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ON LINEAR CONNECTION RELATING TO A GIVEN CURVATURE

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Abstract: We study the existence and the behavior of a linear connection from a curvature given in a n-dimensional riemannian manifold M. For a polynomial section of the dual space of TM on $\mathbb{R}^n$, in particular, we find that there is a polynomial linear connection on $\mathbb{R}^n$. We prove that if the nullity space of the Ricci tensor is equal to that of the curvature, then the Ricci tensor and the curvature coincide.

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1. Introduction

Several researchers worked on connections in the differential manifolds, namely, the analytical notion of the geodesic coordinates, of parallel displacement as well as the birth of the finsler geometry. In 1979, [1] studied the existence and the analytical behavior of a 1-vector valued form from a 1-differential valued form by using an analytical method to solve a system of non-linear differential equations of first order. His work aims at finding the existence of a linear connection from the given Ricci curvature. More recently, [5] studied the Lie algebras

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associated to a connection in the sense of Grifone and he gave some properties of connection by studying the associated Lie algebras, he also computed the first space of Chevalley Eilenberg cohomology of the horizontal part of the Lie algebra of the vector fields and the horizontal nullity space of the curvature. In this paper, we will study, by a geometrical approach, not only the existence of a linear connection from a given Ricci curvature in a n-dimensional riemannian manifolds, but also to examine the algebraic behavior and some structures of obtained connection. Moreover, we prove that with any diagonal coefficients curvature we obtain a diagonal connection. The same applies in a polynomial section of a tensor of $(m+1)$-degree (resp. infinite) there is a polynomial linear connection of m-degree (infinite resp. by using its entire series development in its disc of convergence). While considering, a connection Γ in the sense of Grifone, we can give some algebraic structures of connections studied above. We find that all Lie algebras obtained starting from the linear connection to polynomial coefficients are Lie algebras of the polynomial vector fields. These latter check all results mentioned in [7]. Examples have been given to illustrate our results and as for the summation of index, the Einstein convention has been adopted.

2. Preliminary

Definition 2.1. We have an exact sequence of vector bundle on TM cf. [2]

$$0 \to \pi^*(TM) \xrightarrow{i} TTM \xrightarrow{j} \pi^*(TM) \to 0$$

$\pi : TM \to M$ the projection of bundle tangent with M; $P : TTM \to TM$ the projection of bundle tangent with TM, i the natural injection; $j = (P, \pi_*)$ où π_* is the tangent linear application of π. The $J = i \circ j$ application is the almost tangent structure on TM.

Definition 2.2. We called connection in the sense of Grifone on M cf.[2] a $1-$vector form Γ of TM, of class $\mathcal{C}^\infty$ on $TM - \{0\}$ such that $J\Gamma = J$, $\Gamma J = -J$.

There is the equality $\Gamma^2 = I$ where I is the identity application of $\chi(TM)$ and, Γ has two eigenvalues associated 1 and -1. Connections thus defined is a structure almost-product on TM. The horizontal projector (resp. vertical) of Γ is defined by $h = \frac{1}{2}(I + \Gamma)$ (resp. $v = \frac{1}{2}(I - \Gamma)$).

A connection Γ makes it possible to obtain a decomposition of TTM, the tangent bundle of TM in nap of spaces horizontal and vertical:

$$TTM = H(TM) \oplus V(TM)$$

with $H(TM) = Im(h) = Ker(v)$ et $V(TM) = Im(v) = Ker(h)$.

For the continuation, Γ indicated a Grifone connection, except for explicit mentions.

Definition 2.3. The expressions in local coordinates of J, h, v et Γ are respectively:

$$\begin{aligned} J &= dx^i \otimes \frac{\partial}{\partial y^i}; \\ h &= dx^i \otimes \frac{\partial}{\partial x^i} - \Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j}; \\ v &= dy^i \otimes \frac{\partial}{\partial y^i} + \Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^i}; \\ \Gamma &= dx^i \otimes \frac{\partial}{\partial x^i} - 2\Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j} - dy^i \otimes \frac{\partial}{\partial y^i}. \end{aligned}$$

Then a linear connection being a horizontal distribution thus its local and dual basis is carried by $\frac{\partial}{\partial x^i}$ et dx^i respectively, and in local coordinates $\Gamma = \Gamma_i^j dx^i \otimes \frac{\partial}{\partial x^j}$ où $\Gamma_j^i = y^a \Gamma_{aj}^i$.

Definition 2.4. The curvature of the connection Γ is defined by the 2−vector form $R = \frac{1}{2}[h,h]$ where $\frac{1}{2}[h,h](X,Y) = [hX,hY] + h[X,Y] - h[hX,Y] - h[X,hY]$, for all $X, Y \in \chi(TM)$.

In local coordinates $(x^i, y^j)_{1\leq i\leq n}$ of TM. $\Gamma = dx^i \otimes \frac{\partial}{\partial x^i} - 2\Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j} - dy^i \otimes \frac{\partial}{\partial y^i}$ cf. [2]:

$R = \frac{1}{2} R_{jk}^i dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^i}$ where $R_{jk}^i = \frac{\partial \Gamma_k^i}{\partial x^j} - \frac{\partial \Gamma_j^i}{\partial x^k} + \Gamma_k^l \frac{\partial \Gamma_j^i}{\partial y^l} - \Gamma_j^l \frac{\partial \Gamma_k^i}{\partial y^l}$.

For a linear connection Γ is written $\Gamma_{jk}^i dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^i}$ where $\Gamma_j^i = y^k \Gamma_{jk}^i$ and its curvature is $R = \frac{1}{2} R_{jkl}^i dx^j \wedge dx^k \wedge dx^l \otimes \frac{\partial}{\partial x^i}$ where $R_{jkl}^i = \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{jk}^h \Gamma_{hl}^i - \Gamma_{jl}^h \Gamma_{hk}^i$.

Definition 2.5. Into a linear connection, we introduce the notion of Ricci tensor which is the contraction of curvature tensor on the first and third indexes, i.e. $Ric_{ij} = R_{ikj}^k$, and symmetrical at the same time i.e. $Ric_{ij} = Ric_{ji}$. Locally Ric_{jk} has as expression

$$Ric_{jk} = R_{jk} = R_{jik}^i = \frac{\partial \Gamma_{ik,j}^i}{\partial x^i} - \frac{\partial \Gamma_{ik}^i}{\partial x^j} + \Gamma_{jk}^m \Gamma_{im}^i - \Gamma_{ik}^m \Gamma_{jm}^i.$$

or

$$Ric_{jk} = Trace(R(e_i, e_j)e_k).$$

Subsequently, M denotes a n-dimensional riemannian manifolds, except for explicit mentions.

3. Local Study of Some Curvature Tensor of an Unspecified Connection

Definition 3.1. A linear connection is known as diagonal if $\forall i \in [1, n]$

$$\left(\Gamma^i_{jk}\right)_{i,j,k} = \begin{cases} \Gamma^i_{jj} & \text{si } j = k \\ 0 & \text{si } j \neq k. \end{cases}$$

Its Ricci tensor is defined as follows

$$R_{jk} = \delta_{jk} \frac{\partial \Gamma^i_{ik}}{\partial x^i} - \delta_{ik} \frac{\partial \Gamma^i_{ik}}{\partial x^j} + \delta_{jk} \delta_{im} \Gamma^m_{jk} \Gamma^i_{im} - \delta_{ik} \delta_{jm} \Gamma^m_{ik} \Gamma^i_{jm}. \tag{3.1}$$

où δ_{ij} is the Kronecker symbol.

Definition 3.2 (Polynomial linear connection)**.** A linear connection Γ is known polynomial of degree m if the symbols of Christoffel are polynomials of degree $\leq m$ on $\mathbb{R}^n$ and $\deg(\Gamma) = \max_{1 \leq i,j,k \leq n}(\deg(\Gamma^i_{jk}))$.

In particular, a connection is polynomial if it is $\mathcal{C}^\infty - class$ on $\mathbb{R}^n$, i.e. we can develop it in whole series in its disc of convergence.

Theorem 3.3. *Let Ω be a $(m+1)$-degree polynomial section of the dual space $\bigotimes^2 T^*M$ on $\mathbb{R}^n$, with $m \geq 1$. For all $x \in \mathbb{R}^n$, there is a m-degree polynomial linear connection on $\mathbb{R}^n$ and that $\deg(\Gamma^s_{jk}\Gamma^i_{is} - \Gamma^s_{ik}\Gamma^i_{js}) = m+1$, defined in the neighborhood of x which the curvature is equal to Ω.*

Proof. Let us suppose that

$$\Gamma^i_{jj} = \begin{cases} \Gamma^1_{22} & \\ \Gamma^2_{11} & \\ 0 & \text{elsewhere.} \end{cases}$$

Let be $Ric_{jk} = \frac{\partial \Gamma^i_{jk}}{\partial x^i} - \frac{\partial \Gamma^i_{ik}}{\partial x^j} + \Gamma^m_{jk}\Gamma^i_{im} - \Gamma^m_{ik}\Gamma^i_{jm}$ and let us pose that

$$(Ric_{jk})_{j,k} = \begin{cases} \Omega_{11} = \sum_{i+j+k=1}^{m} \alpha_{ijk} (x^1)^i (x^2)^j (x^3)^k & \\ \Omega_{22} = \sum_{i+j+k=1}^{m} \beta_{ijk} (x^1)^i (x^2)^j (x^3)^k & \\ \Omega_{12} = \Omega_{21} = \sum_{i+j+k=1}^{m} \gamma_{ijk} (x^1)^i (x^2)^j (x^3)^k & \\ 0 & \text{elsewhere,} \end{cases}$$

according to the Ricci curvature equations R_{jk}. As Ric_{jk} is polynomial of m-degree then $Ric_{ts} = \sum_{i+j+k=1}^{m} a_{ijk}(x^1)^i(x^2)^j(x^3)^k$ for all t and $s = 1, 2, 3$. According to the rule of resolution of a system of partial differential equations, the solution must be form $\sum_{i+j+k=1}^{m-1} b_{ijk}(x^1)^i(x^2)^j(x^3)^k$. We will search b_{ijk} in function to a_{ijk}. Let be $\Gamma_{22}^1 = \sum_{i+j+k=1}^{m-1} A_{ijk}(x^1)^i(x^2)^j(x^3)^k$ and $\Gamma_{11}^2 = \sum_{i+j+k=1}^{m-1} B_{ijk}(x^1)^i(x^2)^j(x^3)^k$. Hence

$$\begin{cases} \Omega_{11} = \frac{\partial \Gamma_{11}^2}{\partial x^2} \\ \Omega_{22} = \frac{\partial \Gamma_{22}^1}{\partial x^1} \\ \Omega_{33} = 0 \\ \Omega_{12} = \Omega_{21} = -\Gamma_{22}^1 \Gamma_{11}^2 \\ \Omega_{13} = \Omega_{31} = 0 \\ \Omega_{23} = \Omega_{32} = 0. \end{cases}$$

Then

$$\begin{cases} \sum_{i+j+k=1}^{m} \alpha_{ijk}(x^1)^i(x^2)^j(x^3)^k = \frac{\partial(\sum_{i+j+k=1}^{m-1} B_{ijk}(x^1)^i(x^2)^j(x^3)^k)}{\partial x^2} \\ \sum_{i+j+k=1}^{m} \beta_{ijk}(x^1)^i(x^2)^j(x^3)^k = \frac{\partial(\sum_{i+j+k=1}^{m-1} A_{ijk}(x^1)^i(x^2)^j(x^3)^k)}{\partial x^1} \\ \Omega_{33} = 0 \\ \sum_{i+j+k=1}^{m} \gamma_{ijk}(x^1)^i(x^2)^j(x^3)^k = -(\sum_{i+j+k=1}^{m-1} A_{ijk}(x^1)^i(x^2)^j(x^3)^k) \\ \qquad (\sum_{i+j+k=1}^{m-1} B_{ijk}(x^1)^i(x^2)^j(x^3)^k) \\ \Omega_{13} = 0 \\ \Omega_{23} = 0. \end{cases}$$

Hence,

$$\begin{cases} \sum_{i+j+k=1}^{m} \alpha_{ijk}(x^1)^i(x^2)^j(x^3)^k = \sum_{i+j+k=1}^{m-1} jB_{ijk}(x^1)^i(x^2)^{j-1}(x^3)^k \\ \sum_{i+j+k=1}^{m} \beta_{ijk}(x^1)^i(x^2)^j(x^3)^k = \sum_{i+j+k=1}^{m-1} iA_{ijk}(x^1)^{i-1}(x^2)^j(x^3)^k \\ \Omega_{33} = 0 \\ \sum_{i+j+k=1}^{m} \gamma_{ijk}(x^1)^i(x^2)^j(x^3)^k = -(\sum_{i+j+k=1}^{m-1} A_{ijk}(x^1)^i(x^2)^j(x^3)^k) \\ \qquad (\sum_{i+j+k=1}^{m-1} B_{ijk}(x^1)^i(x^2)^j(x^3)^k) \\ \Omega_{13} = 0 \\ \Omega_{23} = 0. \end{cases}$$

By proceeding to a change of index, we obtain

$$\begin{cases} \sum_{i+j+k=1}^{m} \alpha_{ijk}(x^1)^i(x^2)^j(x^3)^k = \sum_{i+j+k=1}^{m}(j+1)B_{i(j+1)k}(x^1)^i(x^2)^j(x^3)^k, \\ \sum_{i+j+k=1}^{m} \beta_{ijk}(x^1)^i(x^2)^j(x^3)^k = \sum_{i+j+k=1}^{m}(i+1)A_{(i+1)jk}(x^1)^i(x^2)^j(x^3)^k, \\ \Omega_{33} = 0, \\ \sum_{i+j+k=1}^{m} \gamma_{ijk}(x^1)^i(x^2)^j(x^3)^k = \\ \qquad -(\sum_{i+j+k=1}^{m} \sum_{s=0}^{i+j+k} A_s B_{i+j+k-s}(x^1)^i(x^2)^j(x^3)^k), \\ \Omega_{13} = 0, \\ \Omega_{23} = 0. \end{cases}$$

By the definition of the product series, we deduce the coefficients of the following diagonal linear connection:

$$\begin{aligned} A_{(i+1)jk} &= \frac{1}{i+1}\beta_{ijk} \\ B_{i(j+1)k} &= \frac{1}{j+1}\alpha_{ijk} \\ -\sum_{s=0}^{i+j+k} A_s B_{i+j+k-s} &= \gamma_{ijk}. \end{aligned}$$

□

Corollary 3.4. *Let Ω be a polynomial section of $\bigotimes^2 T^*M$ on $\mathbb{R}^n$. for all $x \in \mathbb{R}^n$, there is a polynomial linear connection on $\mathbb{R}^n$, defined in the neighborhood x which the curvature is equal to Ω.*

Proof. We make the same reasoning than that of the Theorem 3.3 while setting out $m = +\infty$. □

Example 3.5. Let be

$$(\Omega_{jk})_{j,k} = \begin{pmatrix} 0 & (1+x^1)^2 & 1+(x^1)^2 \\ (1+x^1)^2 & 1+x^1+(x^1)^2 & 1+(x^1)^2 \\ 1+(x^1)^2 & 1+(x^1)^2 & 1+x^1+(x^1)^2 \end{pmatrix}$$

For this we set out $\Gamma^i_{jk} = a_p + b_p x^1$, where a_p and b_p are real, then we make a member to member identification. Its corresponding linear connection is

$$(\Gamma^i_{jj})_{i,j} = \begin{pmatrix} 1+x^1 & 1+x^1 & 1+x^1 \\ -1-x^1 & -1 & 1+x^1 \\ -1-x^1 & -1-x^1 & 1 \end{pmatrix}$$

Example 3.6. We consider

$$(\Omega_{jk})_{j,k} = e^{x^1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-e^{x^1} & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

In this case, we develop in entire series each Christoffel symbol, and then we search for every term which corresponds to it after a member to member identification. Its matching connection is

$$(\Gamma^i_{jj})_{i,j} = \begin{pmatrix} -e^{x^1}-1 & e^{x^1} & -1 \\ -1 & 1 & -1 \\ e^{x^1}+1 & e^{x^1} & 1 \end{pmatrix}$$

Remark 3.7. Given a Ricci curvature of $(m+1)$-degree, we can have a diagonal linear connection of m-degree except in the case where the linear connection, these coefficients included, have the fixed degree because the compatibility condition on the degree of the double product has not been checked.

Example 3.8. Let's take the case of a quadratic linear connection (where the degree of any coefficients is 2).

If we have

$$(R_{ij})_{ij} = \begin{pmatrix} (x^1)^2x^2x^3 & -x^2-4(x^1)^3 & -x^1(x^2)^3 \\ -x^2-4(x^1)^3x^3 & 2(x^1)^2((x^2)^2+4) & 0 \\ -x^1(x^2)^3 & 0 & (x^2)^4 \end{pmatrix}$$

We obtain then

$$(\Gamma^i_{jj})_{ij} = \begin{pmatrix} \frac{(x^2)^2}{2} & 4(x^1)^2 & (x^2)^2 \\ x^1x^3 & x^1x^2 & 0 \\ x^1x^2 & 0 & 0 \end{pmatrix}$$

Remark 3.9. We consider a metric g of riemannian manifolds M. According to the Koszul formula in local coordinates on the metricity of linear connection, it easy to determine a corresponding linear connection to this metric. We can also find coefficients of linear connection from the geodesic coordinates of the trajectory cf.[4].

4. Algebraic Properties Associated a Connection

Definition 4.1. The nullity space of the curvature R given by:

$$\mathcal{N}_R = \{X \in \chi(TM)/\ R(X,Y) = 0\ \forall\ Y \in \chi(TM)\}.$$

Locally, to determine this space, one have just to solve the following system of equation according to

$$X^i R^k_{ij} = 0.$$

where $X = (X^i)^h \frac{\partial}{\partial x^i} + (X^i)^v \frac{\partial}{\partial y^i}$.

Space $\mathcal{N}_R$ is a distribution of TM. As the R curvature is semi-basic, the vertical space is included in $\mathcal{N}_R$. In general, the nullity space $\mathcal{N}_R$ is not involutive. For a linear connection $X^l R^i_{ljk} = 0$, where $X = X^i \frac{\partial}{\partial x^i}$.

For the Ricci tensor, its nullity space is as follows

$$\mathcal{N}_{Ric} = \{X \in \chi(TM)/\ Ric(X,Y) = 0\ \forall\ Y \in \chi(TM)\}.$$

From where the following system

$$X^l Ric_{lk} = 0 \ \text{ où } X = X^i \frac{\partial}{\partial x^i}. \tag{4.1}$$

Theorem 4.2. *For a linear connection, if $\mathcal{N}_{Ric} = \mathcal{N}_R$ then $R = Ric = f$, where $f \in \mathcal{F}(M)$ and fixed for all i, j, k and $l \in [1, n]$.*

Proof. "$\Rightarrow$": Let $R = R^i_{jkl} dx^j \wedge dx^k \wedge dx^l \otimes \frac{\partial}{\partial x^i}$ be a curvature tensor of a linear connection, and $Ric = R_{jk} dx^j \wedge dx^k$ its Ricci tensor. By the definition of the nullity space of the curvature 4.1, we analytically deduce the following system equations for $\mathcal{N}_R$, we have $X^m R^i_{jml} = 0$ and $X^m R_{mk} = 0$ that of $\mathcal{N}_{Ric}$ where $X = X^i \frac{\partial}{\partial x^i} \in \chi(TM)$. As R_{mk} is the contraction of R^i_{jml} (in other words of the endomorphism trace of R), then $X^m R^i_{jml} = a X^m R_{mk}$ with a is a constant depending on n which is the dimension of the manifold M. If $X^m R^i_{jml} = 0$ then we deduce some $X^m R_{mk} = 0$. Consequently, $\mathcal{N}_R = \mathcal{N}_{Ric}$. By adopting the Einstein convention, we then obtain $R^i_{jml} = R_{ml} = f$.

"$\Leftarrow$": Immediate, by using the definition of $\mathcal{N}_R$ and $\mathcal{N}_{Ric}$. □

Proposition 4.3. *Let Ric be a Ricci tensor of a linear connection Γ, if*

$$R_{jk} = \begin{cases} R_{jj} \neq 0 & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

Then $\mathcal{N}_{Ric} = \{0\}$. And if $R_{jk} = 0\ \forall j, k$, we have then $\mathcal{N}_{Ric} = \chi(TM)$.

Proof. Let Ric be a Ricci tensor of the linear connection Γ, according to the definition of $\mathcal{N}_{Ric}$, in local coordinates, we have the following system:

$$X^i R_{ik} = X^1 R_{1k} + X^2 R_{2k} + \ldots + X^n R_{nk} = 0$$

While clarifying according to the index k where $1 \leq k \leq n$, we obtain

$$\begin{pmatrix} R_{11} & \ldots & R_{1j} & \ldots & R_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ R_{j1} & \ldots & R_{jj} & \ldots & R_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ R_{n1} & \ldots & R_{nj} & \ldots & R_{nn} \end{pmatrix} \begin{pmatrix} X^1 \\ \vdots \\ X^j \\ \vdots \\ X^n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Like

$$R_{jk} = \begin{cases} R_{jj} \neq 0 & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

We obtain

$$\begin{pmatrix} R_{11} & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & R_{jj} & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & R_{nn} \end{pmatrix} \begin{pmatrix} X^1 \\ \vdots \\ X^j \\ \vdots \\ X^n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By solving this this system, we find that $X^i = 0\ \forall i \in [1, n]$. Therefore, $\mathcal{N}_{Ric} = \{0\}$. And for $R_{jk} = 0\ \forall j, k \in [1, n]$, we have

$$\begin{pmatrix} 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} X^1 \\ \vdots \\ X^j \\ \vdots \\ X^n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence $\mathcal{N}_{Ric} = \chi(TM)$. □

Definition 4.4. We define the Lie algebras associated with connection Γ

$$\mathcal{A}_\Gamma \;=\; \{X \in \chi(TM) /\ [X, \Gamma] = 0\}$$

$$= \quad \{X \in \chi(TM)/\ \Gamma[X,Y] = [X,\Gamma Y]\ \forall\ Y \in \chi(TM)\}.$$

Locally, if we adopt $(x^i)_{i=1,\ldots,2n}$ the natural coordinates on TM with $\Gamma = \Gamma^i_j dx^j \otimes \frac{\partial}{\partial x^i}$, we obtains, for all $X = X^i \frac{\partial}{\partial x^i} \in \mathcal{A}_\Gamma$, the system with $4n^2$ linear equations with partial derivative, cf.[3]:

$$X^i \frac{\partial \Gamma^j_k}{\partial x^i} - \Gamma^i_j \frac{\partial X^i}{\partial x^i} + \Gamma^i_k \frac{\partial X^j}{\partial x^i} = 0.$$

For a linear connection, by using the relation $\Gamma^j_k = y^a \Gamma^j_{ak}$ and that of diagonal linear connection we find that $\Gamma^j_k = \delta_{ak} y^a \Gamma^j_{ak}$ with δ_{ak} indicates the Kronecker symbol. By applying the preceding equation to the system and we have

$$X^i y^a \frac{\partial \Gamma^j_{ak}}{\partial x^i} - y^a \Gamma^j_{ai} \frac{\partial X^i}{\partial x^k} + y^a \Gamma^i_{ak} \frac{\partial X^j}{\partial x^i} = 0. \tag{4.2}$$

Proposition 4.5. *Let us consider a n-dimensional riemannian manifolds M. Let Ric be the Ricci curvature of a diagonal linear connection Γ. If Ric $= 0$ then $\mathcal{A}_\Gamma$ coincides with $\chi(TM)$ and is sometimes permutable and is necessarily a Lie algebra.*

Proof. Let Ric be a Ricci curvature tensor of a diagonal linear connection. Like $Ric = 0$, i.e. $\forall j,k \in [1,n]$ the coefficients R_{jk} of this tensor are all null. According to the resolution of the system of non-linear differential equation of first order (3.1), we find

$$(\Gamma^i_{jj})_{ij} = \begin{pmatrix} a_1 & 0 & \ldots & 0 & 0 \\ 0 & a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & a_n \end{pmatrix}$$

where a_i are constants $\forall i \in [1,n]$. However, locally, $\mathcal{A}_\Gamma$ is given by (4.2). Considering this system and by introducing all the Christoffel symbol for the connection Γ, we get the coefficients of the vector fields $X = X^i \frac{\partial}{\partial x^i} \in \chi(TM)$ where $X^i = X^i(x^i)$ $\forall i \in [1,n]$. Then $\mathcal{A}_\Gamma = \chi(TM)$, which is inevitable a Lie algebra because it's stable by bracket, except for explicit mentions. $\square$

Remark 4.6. If $\mathcal{A}_\Gamma$ is a Lie algebra of polynomial vector fields. The characterization f $\mathcal{A}_\Gamma$ follows from Theorem 2.21 in [6].

Proposition 4.7. *If Ric is a Ricci curvature with polynomial coefficients then the corresponding linear connection (resp. Lie algebra) is with polynomial coefficients (resp. Lie algebra of polynomial vector fields).*

Proof. It is a classical proof. From the Theorem 3.3 and the Corollary 3.4 we find the relation between the Ricci curvature of a linear connection with polynomial coefficients. To the benefit of the corresponding linear connection to this Ricci curvature, we just apply the Definition 4.4 of $\mathcal{A}_\Gamma$ and we obtain an Lie algebra of polynomial vector fields. □

Remark 4.8. The Lie algebra corresponding to the linear connection with polynomial coefficients necessarily verifies the results found in [7].

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