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# $\phi$-PRIMARY SUBTRACTIVE IDEALS IN SEMIRINGS

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**Abstract:** Let $R$ be a commutative semiring with identity. Let $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function where $I(R)$ is the set of ideals of $R$. A proper ideal $I$ of $R$ is called $\phi$-primary if whenever $a, b \in R$, $ab \in I - \phi(I)$ implies that either $a \in I$ or $b \in \sqrt{I}$. So if we take $\phi_{\emptyset}(I) = \emptyset$ (resp., $\phi_0(I) = 0$), a $\phi$-primary ideal is primary (resp., weakly primary). In this paper we study the properties of several generalizations of primary ideals of $R$.

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**Key Words:** semiring, $\phi$-primary ideal, subtractive ideal

## 1. Introduction

A commutative semiring $R$ is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b+c) = ab + ac$ for all $a, b, c \in R$. A non-empty subset $I$ of a semiring $R$ will be called an ideal if $a, b \in I$ and $r \in R$ imply that $a + b \in I$, $ra \in I$.Throughout this paper we let the semiring $R$ commutative with identity 1. An ideal $I$ of a semiring $R$ is called subtractive ideal if $a, a + b \in I$, then $b \in I$. An ideal $I$ of a semiring $R$ is called primary, for $a, b \in R$ with $\sqrt{I} = \{r \in R |\ r^n \in I \text{ for some n} \geq 1\}$, then $ab \in I$ implies $a \in I$ or $b \in \sqrt{I}$

**Definition 1.** Let $I(R)$ be the set of ideals of $R$ and $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. The proper ideal $I$ of $R$ is called a $\phi$-prime ($\phi$-primary) if for all $a, b \in R$, with $ab \in I - \phi(I)$ implies $a \in I$ or $b \in I$ ($a \in I$ or $b \in \sqrt{I}$).

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Clearly every primary ideal is a $\phi$-primary ideal, but the inverse case is not true. for exampel, let $R = \mathbb{Z}_6$. Then 0 is a $\phi$-primary ideal where $\phi(I) = 0$. But is not a primary ideal.

**Example 2.** Let $R$ be a commutative semiring. Define the following functions $\phi_\alpha : I(R) \to I(R) \cup \{\emptyset\}$ and the corresponding $\phi_\alpha$-primary ideal:

| | | | |
|---|---|---|---|
| (1) | $\phi_\emptyset$ | $\phi(I) = \emptyset$ | primary ideal |
| (2) | $\phi_0$ | $\phi(I) = 0$ | weakly primary ideal |
| (3) | $\phi_2$ | $\phi(I) = I^2$ | almost primary ideal |
| (4) | $\phi_n (n \geq 2)$ | $\phi(I) = I^n$ | $n-$ almost primary ideal |
| (5) | $\phi_w$ | $\phi(I) = \cap_{n=1}^{\infty} I^n$ | $w-$ primary ideal |
| (6) | $\phi_1$ | $\phi(I) = I$ | any ideal |

**Remark 3.** 1. Every $\phi$-prime ideal of $R$ is $\phi$-primary.

2. Given two functions $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$, for each $I \in I(R)$. Then $I$ $\psi_1$-primary implies $I$ $\psi_2$-primary. Note in this case that

$I$ primary $\Rightarrow$ $I$ weakly primary $\Rightarrow$ $I$ $w$-primary $\Rightarrow$ $I$ $(n+1)$-almost primary $\Rightarrow$ $I$ $n$-almost primary $(n \geq 2)$ $\Rightarrow$ $I$ almost primary.

3. Since $I - \phi(I) = I - (I \cap \phi(I))$, without loss of generality we may assume that $\phi(I) \subseteq I$. We henceforth make this assumption.

The notion of factor or quotient semiring was defined by Atani and Golan in [2], [3].

Let $J$ be an ideal of $R$ and $\phi : I(R) \to I(R) \cup \{\emptyset\}$ a function. Define $\phi_J : I(\frac{R}{J}) \to I(\frac{R}{J}) \cup \{\emptyset\}$ by $\phi_J(\frac{P}{J}) = \frac{(\phi(P)+J)}{J}$ for every ideal $P \in I(R)$ with $J \subseteq P$ (and $\phi_J(\frac{P}{J}) = \emptyset$ if $\phi(P) = \emptyset$).

**Theorem 4.** *Let $R$ be a commutative semiring and Let $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. Let $P$ be a $\phi$-primary subtractive ideal of $R$. Then*

*(i) If $J$ is an ideal of $R$ with $J \subseteq P$, then $\frac{P}{J}$ is a $\phi_J$-primary ideal of $\frac{R}{J}$.*

*(ii) If $J \subseteq \phi(P)$ such that $\phi(P)$ is a subtractive and $\frac{P}{J}$ is $\phi_J$-primary ideal of $\frac{R}{J}$, then $P$ is a $\phi$-primary.*

*(iii) If $\phi(P) \subseteq J$ and $P$ is a $\phi$-primary ideal of $R$, then $\frac{P}{J}$ is a weakly primary ideal of $\frac{R}{J}$.*

*(iv) If $\phi(J) \subseteq \phi(P)$, $J$ is a $\phi$-primary ideal of $R$ and $\frac{P}{J}$ is a weakly primary ideal of $\frac{R}{J}$, then $P$ is a $\phi$-primary.*

*Proof.* $(i)$ Let $a, b \in R$. Suppose that $(a+J)(b+J) \in \frac{P}{J} - \phi_J(\frac{P}{J})$ so $ab + J \in \frac{P}{J} - (\phi(P)+J)/J$. $(Hence\ ab \in P - \phi(P)+J)$. Thus $ab \in P - \phi(P)$ and so $a \in P$ or $b^k \in P$ for some $k \in \mathbb{N}$. Therefore $(a+J) \in \frac{P}{J}$ or $(b+J) \in \sqrt{\frac{P}{J}}$, So $\frac{P}{J}$ is $\phi_J$-primary.

$(ii)$ Suppose that $ab \in P - \phi(P)$ for some $a,\ b \in R$. Then $(a+J)(b+J) \in \frac{P}{J} - \frac{\phi(P)}{J} = \frac{P}{J} - \phi_J(\frac{P}{J})$. Since $\frac{P}{J}$ is assumed to be $\phi_J$-primary, we get $a+J \in \frac{P}{J}$ or $b+J \in \sqrt{\frac{P}{J}} = \frac{\sqrt{P}}{J}$. Consequently, either $a \in P$ or $b \in \sqrt{P}$, that is $P$ is $\phi$-primary.

$(iii)$ is a direct consequence of part $(i)$.

$(iv)$ Let $a_1a_2 \in P - \phi(P)$ where $a_1, a_2 \in R$. Note that $a_1a_2 \notin \phi(J)$ because $\phi(J) \subseteq \phi(P)$. If $a_1a_2 \in J$, then either $a_1 \in J \subseteq P$ or $a_2 \in \sqrt{J} \subseteq \sqrt{P}$, since $J$ is $\phi$-primary. If $a_1a_2 \notin J$, then $(a_1+J)(a_2+J) \in \frac{P}{J} - \{0\}$ and so either $(a_1+J) \in \frac{P}{J}$ or $(a_2+J) \in \sqrt{\frac{P}{J}} = \frac{\sqrt{P}}{J}$. Therefore, $a_1 \in P$ or $a_2 \in \sqrt{P}$. Consequently $P$ is a $\phi$-primary ideal of $R$. □

**Corollary 5.** *Let $R$ be a commutative semiring, and $\phi : I(R) \to I(R) \cup \{\phi\}$ be a function. Suppose that $I$ and $\phi(I)$ are subtractive ideals of $R$. Then $I$ is a $\phi$-primary if and only if $\frac{I}{\phi(I)}$ is a weakly primary ideal of $\frac{R}{\phi(I)}$.*

*Proof.* In parts $(ii)$ and $(iii)$ of Theorem 4 set $J = \phi(I)$. □

**Lemma 6.** *Let $Q$ be a subtractive ideal of a semiring $R$. If $a \in Q$ and $a+b \in \sqrt{Q}$, then $b \in \sqrt{Q}$.*

*Proof.* Let $a \in Q$, $a+b \in \sqrt{Q}$. Therefore $(a+b)^m \in Q$ for some $m \in \mathbb{N}$. So $c + b^m \in Q$ for some $c \in Q$. Since $Q$ is a subtractive ideal, $b \in \sqrt{Q}$. □

**Proposition 7.** *Let $R$ be a commutative semiring, and $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function, and let $Q$ be a $\phi$-primary subtractive ideal of $R$.*

*$(i)$ If $\phi(Q)$ is a primary ideal, then $Q$ is a primary ideal of $R$.*

*$(ii)$ If $Q$ is not primary, then $Q^2 \subseteq \phi(Q)$.*

*$(iii)$ If $Q$ is not primary, then $\sqrt{Q} = \sqrt{\phi(Q)}$.*

*Proof.* $(i)$ Let $a, b \in R$. If $ab \in I - \phi(I)$, since $I$ is a $\phi$-primary ideal, then $a \in I$ or $b^n \in I$. and if $ab \in \phi(I)$ then $a \in \phi(I) \subseteq I$ or $b^n \in \phi(I) \subseteq I$.

$(ii)$ Assume that $Q^2 \nsubseteq \phi(Q)$. Suppose that $a, b \in R$ are such that $ab \in Q$. If $ab \notin \phi(Q)$, since $Q$ is $\phi$-primary subtractive, either $a \in Q$ or $b \in \sqrt{Q}$. This is contradiction with $Q$ is not primary. Hence we may assume that $ab \in \phi(Q)$. If $aQ \nsubseteq \phi(Q)$, then there exists an element $a_0 \in Q$ such that $aa_0 \notin \phi(Q)$. Now $a(a_0+b) = aa_0+ab \in Q-\phi(Q)$ and $Q$ $\phi$-primary subtractive imply that either $a \in Q$ or $a_0+b \in \sqrt{Q}$. But $a_0 \in Q$. So, by Lemma 6, either $a \in Q$ or $b \in \sqrt{Q}$. Similarly, if $bQ \nsubseteq \phi(Q)$, we can show that either $a \in Q$ or $b \in \sqrt{Q}$. This is contradiction with $Q$ is not primary. So we may assume that $aQ \subseteq \phi(Q)$ and $bQ \subseteq \phi(Q)$. Since $Q^2 \nsubseteq \phi(Q)$, there exist $c, d \in Q$ with $cd \notin \phi(Q)$. Now $(a+c)(b+d) = ab+ad+bc+cd \in Q-\phi(Q)$, imply that either $a+c \in Q$ or $b+d \in \sqrt{Q}$. Therefore, either $a \in Q$ or $b \in \sqrt{Q}$, this is contradiction with assume. Consequently, $Q^2 \subseteq \phi(Q)$ .

*(iii)* Since $\phi(Q) \subseteq Q$, we have $\sqrt{\phi(Q)} \subseteq \sqrt{Q}$. On the other hand, it follows from part *(ii)* that $Q^2 \subseteq \phi(Q)$. Hence, $\sqrt{Q} = \sqrt{Q^2} \subseteq \sqrt{\phi(Q)}$; and hence $\sqrt{Q} = \sqrt{\phi(Q)}$. □

Recall that a proper ideal $I$ of a semiring $R$ is said to be strong ideal, if for each $a \in I$ there exists $b \in I$ such that $a+b=0$ (see [1]).

**Theorem 8.** *Let $R$ and $R'$ be semirings, $f : R \to R'$ be an epimorphism such that $f(0)=0$ and $I$ be a strong ideal of $R$. If $I$ is an $\phi$-primary ideal of $R$ with $I^2 \nsubseteq \phi(I)$ and $ker f \subseteq I$, then $f(I)$ is an $\phi$-primary ideal of $R'$.*

*Proof.* Let $I$ be a $\phi$-primary ideal of $R$ with $I^2 \nsubseteq \phi(I)$ and $ab \in f(I) \setminus \phi(f(I))$ where $a, b \in R'$. Since $a, b \in f(I)$, therefore there exists an element $m \in I$ such that $ab = f(m)$. Since $f$ is an epimorphism and $a, b \in R'$, then there exist $x, y \in R$ such that $a = f(x)$, $b = f(y)$. As $m \in I$ and $I$ is a strong ideal of $R$, there exists $l \in I$ such that $m+l=0$, which implies $f(m+l)=0$. This gives that $f(xy+l)=0$ implies $xy+l \in ker f \subseteq I$.This implies $xy \in I$, since $I$ is subtractive. Since $I$ is a $\phi$-primary ideal with $I^2 \nsubseteq \phi(I)$, we have that $I$ is a primary ideal by Proposition 7. Therefore, we have $x \in I$ or $y^n \in I$. Thus $a \in f(I)$ or $b^n \in f(I)$. Hence $f(I)$ is a $\phi$-primary ideal of $R'$. □

**Theorem 9.** *Let $R$ be a semiring and let $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. Suppose that $\{I_\lambda\}_\lambda \in \Lambda$ is a family of ideals of $R$ such that for every $\lambda,\ \lambda' \in \Lambda$, $\sqrt{\phi(I_\lambda)} = \sqrt{\phi(I_{\lambda'})}$ and $\phi(I_\lambda) \subseteq \phi(I)$. If for every $\lambda \in \Lambda$, $I_\lambda$ is a $\phi$-primary ideal of $R$ that is not primary, then $I = \bigcap_{\lambda\in\Lambda} I_\lambda$ is $\phi$-primary ideal of $R$.*

*Proof.* Since $I_\lambda$'s are $\phi$-primary but are not primary, then for every $\lambda \in \Lambda$, $\sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$, by Proposition 7. On the other hand $\phi(I_\lambda) \subseteq \phi(I)$ for every $\lambda \in \Lambda$, and so $\sqrt{\phi(I_\lambda)} \subseteq \sqrt{I}$. Hence $\sqrt{I} = \sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$ for every $\lambda \in \Lambda$. Let $a_1a_2 \in I-\phi(I)$ for some $a_1, a_2 \in R$, and let $a_1 \notin I$. Therefore there is a

$\lambda \in \Lambda$ such that $a_1 \notin I_\lambda$. Since $I_\lambda$ is $\phi$-primary and $a_1 a_2 \in I_\lambda - \phi(I_\lambda)$, then $a_1 \in \sqrt{I_\lambda} = \sqrt{I}$. Consequently $I$ is a $\phi$-primary ideal of $R$. □

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