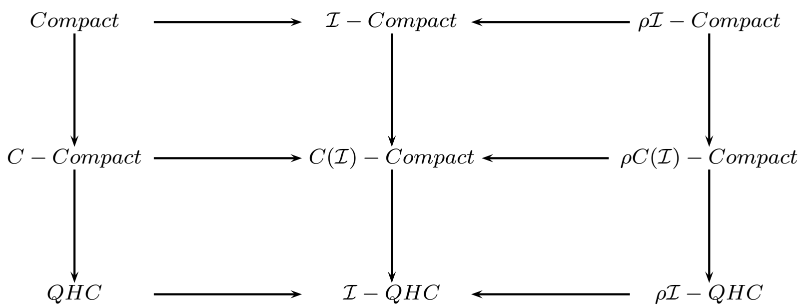
However, $(X,\tau,\mathcal{I})$ is ρC($\mathcal{I}$)-compact, and so $\rho\mathcal{I}$-QHC, since if $V\in\tau\backslash\{\varnothing\}$, $\overline{V}=X$.

The following example shows that QHC and $\rho\mathcal{I}$-QHC are independent concepts, as well as C-compact and ρC($\mathcal{I}$)-compact.

Example 3.3 1) The space $(\mathbb{Z}, \tau, \mathcal{I})$ of Example 2.1 is not $\rho\mathcal{I}$-QHC, but $(\mathbb{Z}, \tau)$ is compact. This implies that Compact $\not\Rightarrow$ $\rho\mathcal{I}$-compact, C-compact $\not\Rightarrow$ ρC($\mathcal{I}$)-compact and QHC$\not\Rightarrow$ $\rho\mathcal{I}$-QHC.

2) If $\mathcal{U}$ is the usual topology for $\mathbb{R}$, then clearly $(\mathbb{R}, \mathcal{U}, \mathcal{P}(\mathbb{R}))$ is $\rho\mathcal{P}(\mathbb{R})$-compact, but $(\mathbb{R}, \mathcal{U})$ is not QHC. This implies that $\rho\mathcal{I}$-compact $\not\Rightarrow$ compact, ρC($\mathcal{I}$)-compact $\not\Rightarrow$ C-compact and $\rho\mathcal{I}$-QHC $\not\Rightarrow$ QHC.

Then we have the following diagram:



Theorem 3.2 *The ideal space $(X, \tau, \mathcal{I})$ is $\rho C(\mathcal{I})$-compact if and only if, for each closed subset F of X, and each open filter base Ω on X such that $\{V \cap F : V \in \Omega\} \subseteq \mathcal{P}(X) \backslash \mathcal{I}$, one has $\bigcap_{V \in \Omega} \overline{V} \cap F \notin \mathcal{I}$.*

Proof. $(\Rightarrow)$ Suppose that $(X, \tau, \mathcal{I})$ is ρC($\mathcal{I}$)-compact and that there are $F \subseteq X$, closed, and an open filter base Ω on X such that $\{V \cap F : V \in \Omega\} \subseteq \mathcal{P}(X) \backslash \mathcal{I}$ and $\bigcap_{V \in \Omega} \overline{V} \cap F \in \mathcal{I}$.

Since $F \backslash \bigcup_{V \in \Omega} \left(X \backslash \overline{V}\right) \in \mathcal{I}$, there is, $\{V_1, V_2, ..., V_n\} \subseteq \Omega$ with $F \backslash \bigcup_{i=1}^{n} \overline{X \backslash \overline{V_i}} \in \mathcal{I}$, or equivalently, $F \backslash \bigcup_{i=1}^{n} \left(X \backslash \overset{0}{\overline{V_i}} \right) \in \mathcal{I}$.

Since $F \backslash \bigcup_{i=1}^{n} (X \backslash V_i) \subseteq F \backslash \bigcup_{i=1}^{n} \left(X \backslash \overset{0}{\overline{V_i}} \right)$, we have that

$$\left(\bigcap_{i=1}^{n} V_i \right) \cap F = F \backslash \bigcup_{i=1}^{n} (X \backslash V_i) \in \mathcal{I}.$$

But there exists $V \in \Omega$ with $V \subseteq \bigcap_{i=1}^{n} V_i$, and so $V \cap F \in \mathcal{I}$. This contradicts that $\{V \cap F : V \in \Omega\} \subseteq \mathcal{P}(X) \backslash \mathcal{I}$.

($\Leftarrow$) Suppose that $(X,\tau,\mathcal{I})$ is not ρC($\mathcal{I}$)-compact. There exist $F \subseteq X$, closed, and a family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, such that $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, but for each $\Lambda_0 \subseteq \Lambda$, finite, $F\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \notin \mathcal{I}$. In particular, for all $\alpha \in \Lambda$, $F\backslash\overline{V_\alpha} \notin \mathcal{I}$. We may assume that $\{V_\alpha\}_{\alpha\in\Lambda}$ is closed for finite unions, because otherwise we can replace $\{V_\alpha\}_{\alpha\in\Lambda}$ by the family of all finite unions of elements in $\{V_\alpha\}_{\alpha\in\Lambda}$.

Then the set $\mathcal{B} = \left\{X\backslash\overline{V_\alpha} : \alpha \in \Lambda\right\}$ is an open filter base on X, and

$$\{B \cap F : B \in \mathcal{B}\} \subseteq \mathcal{P}(X)\backslash\mathcal{I}.$$

The hypothesis implies that $\bigcap_{B\in\mathcal{B}} \overline{B}\cap F \notin \mathcal{I}$, this is $\bigcap_{\alpha\in\Lambda} \overline{X\backslash\overline{V_\alpha}}\cap F \notin \mathcal{I}$. But for each $\alpha \in \Lambda$, $\overline{X\backslash\overline{V_\alpha}} = X\backslash\overset{0}{\overline{V_i}} \subseteq X\backslash V_\alpha$, and so $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha = \bigcap_{\alpha\in\Lambda} (X\backslash V_\alpha) \cap F \notin \mathcal{I}$, contradiction.

Next we present other interesting characterizations of ρC($\mathcal{I}$)-compactness.

Theorem 3.3 *For an ideal space $(X,\tau,\mathcal{I})$, the following statements are equivalents:*

1) $(X,\tau,\mathcal{I})$ is $\rho C(\mathcal{I})$-compact.

2) For all closed subset F and all family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, if $\bigcap_{\alpha\in\Lambda} (F \cap F_\alpha) \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap_{\alpha\in\Lambda_0} (F \cap \overset{0}{F_\alpha}) \in \mathcal{I}$.

3) For each closed subset F and each family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, if $\left\{F \cap \overset{0}{F_\alpha} : \alpha \in \Lambda\right\}$ has the finite-intersection property modulo $\mathcal{I}$, then $\bigcap_{\alpha\in\Lambda} (F \cap F_\alpha) \notin \mathcal{I}$.

4) For all closed subset F and all family $\{V_\alpha\}_{\alpha\in\Lambda}$ of regular open subsets of X, if $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $F\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

5) For each closed subset F of X and each family $\{F_\alpha\}_{\alpha\in\Lambda}$ of regular closed subsets of X, if $\bigcap_{\alpha\in\Lambda} (F\cap F_\alpha) \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap_{\alpha\in\Lambda_0} (F\cap\overset{0}{F_\alpha}) \in \mathcal{I}$.

6) For each closed subset F and each family $\{F_\alpha\}_{\alpha\in\Lambda}$ of regular closed subsets of X, if $\left\{F \cap \overset{0}{F_\alpha} : \alpha \in \Lambda\right\}$ has the finite-intersection property modulo $\mathcal{I}$, then $\bigcap_{\alpha\in\Lambda} (F \cap F_\alpha) \notin \mathcal{I}$.

7) If $F \subseteq X$ is closed, $W \subseteq X$ is open with $F \backslash W \in \mathcal{I}$, and if $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X, such that $(X\backslash F) \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X\backslash \left(W \cup \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \right) \in \mathcal{I}$.

Proof. The implications 1)⇒2), 2)⇒3), 5)⇒6) are easy to be established.

3)⇒4) Let F a closed subset of X and $\{V_\alpha\}_{\alpha\in\Lambda}$ a family of regular open subsets of X with $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, or equivalently, $\bigcap_{\alpha\in\Lambda} (F \cap (X\backslash V_\alpha)) \in \mathcal{I}$. Then the family $\{F \cap int\,(X\backslash V_\alpha) : \alpha \in \Lambda\}$ has no the finite-intersection property modulo $\mathcal{I}$, and so there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\bigcap_{\alpha\in\Lambda_0} (F \cap int(X\backslash V_\alpha)) \in \mathcal{I}$, or equivalently, $F\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

4)⇒5) It is sufficient to note that the complement of a regular closed subset of X is regular open.

6)⇒1) Let F a closed subset of X and $\{V_\alpha\}_{\alpha\in\Lambda}$ a family of open subsets of X with $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, that is, $\bigcap_{\alpha\in\Lambda} (F \cap (X\backslash V_\alpha)) \in \mathcal{I}$. Since $\overline{int\,(X\backslash V_\alpha)} \subseteq X\backslash V_\alpha$, for all $\alpha \in \Lambda$, then $\bigcap_{\alpha\in\Lambda} (F \cap \overline{int(X\backslash V_\alpha)}) \in \mathcal{I}$. But $\overline{int\,(X\backslash V_\alpha)}$ is regular closed, for all $\alpha \in \Lambda$. By the hypothesis there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap_{\alpha\in\Lambda_0} (F \cap int\left(\overline{int(X\backslash V_\alpha)}\right)) \in \mathcal{I}$, and so $\bigcap_{\alpha\in\Lambda_0} (F \cap int(X\backslash V_\alpha)) \in \mathcal{I}$, that is, $F\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

7)⇒1) Suppose that $F \subseteq X$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X, with $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Let $W = \bigcup_{\alpha\in\Lambda} V_\alpha$ and $K = X\backslash W$. We have that $K\backslash (X\backslash F) = (X\backslash W) \backslash (X\backslash F) = F\backslash W \in \mathcal{I}$, and that $(X\backslash K) \backslash \bigcup_{\alpha\in\Lambda} V_\alpha = \varnothing \in \mathcal{I}$. The hypothesis implies that there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X\backslash \left[(X\backslash F) \cup \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \right] \in \mathcal{I}$, this is, $F\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

1)⇒7) Suppose that $F \subseteq X$ is closed, $W \subseteq X$ is open with $F\backslash W \in \mathcal{I}$, and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X, with $(X\backslash F) \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$.

Since $(X\backslash W) \backslash \bigcup_{\alpha\in\Lambda} V_\alpha = X\backslash \left[W \cup \bigcup_{\alpha\in\Lambda} V_\alpha \right] \subseteq \left[(X\backslash F) \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \right] \cup (F\backslash W) \in \mathcal{I}$, and $X\backslash W$ is closed, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $(X\backslash W) \backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$, this is, $X \backslash \left[W \cup \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \right] \in \mathcal{I}$.

In the following theorem we review the behavior of ρC($\mathcal{I}$)-compact spaces under continuous or open functions.

Theorem 3.4 *1) If $(X,\tau,\mathcal{I})$ is $\rho C(\mathcal{I})$-compact and if $f:(X,\tau)\to(Y,\beta)$ is a continuous biyective function, then $(Y,\beta,f(\mathcal{I}))$ is $\rho C(f((\mathcal{I}))$-compact.*

2) If $(X,\tau,\mathcal{I})$ is $\rho C(\mathcal{I})$-compact, $f:(X,\tau)\to(Y,\beta)$ is a continuous function and if $\mathcal{J}$ is the ideal $\left\{V\subseteq Y : f^{-1}(V)\in\mathcal{I}\right\}$, then $(Y,\beta,\mathcal{J})$ is $\rho C(\mathcal{J})$-compact.

3) If $(Y,\beta,\mathcal{J})$ is $\rho C(\mathcal{J})$-compact and if $f:(X,\tau)\to(Y,\beta)$ is an open and biyective function, then $\left(X,\tau,f^{-1}(\mathcal{J})\right)$ is $\rho C(f^{-1}(\mathcal{J}))$-compact, where $f^{-1}(\mathcal{J})$ is the ideal $\left\{f^{-1}(J) : J\in\mathcal{J}\right\}$.

Proof. 1) Let B a closed subset of Y, and $\{V_\alpha\}_{\alpha\in\Lambda}$ a family of open subsets of Y, with $B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{I})$. There exists $I\in\mathcal{I}$ such that $B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha = f(I)$.

Since $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}(f(I)) = I \in \mathcal{I}$, there exists $\Lambda_0\subseteq\Lambda$, finite, with $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} \overline{f^{-1}(V_\alpha)} \in \mathcal{I}$.

But $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} f^{-1}(\overline{V_\alpha}) \subseteq f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} \overline{f^{-1}(V_\alpha)}$, and this implies that $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} f^{-1}(\overline{V_\alpha}) \in \mathcal{I}$.

Then $B\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} = f\left(f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} f^{-1}\left(\overline{V_\alpha}\right)\right) \in f(\mathcal{I})$.

2) Suppose that $B\subseteq Y$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of Y, with $B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{J}$.

Given that $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}\left(B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha\right) \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} \overline{f^{-1}(V_\alpha)} \in \mathcal{I}$, and given that for all $\alpha\in\Lambda_0$, $\overline{f^{-1}(V_\alpha)} \subseteq f^{-1}\left(\overline{V_\alpha}\right)$, we have that $f^{-1}(B)\backslash \bigcup_{\alpha\in\Lambda_0} f^{-1}\left(\overline{V_\alpha}\right) \in \mathcal{I}$, this is, $f^{-1}\left(B\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right) \in \mathcal{I}$. Hence $B\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{J}$.

3) Note that since f is biyective and open then f is closed. Suppose that $A\subseteq X$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X, with

$A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{J})$. There exists $J \in \mathcal{J}$ such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(J)$, and so $f(A) \setminus \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f\left(A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha\right) = f\left(f^{-1}(J)\right) = J \in \mathcal{J}$. Since $f(A)$ is closed in Y, there is $\Lambda_0 \subseteq \Lambda$, finite, with $f(A) \setminus \bigcup_{\alpha\in\Lambda_0} \overline{f(V_\alpha)} \in \mathcal{J}$. Given that f is closed, $\overline{f(V_\alpha)} \subseteq f\left(\overline{V_\alpha}\right)$ for all $\alpha \in \Lambda_0$, and so $f\left(A \setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right) = f(A) \setminus \bigcup_{\alpha\in\Lambda_0} f\left(\overline{V_\alpha}\right) \in \mathcal{J}$. Then $A \setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} = f^{-1}\left(f\left(A \setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right)\right) \in f^{-1}(\mathcal{J})$.

Next we consider some special subsets of ρC($\mathcal{I}$)-compact spaces.

Definition 3.2 If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, A is said to be $\rho C(\mathcal{I})$-*compact* if for each $F \subseteq A$, closed in A, and for each family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $F \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

Example 3.4 In the following ideal spaces, each subset it is ρC($\mathcal{I}$)-compact.

1) $(X, \tau, \mathcal{P}(X))$, where (X, τ) is any topological space.

2) $(X, \beta, \mathcal{I})$, where X is an infinite set, β is the cofinite topology on X, and $\mathcal{I}$ is any ideal in X.

3) $(\mathbb{Z}, \tau, \mathcal{I})$, where $\mathcal{I} = \mathcal{P}(2\mathbb{Z}+1)$ and τ is the topology on $\mathbb{Z}$ given by: $V \in \tau \Leftrightarrow$ [for each $n \in \mathbb{Z}$, if $n \in V$ then $[n]_2 \in V$]. Here $[n]_2 = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd.} \end{cases}$

Theorem 3.5 *1) If $(X, \tau, \mathcal{I})$ is $\rho C(\mathcal{I})$-compact and $A \subseteq X$ is closed, then A is $\rho C(\mathcal{I})$-compact.*

2) If $(X, \tau, \mathcal{I})$ is an ideal space and $A_1 \subseteq X$ and $A_2 \subseteq X$ are $\rho C(\mathcal{I})$-compact, then $A_1 \cup A_2$ is $\rho C(\mathcal{I})$-compact.

Proof. 1) It is clear because if B is closed in A, then B is closed in X.

2) Suppose that B is closed in $A_1 \cup A_2$, and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of X with $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. There exists $G \subseteq X$, closed, such that $B = (A_1 \cup A_2) \cap G = (A_1 \cap G) \cup (A_2 \cap G)$. Since $A_i \cap G$ is closed in A_i and $(A_i \cap G) \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, for each $i \in \{1, 2\}$, there exists $\Lambda_i \subseteq \Lambda$, finite, with $(A_i \cap G) \setminus \bigcup_{\alpha\in\Lambda_i} \overline{V_\alpha} \in \mathcal{I}$, for each $i \in \{1, 2\}$.

Thus $(A_i \cap G) \setminus \bigcup_{\alpha \in \Lambda_1 \cup \Lambda_2} \overline{V_\alpha} \in \mathcal{I}$, and $[(A_1 \cup A_2) \cap G] \setminus \bigcup_{\alpha \in \Lambda_1 \cup \Lambda_2} \overline{V_\alpha} \in \mathcal{I}$, this is $B \setminus \bigcup_{\alpha \in \Lambda_1 \cup \Lambda_2} \overline{V_\alpha} \in \mathcal{I}$.

In the next result we present a new characterization of ρC($\mathcal{I}$)-compactness, in terms of some special open subsets.

Definition 3.3 If $(X,\tau,\mathcal{I})$ is an ideal space and $Y \subseteq X$, then Y is *closure $\rho C(\mathcal{I})$-compact* if for all $K \subseteq Y$, closed in Y, and all family $\{V_\alpha\}_{\alpha \in \Lambda}$ of open subsets of X, if $\overline{K} \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $K \setminus \bigcup_{\alpha \in \Lambda_0} adh_{\tau_Y}(V_\alpha \cap Y) \in \mathcal{I}$.

Example 3.5 Let $\mathcal{U}$ the usual topology for $X = [0,1]$, $Y = (0,1]$ and $K \subseteq Y$, closed in Y.

(i) Suppose that $\{V_\alpha\}_{\alpha \in \Lambda}$ is a $\mathcal{U}$-open cover of $\overline{K}$. Since $\overline{K}$ is compact in X, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\overline{K} \subseteq \bigcup_{\alpha \in \Lambda_0} V_\alpha$, and so $K \subseteq \bigcup_{\alpha \in \Lambda_0} \left(\overline{V_\alpha} \cap Y\right)$. But, for all $\alpha \in \Lambda_0$, $adh_{\mathcal{U}_Y}(V_\alpha \cap Y) = \overline{V_\alpha \cap Y} \cap Y = \overline{V_\alpha} \cap Y$, because Y is open. Therefore Y is closure ρC($\{\varnothing\}$)-compact.

(ii) Y is not ρC($\{\varnothing\}$)-compact, because $Y \subseteq \bigcup_{0<r<1} (r,1]$, but if $0 < r_1 < r_2 < \cdots < r_n < 1$ then $Y \nsubseteq \bigcup_{i=1}^{n} \overline{(r_i,1]} = \bigcup_{i=1}^{n} [r_i,1] = [r_1,1]$.

Theorem 3.6 *The ideal space $(X,\tau,\mathcal{I})$ is $\rho C(\mathcal{I})$-compact if and only if each $Y \in \tau$ is closure $\rho C(\mathcal{I})$-compact .*

Proof. ($\Rightarrow$) Suppose that $(X,\tau,\mathcal{I})$ is ρC($\mathcal{I}$)-compact and that $Y \in \tau$.

Let $K \subseteq Y$, closed in Y, and $\{V_\alpha\}_{\alpha \in \Lambda}$ a family of open subsets of X with $\overline{K} \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}$. Since $\overline{K}$ is closed in X and $(X,\tau,\mathcal{I})$ is ρC($\mathcal{I}$)-compact, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\overline{K} \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \in \mathcal{I}$, and so $K \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \in \mathcal{I}$. Given that Y is open in X, $adh_{\tau_Y}(V_\alpha \cap Y) = \overline{V_\alpha} \cap Y$, for all $\alpha \in \Lambda_0$.

But $K \setminus \bigcup_{\alpha \in \Lambda_0} (\overline{V_\alpha} \cap Y) = K \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

Thus $K \setminus \bigcup_{\alpha \in \Lambda_0} adh_{\tau_Y}(V_\alpha \cap Y) \in \mathcal{I}$.

($\Leftarrow$) Suppose that F is closed in X, and that $\{V_\alpha\}_{\alpha \in \Lambda}$ is a family of open subsets of X with $F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}$. Let $\alpha_0 \in \Lambda$. The set $Y = X \setminus \overline{V_{\alpha_0}}$ is open in X and $F \cap Y$ is closed in Y.

Since $\overline{F\cap Y}\subseteq F$ we have that $\overline{F\cap Y}\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$.

Now, $\overline{F\cap Y}\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha=\overline{F\cap Y}\setminus\bigcup\limits_{\alpha\in\Lambda\setminus\{\alpha_0\}}V_\alpha$. Thus there exists $\Lambda_0\subseteq\Lambda\setminus\{\alpha_0\}$, finite, such that $(F\cap Y)\setminus\bigcup\limits_{\alpha\in\Lambda_0}adh_{\tau_Y}(V_\alpha\cap Y)\in\mathcal{I}$.

Given that $Y\in\tau$, $adh_{\tau_Y}(V_\alpha\cap Y)=\overline{V_\alpha}\cap Y\subseteq\overline{V_\alpha}$, and so $(F\cap Y)\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$, this is, $\left[F\cap\left(X\setminus\overline{V_{\alpha_0}}\right)\right]\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$.

Therefore $F\setminus\bigcup\limits_{\alpha\in\Lambda_0\cup\{\alpha_0\}}\overline{V_\alpha}\in\mathcal{I}$.

Finally, we show an additional characterization of ρC($\mathcal{I}$)- compactness, by means of pre-open and α-open subsets.

Theorem 3.7 *If* $(X,\tau,\mathcal{I})$ *is an ideal space, the following statements are equivalents:*

1) $(X,\tau,\mathcal{I})$ *is* $\rho C(\mathcal{I})$*-compact.*

2) For each $F\subseteq X$*, closed, and each family* $\{V_\alpha\}_{\alpha\in\Lambda}$ *of pre-open subsets of* X*, if* $F\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$ *then there exists* $\Lambda_0\subseteq\Lambda$*, finite, with* $F\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$.

3) For each $F\subseteq X$*, closed, and each family* $\{V_\alpha\}_{\alpha\in\Lambda}$ *of* α*-open subsets of* X*, if* $F\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$ *then there exists* $\Lambda_0\subseteq\Lambda$*, finite, with* $F\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$.

Proof. It is sufficient to show that 1)⇒2), since open⇒ α-open⇒pre-open.

1) ⇒ 2) Suppose that $F\subseteq X$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of pre-open subsets of X, with $F\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$. Given that $V_\alpha\subseteq\overset{0}{\overline{V_\alpha}}$, for each $\alpha\in\Lambda$, we have that $F\setminus\bigcup\limits_{\alpha\in\Lambda}\overset{0}{\overline{V_\alpha}}\in\mathcal{I}$, and then there exists $\Lambda_0\subseteq\Lambda$, finite, such that $F\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{\overset{0}{\overline{V_\alpha}}}\in\mathcal{I}$. But $\overline{\overset{0}{\overline{V_\alpha}}}\subseteq\overline{V_\alpha}$, for all $\alpha\in\Lambda_0$. Thus $F\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$.

References

[1] J. Dontchev, M. Ganster and D. A. Rose, Ideal resolvability, *Top. and its Appl.*, **93**, No 1 (1999), 1-16. DOI:10.1016/S0166-8641(97)00257-5

[2] M. K. Gupta and T. Noiri, C-compactness modulo an ideal, *Int. Jour. of Math. and Math. Sci.*, **2006** (2006), 1-12.DOI: 10.1155/IJMMS/2006/78135

[3] T. R. Hamlett and D. Jancovic, Compactness with respect to an ideal, *Boll. Un. Mat. Ital.*, **7**, No 4B (1990), 849-861.

[4] T. R. Hamlett, D. Jancovic and D. Rose, Countable compactness with respect to an ideal, *Math. Chronicle*, **20** (1991), 109-126.

[5] K. Kuratowski, Topology, Vol. I, *Academic Press*, New York, 1966.

[6] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontinuous and weak precontinuous mappings, *Proc. Math. and Phys. Soc. of Egypt*, **53** (1982), 47-53.

[7] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph. Dissertation. Thesis, *Univ. of Calif. at Santa Barbara*, California, 1967.

[8] O. Njastad, On some classes of nearly open subsets, *Pacific J. Math.*, 15, (1965), 961-970. DOI: 10.2140/pjm.1965.15.961

[9] N. R. Pachón, New forms of strong compactness in terms of ideals, *Int. Jour. of Pure and App. Math.*, **106**, No 2 (2016), 481-493. DOI: 10.12732/ijpam.v106i2.12

[10] J. Porter and J. Thomas, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.*, **138** (1969), 159-170. DOI: 10.1090/S0002-9947-1969-0238268-4

[11] V. Renuka Devi, D. Sivaraj and T. Tamizh Chelvam, Codense and completely codense ideals, *Acta Mathematica Hungarica*, **108**, No 3 (2005), 197-205. DOI: 10.1007/s10474-005-0220-0

[12] R. Vaidyanathaswamy, Set Topology, 2nd ed., *Chelsea*, New York, 1960.

[13] G. Viglino, C-compact spaces, *Duke Mathematical Journal*, **36**, No 4 (1969), 761-764. DOI: 10.1215/S0012-7094-69-03691-6