THE LABOR IMMIGRATION CONTROL

Alexandra Lukina¹§, Alexander Prasolov²

¹,²Faculty of Applied Mathematics and Control Processes
Saint Petersburg State University
Universitetskii Pr. 35, Petergof, Saint Petersburg, 198504, RUSSIA

Abstract: In this paper we propose a dynamic model of controlled labor immigration. The model can be used as a framework to study the economic impacts of immigration for host countries. The system’s trajectories are analyzed for different parameter values. A control problem formalization is given. Existence of optimal immigration quota is proved.

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1. Introduction

International migration is becoming a global phenomenon which has far-reaching social and economic consequences. We focus on the impact of labor immigration on the recipient and donor countries economic growth. Traditionally this problem is studied within the bounds of an econometric [1] or a simulation-based [2, 3, 4] analysis. To our knowledge, there are only few mathematical models urged to estimate the effect of migration on the economic growth [5, 6, 7, 8]. We construct the dynamic model of labor immigration which allows to evaluate the dependence of the recipient country economic growth rate on recruiting foreign
2. The Model

Let us consider the country receiving labor migrants from the “outer world” (it has to be interpreted as a consolidation of all donor countries). The main hypotheses of our model are the following:

1. In the absence of government regulations the migration rate is assumed to be proportional to the difference between the recipient country and the “outer world” labor productivities. If we deal with the process of immigration the labor productivity in the country will be always higher than the labor productivity in the “outer world”.

2. The government establishes the annual quota on the number of immigrants who have a permission to enter the country during the year. If the number of immigrants accumulated from the beginning of the year according to the rule 1 runs up to the quota then the migration process will be stopped immediately by the government until the turn of the year.

The first assumption is consistent with the neoclassical economics theory of migration (mainframe papers are [12, 13]) which picks out a difference in average wage rates as a basic reason of moving from one region to another.

Let us denote the whole population of the recipient country (we interpret the term “population” as the working population), composed of the native population \( N \) and the number of immigrants \( M \), by \( E \). Hence, \( E = N + M \). Then, assuming that all these values depend on time \( t \), \( \dot{E} = \dot{N} + \dot{M} \), where “over-dot” denotes time derivative \( \left( \dot{E} = \frac{dE}{dt} \right) \). Assumptions (1)-(2) can be formally expressed as

\[
\dot{M} = \begin{cases} 
\alpha(z - z_{ex}), & \text{if } \alpha \int_{[t]}^{t}(z - z_{ex})dx \leq \bar{M}, \\
0, & \text{otherwise},
\end{cases}
\]

where \( z = \frac{F(K, E)}{E} \) is the country labor productivity; \( Y = F(K, E) = aK^\beta E^{1-\beta} \) is the output level represented by two factor Cobb-Douglas production function; \( K \) is the capital input; \( z_{ex} > 0 \) is the “outer world” labor productivity (here we
assume that it is constant); \(a > 0\) is the total factor productivity; \(0 < \beta < 1\) is the output elasticity of capital; the coefficient \(\alpha > 0\) characterizes an attraction to the higher labor productivity; \([t]\), i.e. integer part of \(t\), corresponds to the beginning of the year; \(\bar{M} \geq 0\) is the government annual quota.

Therefore, from the beginning of the year the number of new immigrants is determined according to the rule of “attraction” to the higher labor productivity. And when the number of immigrants accumulated from the beginning of the year runs up to the quota \(\bar{M}\), the process of immigration is stopped by the government until the turn of the year. The process is analyzed while \(z(t) > z_{ex}\). Capital is an endogenous variable determined by standard Solow model equation \(\dot{K} = -\delta K + pF(K, E)\), where \(0 < \delta < 1\) is the depreciation rate, \(0 < p < 1\) is the fraction of output devoted to investment. This completes the description of the model.

Thus, the process is described by the system

\[
\dot{E} = \begin{cases} 
\dot{N} + \alpha \left( a \left( \frac{K}{E} \right)^\beta - z_{ex} \right), & \text{if } J(t) \leq \bar{M}, \\
\dot{N}, & \text{otherwise,}
\end{cases} \quad (1)
\]

\[
\dot{K} = -\delta K + paK^\beta E^{1-\beta}, \quad (2)
\]

where

\[
J(t) = \alpha \int_{[t]}^t \left( a \left( \frac{K(x)}{E(x)} \right)^\beta - z_{ex} \right) dx.
\]

Note that equation (1) has a discontinuous second member even if function \(N\), characterizing the native population dynamics, is sufficiently smooth. From the condition

\[
\alpha \int_{[t]}^t \left( a \left( \frac{K(x)}{E(x)} \right)^\beta - z_{ex} \right) dx = \bar{M}
\]

one obtains the switchover moments of time (there are two regimes: immigrants reception regime and enter prohibited regime). More precisely, let us consider \(t_i \in \mathbb{Z}\) and specific year \([t_i, t_i + 1]\). From the moment \(t_i\) till the moment \(\bar{t}\) the reception regime is switched on, i.e. \(\dot{E} = \dot{N} + \alpha \left( a \left( \frac{K}{E} \right)^\beta - z_{ex} \right)\), where the moment of time \(\bar{t} \in (t_i, t_i + 1)\) satisfies

\[
\alpha \int_{[t]}^{\bar{t}} \left( a \left( \frac{K(x)}{E(x)} \right)^\beta - z_{ex} \right) dx = \bar{M}.
\]
From the moment \( t \) till the moment \( t_{i}+1 \) enter prohibited regime is switched on, i.e. \( \dot{E} = \dot{N} \). If for all \( t \in [t_{i}, t_{i}+1] \) condition
\[
\alpha \int_{[t]}^{t} \left( a \left( \frac{K(x)}{E(x)} \right)^{\beta} - z_{ex} \right) dx < \bar{M}
\]
is fullfilled, then the immigrants reception regime is switched on during the whole year \( [t_{i}, t_{i}+1] \). If \( \bar{t} \) exists and switching occurs, then inverse switching from enter prohibited regime to the reception regime occurs at the beginning of the next year, i.e. at the moment \( t_{i}+1 \).

If continuously differentiable function \( N(t) \) and initial values \( E(0) = N(0) = N_{0}, K(0) = K_{0} \) are given, then the continuous solution of the system (1)-(2) that satisfies initial conditions will be unique. Solving equation (1), one integrates two different equations according to switching regimes, thus, there are two different regimes in (2), since the second member of (2) includes \( E(t) \). Primary arbitrary constants are found from initial conditions, following arbitrary constants are determined according to combining of constituent solutions (corresponding to different regimes) into the continuous general solution. Note that function \( K(t) \) is continuously differentiable since the second member of (2) is continuous.

Given continuously differentiable function \( N(t) \) consider the system (1)-(2) and initial values \( E(0) = N_{0}, K(0) = K_{0} \). Let \( E_{1}(t), K_{1}(t) \) be the solution to the system (1)-(2) and initial value problem for the parameter value \( \bar{M} = 0 \) and \( E_{2}(t), K_{2}(t) \) be the solution to the same problem for \( \bar{M} \gg 1 \). Then we have

**Theorem 1.** Let \( E(t), K(t) \) be the solution to (1)-(2) and initial value problem for \( \bar{M} > 0 \). Then for all \( t \geq 0 \) that satisfies \( z(t) > z_{ex} \) we have the following inequalities
\[
E_{1}(t) \leq E(t) \leq E_{2}(t), \quad K_{1}(t) \leq K(t) \leq K_{2}(t), \quad Y_{1}(t) \leq Y(t) \leq Y_{2}(t), \quad Y_{i}(t) = F(K_{i}(t), E(t)), \quad Y_{i}(t) = F(K_{i}(t), E_{i}(t)), \quad i = 1, 2.
\]

Therefore, solutions for \( \bar{M} = 0 \) and for unrestrictedly large \( \bar{M} \), i.e. for banned and unrestricted immigration, respectively, define a range of variables variation for quota value \( \bar{M} > 0 \), restraining migrants inflow.

Figure 1 illustrates theorem 1, it represents the output trajectories \( Y(t) \) corresponding to (a) banned immigration \( (\bar{M} = 0) \), (b) quota value \( \bar{M} > 0 \), restraining migrants inflow, (c) unrestricted immigration \( (\bar{M} \gg 1)^{1} \). Curves (a)

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\footnote{Curves are obtained by numerical integration of the system (1)-(2) with \( \dot{N} = \lambda N, \lambda = 0.001, \delta = 0.03, \beta = 0.25, \alpha = 10, p = 0.2, a = 0.19, z_{ex} = 0.016, E(0) = 8, K(0) = 10. \) In case (b) \( \bar{M} = 0.05E(0) \).}
and (c), corresponding to banned and unrestricted immigration, respectively, define $Y(t)$ variation range. Thus, the output grows not worse than (a) pictures, but also not better than (c) pictures.

Consider the case of the autonomous system (1)-(2), i.e. $N(t) \equiv const$. The desired economic development scenario corresponds to the zone $\Lambda$ in the phase space $(E, K)$, where the population and the capital don’t decrease, i.e. $\dot{E} \geq 0$, $\dot{K} \geq 0$.

**Theorem 2.** Suppose that $N(t) \equiv const$. The set $\Lambda = \{(E, K) : \dot{E} \geq 0, \dot{K} \geq 0\}$ in the phase space is non-empty if and only if the saving rate $p$ satisfies the inequality $p \geq \frac{\delta}{a^{\frac{1-\beta}{\beta}}} z_{ex}^{1-\beta}$.

If we have the equality $p = \frac{\delta}{a^{\frac{1-\beta}{\beta}}} z_{ex}^{1-\beta}$ zone $\Lambda$ degenerates into the straight line.

The theorem 2 is an estimation judgment that permits the recipient country, confronting stagnation or a small decreasing of the native population, to conclude whether the immigration could serve the economy expansion and to determine which saving rate should be chosen. Mathematically theorems 1 and 2 are rather trivial but they are very substantial as a matter of the task. The theorem 1 will be evident from the following reasoning. The proof of the theorem 2 is omitted because of its simplicity.
Remark. In case \( N(t) \equiv N_0 \) without immigration, i.e. for \( \bar{M} = 0 \), the system could be solved analytically. Here we have \( E(t) \equiv N_0 \). With constant function \( E(t) \) equation (2) is Bernoulli equation, which could be solved using the substitution \( y = K^{1-\beta} \) leading equation (2) to a linear equation. In this case we have

\[
K(t) = \left( \frac{N_0}{\delta} + ((K(0))^{1-\beta} - \frac{N_0}{\delta}) \exp (\delta (\beta - 1)t) \right)^{\frac{1}{1-\beta}}.
\]

The model of controlled labor immigration could be used as a framework to analyze the following situation. The recipient country confronts the labor shortage because of, for example, some demographical problems. It’s assumed that the recipient country has nonpositive or insufficient economic growth. Using the constructed model we can answer the following questions:

1. Could the recipient country, that is enough attractive for immigrants, achieve desired economic growth due to labor reinforcement by immigrants (other economic indicators being equal)?

2. If we could answer the first question affirmatively and, moreover, the number of immigrants wishing to enter the country is greater than the number necessary for achievement desired growth, what annual quota should be established? Immigration restrictions are most of all associated with deleterious effects of immigration such as increasing inter-ethnic tension. Also recruiting foreign labor force on a substantial scale leads to incidental expenses on sociocultural adaptation, training etc.

3. The labor immigration control problem

Let us pose a labor immigration control problem. Suppose in the system (1)-(2) we are given the parameters \( \delta, \beta, \alpha, p, a, z_{ex} \), initial values \( E(0) = N_0 \), \( K(0) = K_0 \), the time segment \([0, T]\) in future, where the terminal time \( T \) is fixed, and the function \( N(t), t \in [0, T] \), characterizing the future native population dynamics.

The first problem, which could be solved in the framework of the constructed model, is the following. Suppose the parameter \( \bar{M} \), i.e. the annual quota, is fixed by the governing body. Using the model we can find attainable output levels given this annual quota \( \bar{M} \). This problem is trivial, to be more
exact, the system (1)-(2) is numerically integrated on the time interval \([0, T]\) given the parameters \(\delta, \beta, \alpha, p, a, z_{ex}\), initial conditions \(E(0), K(0)\) and the function \(N(t)\). And if we have \(E(t)\) and \(K(t)\), then the output is determined according to the formula \(Y(t) = F(K(t), E(t)) = a(K(t))^{\beta}(E(t))^{1-\beta}\).

If the governing body is satisfied with the resulted output level dynamics, then the parameter \(\bar{M}\) should not be changed. If the output dynamics exceeds or, on the contrary, doesn’t conform to the desired output dynamics, then the governing body needs to change the annual quota.

We now proceed to formulate the second problem, which could be solved in the framework of the model. Suppose that the continuous function \(\hat{Y}(t), t \in [0, T]\), characterizing the desired future output dynamics, is given. Obviously the function \(\hat{Y}(t)\) is inherently nondecreasing.

Having solved the system (1)-(2) for \(\bar{M} = 0\) and \(\bar{M} \gg 1\), we get \(Y_1(t)\), which is the output corresponding to banned immigration, and \(Y_2(t)\), which corresponds to unrestricted immigration. If the function \(\hat{Y}(t)\) satisfies the following requirement: \(\hat{Y}(t) \leq Y_2(t)\) for \(\forall t \in [0, T]\) and at the same time \(\exists t \in [0, T]\) such that \(\hat{Y}(t) > Y_1(t)\), then it’s possible to find an optimal quota ensuring the desired economic growth levels. In the contrary case the labor immigration control problem for the recipient country is not sensible. More specifically, if for some \(t\) the inequality \(\hat{Y}(t) > Y_2(t)\) is fulfilled, then the labor reinforcement is not enough for planned output \(\hat{Y}(t)\) attainment (some other economy stimulating measures should be taken). If \(\hat{Y}(t) \leq Y_1(t)\) for \(\forall t \in [0, T]\), then the desired output levels could be attained without foreign labor force recruiting.

Suppose that conditions \(\hat{Y}(t) \leq Y_2(t)\) for \(\forall t \in [0, T]\) and \(\exists t \in [0, T]\) such that \(\hat{Y}(t) > Y_1(t)\), are fulfilled and define the optimal annual quota. Formalize the control optimal problem in the following way (some other sensible formulations of the problem will be pointed out in remarks). Let the governing body observes the output level only at the end of the period under report and poses a problem to achieve at time \(T\) the desired output level \(\hat{Y}(T)\) (which is fixed).

Therefore, we have the fixed endpoint problem. The dynamic constraints are given by the system (1)-(2), \(\bar{M}\) is the governing parameter. \(\bar{M}\) is admissible control if \(\bar{M} \geq 0\). The initial state \(E(0) = N_0, K(0) = K_0\) is fixed. The problem is to find admissible \(\bar{M}\) such that the system evolves from the initial state to the state \(E(T), K(T)\) satisfying the following condition \(Y(T) = F(K(T), E(T)) = a(K(T))^{\beta}(E(T))^{1-\beta} = \hat{Y}(T)\). We want to minimize the annual quota \(\bar{M}\) (i.e. the cost functional is \(\bar{M}\) himself). Thus, the task is to achieve at time \(T\) prescribed in advance output level \(\hat{Y}(T)\) recruiting the least number of labor immigrants.
Theorem 3. Suppose that in the system (1)-(2)
1. parameters $0 < \delta < 1$, $0 < \beta < 1$, $\alpha > 0$, $0 < p < 1$, $a > 0$, $z_{ex} > 0$ and the initial state $E(0) = N_0$, $K(0) = K_0$ are given,
2. the time segment $[0, T]$ in future, where the terminal time $T$ is fixed, the function $N(t)$, $t \in [0, T]$, the number $\hat{Y}(T) \leq Y_2(T)$, where $Y_2(t)$, $t \in [0, T]$, corresponds to the solution for $M \gg 1$, are given
3. for $\forall t \in [0, T]$ the inequality $a \left( \frac{K(t)}{E(t)} \right)^\beta > z_{ex}$ is fulfilled

Then there exists the governing parameter minimum value $\bar{M}$ such that $Y(T) = \hat{Y}(T)$. If we also have $N(t) \equiv N_0$ and $p \geq \frac{\delta}{a^\beta} z_{ex}^{-\frac{1}{\beta}}$, then the state variables $E(t)$, $K(t)$ and, hence, the output $Y(t)$ don’t decrease for $t \in [0, T]$.

Proof. According to the condition

$$\alpha \int_{[t]}^t \left( a \left( \frac{K(x)}{E(x)} \right)^\beta - z_{ex} \right) dx = \bar{M},$$

the moments of time $[t] \leq \bar{t} \leq [t] + 1$, switching from the reception regimes to the enter prohibited regimes, become greater as the governing parameter $\bar{M}$ increases. This assertion is true since the expression under the integral sign is positive according to the assumption 3. By (1) and (2) during the reception regime the growth rates of variables $E(t)$ and $K(t)$ are greater than during the enter prohibited regime. It follows from here that if $\bar{M} > \bar{M}$, then $\bar{E}(t) > \bar{E}(t)$, $\bar{K}(t) > \bar{K}(t)$ for $\forall t \in [0, T]$, where $\bar{E}(t)$, $\bar{K}(t)$ is the solution for $\bar{M}$ and $\bar{E}(t)$, $\bar{K}(t)$ – for $\bar{M}$. Obviously, in this case we also have $\bar{Y}(t) > \bar{Y}(t)$ for $\forall t \in [0, T]$. Moreover, as long as all the conditions

$$\alpha \int_{[t]}^t \left( a \left( \frac{K(x)}{E(x)} \right)^\beta - z_{ex} \right) dx = \bar{M}$$

are restrictive, i.e. the switchover moments of time exist in the course of every year, curves $E(t)$, $K(t)$ and, hence, $Y(t)$ shift upwards strictly monotonously in axes $(t, E(t))$, $(t, K(t))$, $(t, Y(t))$, respectively. Further, if we increase the parameter $\bar{M}$ continuously, then these curves shift upwards also continuously. This is true since the switchover moments of time and the arbitrary constants, originated from the integration of equations, corresponding to the different regimes, continuously depend on $\bar{M}$. 
Thus, the curve $Y(t)$ shifts upwards in the axes $(t, Y(t))$ continuously as $\bar{M}$ increases continuously. When all the conditions

$$\alpha \int_{[t]}^{t} \left( a \left( \frac{K(x)}{E(x)} \right)^{\beta} - z_{ex} \right) dx = \bar{M}$$

cease to be restrictive (i.e. in case of unrestricted immigration), we have the output $Y_2(t)$. So $Y(t)$ continuously shifts upwards from the curve $Y_1(t)$, corresponding to $\bar{M} = 0$, to the curve $Y_2(t)$, corresponding to $\bar{M} \gg 1$. Since according to the assumption we have $\hat{Y}(T) \leq Y_2(T)$, then there exists such $\bar{M}$ that the equality $Y(T) = \hat{Y}(T)$ is fulfilled.

If the conditions $N(t) \equiv N_0$ and $p \geq \frac{\delta}{a^{\beta}} z_{ex}$ are fulfilled, then the theorem 2 is true. In this case we have $\dot{E} \geq 0$, $\dot{K} \geq 0$, and hence, $\dot{Y} \geq 0$.

The labor immigration control problem, described above, could be solved numerically. The task comes to numerical integration of the system (1)-(2) for different values of $\bar{M}$, which we increase from $\bar{M} = 0$ with the given step. This process should be stopped when the numerically found value $Y(T)$ exceeds given in advance value $\hat{Y}(T)$. The last value of $\bar{M}$ represents the numerical analogue of the optimal quota. The proximity of the numerically found $\bar{M}$ to the optimal $\bar{M}$ from the theorem 1, obviously, depends on the step size and on the numerical method.

**Remark 1.** The control optimal problem could be formulated in a different way. For example, we may consider the fixed time, free endpoint problem, where we choose the output $Y(t)$ deviation from the planned output $\hat{Y}(t)$ in the fixed metric as the cost functional to minimize. Since the functions $Y(t)$ and $\hat{Y}(t)$ are continuous, Chebyshev norm of their difference, i.e. $||Y(t) - \hat{Y}(t)||_{C[0,T]} = \max_{t \in [0,T]} |Y(t) - \hat{Y}(t)|$, could be regarded as the distance measure. If we are additionally given the range $[M_1, M_2]$ of the governing parameter admissible values (where $M_1$ and $M_2$ are, for example, the fixed fractions of the value $E(0)$), then there exists the quota optimal value, since the continuous in $\bar{M}$ function $||Y(t) - \hat{Y}(t)||_{C[0,T]}$ attains a minimum on a compact space $[M_1, M_2]$.

**Remark 2.** We may also consider piecewise continuous functions as admissible controls. For example, it seems sensible to consider establishment of different quotas during different years. In this case we have a step function $M(t) = M_i$ for $t_i < t \leq t_i + 1$, where $t_i \in \mathbb{Z}$. 
Remark 3. Formulation of the problem with several objective functionals is more difficult (for example, [14]). Nevertheless, one may simultaneously consider, for example, the growth rates maximization, expenses on migrants training minimization, inter-ethnic tension minimization etc. Some other additional assumptions are necessary for mathematically strict formalization.

4. Conclusion

International migration is a prominent demographic and economic issue in a present-day world. Most of migrants move from one country to another in search of better jobs, thus, undoubtedly, the problem of labor immigration control is significant. We are especially interested in the macro-effects of labor immigration and the recipient country immigration quotation.

Guided by the neoclassical economics theory of migration we endogenously introduce labor immigration into the Solow growth model. Also we consider annual migration quota as the model governing parameter.

The constructed dynamic model of controlled labor immigration allows to evaluate the dependence of the recipient country economic growth rate on recruiting foreign labor force. More precisely, we find attainable output levels given the annual quota.

Furthermore, the inverse problem could be solved. We find the minimal annual quota such that the predetermined output level at fixed future moment is guaranteed. The control problem is formalized, necessary conditions for the optimal quota existence are found.

References


