Abstract: Let $R$ be a commutative semiring with identity. Let $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function where $I(R)$ is the set of ideals of $R$. A proper ideal $I$ of $R$ is called $\phi$-primary if whenever $a, b \in R$, $ab \in I - \phi(I)$ implies that either $a \in I$ or $b \in \sqrt{I}$. So if we take $\phi_{\emptyset}(I) = \emptyset$ (resp., $\phi_0(I) = 0$), a $\phi$-primary ideal is primary (resp., weakly primary). In this paper we study the properties of several generalizations of primary ideals of $R$.

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1. Introduction

A commutative semiring $R$ is defined as an algebraic system $(R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$. A non-empty subset $I$ of a semiring $R$ will be called an ideal if $a, b \in I$ and $r \in R$ imply that $a + b \in I$, $ra \in I$. Throughout this paper we let the semiring $R$ commutative with identity 1. An ideal $I$ of a semiring $R$ is called subtractive ideal if $a, a + b \in I$, then $b \in I$. An ideal $I$ of a semiring $R$ is called primary, for $a, b \in R$ with $\sqrt{I} = \{r \in R| r^n \in I \text{ for some } n \geq 1\}$, then $ab \in I$ implies $a \in I$ or $b \in \sqrt{I}$.

Definition 1. Let $I(R)$ be the set of ideals of $R$ and $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. The proper ideal $I$ of $R$ is called a $\phi$-prime ($\phi$-primary) if for all $a, b \in R$, with $ab \in I - \phi(I)$ implies $a \in I$ or $b \in I$ ($a \in I$ or $b \in \sqrt{I}$).
Clearly every primary ideal is a $\phi$-primary ideal, but the inverse case is not true. For example, let $R = \mathbb{Z}_6$. Then $0$ is a $\phi$-primary ideal where $\phi(I) = 0$. But is not a primary ideal.

**Example 2.** Let $R$ be a commutative semiring. Define the following functions $\phi_\alpha : I(R) \to I(R) \cup \{\emptyset\}$ and the corresponding $\phi_\alpha$-primary ideal:

1. $\phi_\emptyset$ $\phi(I) = \emptyset$ primary ideal
2. $\phi_0$ $\phi(I) = 0$ weakly primary ideal
3. $\phi_2$ $\phi(I) = I^2$ almost primary ideal
4. $\phi_n(n \geq 2)$ $\phi(I) = I^n$ $n$-almost primary ideal
5. $\phi_w$ $\phi(I) = \cap_{n=1}^\infty I^n$ $w$-primary ideal
6. $\phi_1$ $\phi(I) = I$ any ideal

**Remark 3.** 1. Every $\phi$-prime ideal of $R$ is $\phi$-primary.

2. Given two functions $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$, for each $I \in I(R)$. Then $I$ $\psi_1$-primary implies $I$ $\psi_2$-primary. Note in this case that

$I$ primary $\Rightarrow$ $I$ weakly primary $\Rightarrow$ $I$ $w$-primary $\Rightarrow$ $I$ $(n + 1)$-almost primary $\Rightarrow$ $I$ $n$-almost primary $(n \geq 2)$ $\Rightarrow$ $I$ almost primary.

3. Since $I - \phi(I) = I - (I \cap \phi(I))$, without loss of generality we may assume that $\phi(I) \subseteq I$. We henceforth make this assumption.

The notion of factor or quotient semiring was defined by Atani and Golan in [2], [3].

Let $J$ be an ideal of $R$ and $\phi : I(R) \to I(R) \cup \{\emptyset\}$ a function. Define $\phi_J : I(\frac{R}{J}) \to I(\frac{R}{J}) \cup \{\emptyset\}$ by $\phi_J(P) = (\phi(P) + J)$ for every ideal $P \in I(R)$ with $J \subseteq P$ (and $\phi_J(0) = \emptyset$ if $\phi(P) = \emptyset$).

**Theorem 4.** Let $R$ be a commutative semiring and let $\phi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. Let $P$ be a $\phi$-primary subtractive ideal of $R$. Then

(i) If $J$ is an ideal of $R$ with $J \subseteq P$, then $\frac{P}{J}$ is a $\phi_J$-primary ideal of $\frac{R}{J}$.

(ii) If $J \subseteq \phi(P)$ such that $\phi(P)$ is a subtractive and $\frac{P}{J}$ is a $\phi_J$-primary ideal of $\frac{R}{J}$, then $P$ is a $\phi$-primary.

(iii) If $\phi(P) \subseteq J$ and $P$ is a $\phi$-primary ideal of $R$, then $\frac{P}{J}$ is a weakly primary ideal of $\frac{R}{J}$.

(iv) If $\phi(J) \subseteq \phi(P)$, $J$ is a $\phi$-primary ideal of $R$ and $\frac{P}{J}$ is a weakly primary ideal of $\frac{R}{J}$, then $P$ is a $\phi$-primary.
Proof. (i) Let \(a, b \in R\). Suppose that \((a + J)(b + J) \notin P_J - \phi(J)\) so \(ab + J \notin \sqrt{P_J - (\phi(P) + J)}/J\). (Hence \(ab \in P - \phi(P) + J\). Thus \(ab \in P - \phi(P)\) and so \(a \in P\) or \(b \in P\) for some \(k \in N\). Therefore \((a + J) \notin P_J\) or \((b + J) \notin \sqrt{P_J}\), So \(P_J\) is \(\phi(J)\)-primary.

(ii) Suppose that \(ab \in P - \phi(P)\) for some \(a, b \in R\). Then \((a + J)(b + J) \in P_J - \phi(J) = P_J - \phi(P)\). Since \(P_J\) is assumed to be \(\phi(J)\)-primary, we get \(a + J \notin P_J\) or \(b + J \notin \sqrt{P_J} = \sqrt{P}\). Consequently, either \(a \in P\) or \(b \in \sqrt{P}\), that is \(P\) is \(\phi\)-primary.

(iii) is a direct consequence of part (i).

(iv) Let \(a_1a_2 \in P - \phi(P)\) where \(a_1, a_2 \in R\). Note that \(a_1a_2 \notin \phi(J)\) because \(\phi(J) \subseteq \phi(P)\). If \(a_1a_2 \in J\), then either \(a_1 \in J \subseteq P\) or \(a_2 \in \sqrt{J} \subseteq \sqrt{P}\), since \(J\) is \(\phi\)-primary. If \(a_1a_2 \notin J\), then \((a_1 + J)(a_2 + J) \notin P_J\) and so either \((a_1 + J) \notin P_J\) or \((a_2 + J) \notin \sqrt{P_J} = \sqrt{P}\). Therefore, \(a_1 \in P\) or \(a_2 \notin \sqrt{P}\). Consequently \(P\) is a \(\phi\)-primary ideal of \(R\).

Corollary 5. Let \(R\) be a commutative semiring, and \(\phi : I(R) \to I(R) \cup \{\emptyset\}\) be a function. Suppose that \(I\) and \(\phi(I)\) are subtractive ideals of \(R\). Then \(I\) is a \(\phi\)-primary if and only if \(\frac{I}{\phi(I)}\) is a weakly primary ideal of \(\frac{R}{\phi(I)}\).

Proof. In parts (ii) and (iii) of Theorem 4 set \(J = \phi(I)\).

Lemma 6. Let \(Q\) be a subtractive ideal of a semiring \(R\). If \(a \in Q\) and \(a + b \in \sqrt{Q}\), then \(b \in \sqrt{Q}\).

Proof. Let \(a \in Q\), \(a + b \in \sqrt{Q}\). Therefore \((a + b)^m \in Q\) for some \(m \in N\). So \(c + b^m \in Q\) for some \(c \in Q\). Since \(Q\) is a subtractive ideal, \(b \in \sqrt{Q}\).

Proposition 7. Let \(R\) be a commutative semiring, and \(\phi : I(R) \to I(R) \cup \{\emptyset\}\) be a function, and let \(Q\) be a \(\phi\)-primary subtractive ideal of \(R\).

(i) If \(\phi(Q)\) is a primary ideal, then \(Q\) is a primary ideal of \(R\).

(ii) If \(Q\) is not primary, then \(Q^2 \subseteq \phi(Q)\).

(iii) If \(Q\) is not primary, then \(\sqrt{Q} = \sqrt{\phi(Q)}\).

Proof. (i) Let \(a, b \in R\). If \(ab \in I - \phi(I)\), since \(I\) is a \(\phi\)-primary ideal, then \(a \in I\) or \(b^n \in I\). and if \(ab \in \phi(I)\) then \(a \in \phi(I) \subseteq I\) or \(b^n \in \phi(I) \subseteq I\).

(ii) Assume that \(Q^2 \notin \phi(Q)\). Suppose that \(a, b \in R\) are such that \(ab \in Q\). If \(ab \notin \phi(Q)\), since \(Q\) is \(\phi\)-primary subtractive, either \(a \in Q\) or \(b \in \sqrt{Q}\). This
is contradiction with $Q$ is not primary. Hence we may assume that $ab ∈ φ(Q)$. If $aQ ⊈ φ(Q)$, then there exists an element $a_0 ∈ Q$ such that $aa_0 ⊈ φ(Q)$. Now $a(a_0 + b) = aa_0 + ab ∈ Q − φ(Q)$ and $Q$ $φ$-primary subtractive imply that either $a ∈ Q$ or $a_0 + b ∈ √Q$. But $a_0 ∈ Q$. So, by Lemma 6, either $a ∈ Q$ or $b ∈ √Q$. Similarly, if $bQ ⊈ φ(Q)$, we can show that either $a ∈ Q$ or $b ∈ √Q$. This is contradiction with $Q$ is not primary. So we may assume that $aQ ⊆ φ(Q)$ and $bQ ⊆ φ(Q)$. Since $Q^2 ⊈ φ(Q)$, there exist $c, d ∈ Q$ with $cd ⊈ φ(Q)$. Now $(a + c)(b + d) = ab + ad + bc + cd ∈ Q − φ(Q)$, imply that either $a + c ∈ Q$ or $b + d ∈ √Q$. Therefore, either $a ∈ Q$ or $b ∈ √Q$, this is contradiction with assume. Consequently, $Q^2 ⊆ φ(Q)$.

(iii) Since $φ(Q) ⊆ Q$, we have $√φ(Q) ⊆ √Q$. On the other hand, it follows from part (ii) that $Q^2 ⊆ φ(Q)$. Hence, $√Q = √Q^2 ⊆ √φ(Q)$; and hence $√Q = √φ(Q).$  

Recall that a proper ideal $I$ of a semiring $R$ is said to be strong ideal, if for each $a ∈ I$ there exists $b ∈ I$ such that $a + b = 0$ (see [1]).

**Theorem 8.** Let $R$ and $R'$ be semirings, $f : R → R'$ be an epimorphism such that $f(0) = 0$ and $I$ be a strong ideal of $R$. If $I$ is an $φ$-primary ideal of $R$ with $I^2 ⊈ φ(I)$ and $kerf ⊆ I$, then $f(I)$ is an $φ$-primary ideal of $R'$.

**Proof.** Let $I$ be a $φ$-primary ideal of $R$ with $I^2 ⊈ φ(I)$ and $ab ∈ f(I) \ setminus φ(f(I))$ where $a, b ∈ R'$. Since $a, b ∈ f(I)$, therefore there exists an element $m ∈ I$ such that $ab = f(m)$. Since $f$ is an epimorphism and $a, b ∈ R'$, then there exist $x, y ∈ R$ such that $a = f(x), b = f(y)$. As $m ∈ I$ and $I$ is a strong ideal of $R$, there exists $l ∈ I$ such that $m + l = 0$, which implies $f(m + l) = 0$. This gives that $f(xy + l) = 0$ implies $xy + l ∈ kerf ⊆ I$. This implies $xy ∈ I$, since $I$ is subtractive. Since $I$ is a $φ$-primary ideal with $I^2 ⊈ φ(I)$, we have that $I$ is a primary ideal by Proposition 7. Therefore, we have $x ∈ I$ or $y^n ∈ I$. Thus $a ∈ f(I)$ or $b^n ∈ f(I)$. Hence $f(I)$ is a $φ$-primary ideal of $R'$.

**Theorem 9.** Let $R$ be a semiring and let $φ : I(R) → I(R) \cup \{\emptyset\}$ be a function. Suppose that $\{I_λ\}_λ ∈ Λ$ is a family of ideals of $R$ such that for every $λ, Λ' ∈ Λ$, $\sqrt{φ(I_λ)} = \sqrt{φ(I_{λ'})}$ and $φ(I_λ) ⊆ φ(I)$. If for every $λ ∈ Λ$, $I_λ$ is a $φ$-primary ideal of $R$ that is not primary, then $I = \bigcap_{λ ∈ Λ} I_λ$ is $φ$-primary ideal of $R$.

**Proof.** Since $I_λ$’s are $φ$-primary but are not primary, then for every $λ ∈ Λ$, $\sqrt{I_λ} = \sqrt{φ(I_λ)}$, by Proposition 7. On the other hand $φ(I_λ) ⊆ φ(I)$ for every $λ ∈ Λ$, and so $\sqrt{φ(I_λ)} ⊆ \sqrt{I}$. Hence $\sqrt{I} = \sqrt{T} = \sqrt{φ(I)}$ for every $λ ∈ Λ$. Let $a_1a_2 ∈ I − φ(I)$ for some $a_1, a_2 ∈ R$, and let $a_1 ∉ I$. Therefore there is a
λ ∈ Λ such that \( a_1 \not\in I_\lambda \). Since \( I_\lambda \) is \( \phi \)-primary and \( a_1a_2 \in I_\lambda - \phi(I_\lambda) \), then \( a_1 \in \sqrt{I_\lambda} = \sqrt{I} \). Consequently \( I \) is a \( \phi \)-primary ideal of \( R \).

\[ \square \]

References


