ON PSEUDO B-WEYL AND PSEUDO B-FREDHOLM OPERATORS

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Abstract: In this note, we show that the pseudo B-Fredholm and pseudo B-Weyl spectra, for a bounded linear operator on a Banach space, are compact in the complex plane. Afterwards, we prove that the pseudo B-Fredholm spectrum differs from the Kato spectrum on at most countable many points. Furthermore, if $T$ and $T^*$ have the SVEP at $\lambda_0$, we show that $\lambda_0 I - T$ is a pseudo B-Weyl operator if and only if $\lambda_0 I - T$ is a pseudo B-Fredholm operator.

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1. Introduction

Throughout, $X$ denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on $X$, let $I$ be the identity operator, and for $T \in \mathcal{B}(X)$ we denote by $T^*$, $R(T)$, $R^\infty(T) = \bigcap_{n \geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the adjoint, the range, the hyper-range, the resolvent set, the spectrum, the point spectrum, the approximate point spectrum and the surjectivity spectrum of $T$.

Recall that $T \in \mathcal{B}(X)$ is called a Kato operator or semi-regular if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$. Denote by $\rho_K(T) : \rho_K(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is Kato} \}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C} \setminus \rho_K(T)$ the Kato spectrum of
It is well known that $\rho_K(T)$ is an open subset of $\mathbb{C}$ and may be decomposed in connected disjoint open nonempty components [1]. The Kato spectrum play an important role in local spectral theory, in particular, we have:

$$\partial \sigma(T) \subseteq \sigma_K(T) \subseteq \sigma_{su}(T) \cap \sigma_{ap}(T) \subseteq \sigma(T)$$

The concept of analytical core for an operator has been introduced by Vrbová in [15] and study by Mbekhta [9] [10] [11], that is the following set:

$$K(T) = \{ x \in X : \exists (x_n)_{n \geq 0} \subset X \text{ and } \delta > 0 \text{ such that } x_0 = x, \; T x_n = x_{n-1} \forall n \geq 1 \text{ and } \|x_n\| \leq \delta^n \|x\| \}$$

The quasi-nilpotent part of $T$, $H_0(T)$ is given by:

$$H_0(T) := \{ x \in X ; r_T(x) = 0 \} \text{ where } r_T(x) = \lim_{n \to +\infty} \|T^n x\|^\frac{1}{n}.$$

Next, let $T \in \mathcal{B}(X)$, $T$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of $\lambda_0$, the only analytic function $f : U \rightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. $T$ is said to have the SVEP if $T$ has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, then $T$ and $T^*$ have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. In particular, $T$ and $T^*$ have the SVEP at every isolated point of the spectrum. We have the implication [1]:

$$\sigma(T) \text{ does not cluster at } \lambda \implies T \text{ and } T^* \text{ have the SVEP at } \lambda$$

An operator $T \in \mathcal{B}(X)$ is said to be decomposable if for any open covering $U_1, U_2$ of the complex plane $\mathbb{C}$, there are two closed $T$-invariant subspaces $X_1$ and $X_2$ of $X$ such that $X_1 + X_2 = X$ and $\sigma(T|X_k) \subset U_k$, $k = 1, 2$. Note that $T$ is decomposable implies that $T$ and $T^*$ have the SVEP.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi Fredholm) if $dim N(T) < \infty$ and $R(T)$ is closed (resp, $codim R(T) < \infty$). $T$ is semi-Fredholm if is a lower or upper semi-Fredholm operator. The index of a semi Fredholm operator $T$ is defined by $ind(T) = dim N(T) - codim R(T)$

$T$ is a Fredholm operator if is a lower and upper semi-Fredholm operator, and is called a Weyl operator if it is Fredholm of index zero.

The essential and Weyl spectra of $T$ are closed and defined by:

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator} \}$$

$$\sigma_W(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator} \}$$
Let $T \in B(X)$, $T$ is said to be a Drazin invertible if there exists a positive integer $k$ and an operator $S \in B(X)$ such that

$$ST = TS, \ T^{k+1}S = T^k \text{ and } S^2T = S.$$ 

Which is also equivalent to the fact that $T = T_1 \oplus T_2$; where $T_1$ is invertible and $T_2$ is nilpotent. The concept of Drazin invertible operators has been generalized by Koliha [7]. In fact, $T \in B(X)$ is generalized Drazin invertible if and only if $0 \notin \text{acc}(\sigma(T))$ ( $\text{acc}(\sigma(T))$ is the set of all points of accumulation of $\sigma(T)$), which is also equivalent to the fact that $T = T_1 \oplus T_2$ where $T_1$ is invertible and $T_2$ is quasinilpotent. The generalized Drazin invertible spectrum defined by

$$\sigma_{gD}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin invertible} \}.$$ 

An operator $T \in B(X)$ is said to be B-Fredholm, if for some integer $n \geq 0$ the range $R(T^n)$ is closed and $T_n$, the restriction of $T$ to $R(T^n)$ is a Fredholm operator. This class of operators, introduced and studied by Berkani et al. in a series of papers which extends the class of semi-Fredholm operators. $T$ is said to be a B-Weyl operator if $T_n$ is a Fredholm operator of index zero. The B-Fredholm and B-Weyl spectra are defined by

$$\sigma_{BF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Fredholm} \};$$

$$\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl} \}.$$ 

Note that $T$ is a B-Fredholm operator if there exists two closed invariant subspaces $X_1$ and $X_2$ such that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, $T_1$ is Fredholm and $T_2$ is nilpotent [4, Theorem 2.7]. From [5, Lemma 4.1], $T$ is a B-Weyl operator if there exists two closed invariant subspaces $X_1$ and $X_2$ such that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, $T_1$ is Weyl and $T_2$ is nilpotent

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl [6], [16]. Precisely:

$T$ is a pseudo B-Fredholm operator if there exists two closed $T$-invariant subspaces $X_1$ and $X_2$ such that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, $T_1$ is a Fredholm operator and $T_2$ is a quasi-nilpotent operator. $T$ is said to be pseudo B-Weyl operator if there exists two closed $T$-invariant subspaces $X_1$ and $X_2$ such that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, $T_1$ is a Weyl operator and $T_2$ is a quasi-nilpotent operator. The pseudo B-Fredholm and pseudo B-Weyl spectra are defined by

$$\sigma_{pBF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Fredholm} \};$$

$$\sigma_{pBW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Weyl} \}.$$
It is easy to see that $T$ is a pseudo B-Fredholm (resp. pseudo B-Weyl) operator if and only if $T^*$ is pseudo B-Fredholm (resp. pseudo B-Weyl) operator. Hence $\sigma_{pBF}(T) = \sigma_{pBF}(T^*)$ and $\sigma_{pBW}(T) = \sigma_{pBW}(T^*)$.

$\sigma_{pBW}(T)$ and $\sigma_{pBF}(T)$ is not necessarily non empty. For example, the quasi nilpotent operator has empty pseudo B-Weyl and B-Fredholm spectra. Evidently $\sigma_{pBF}(T) \subset \sigma_{BW}(T) \subset \sigma_{W}(T) \subset \sigma(T)$. Then it is naturel to ask about the defect set $\sigma_{BW}(T) \setminus \sigma_{pBW}(T)$.

In this present paper, we show that the pseudo B-Fredholm spectrum $\sigma_{pBF}(T)$ is a compact set of $\mathbb{C}$ and the set $\sigma_{BW}(T) \setminus \sigma_{pBW}(T)$ is at most countable. Also, if $\lambda_0 \in \sigma(T)$, $T$ and $T^*$ have the SVEP at $\lambda_0$, we show that $\lambda_0 I - T$ is a pseudo B-Weyl operator if and only if $\lambda_0 I - T$ is a pseudo B-Fredholm operator. From this characterization, we explore sufficient conditions which ensures the equalities $\sigma_{pBF}(T) = \sigma_{gD}(T)$.

2. Pseudo B-Fredholm and Pseudo B-Weyl Spectra

Denote the open disc centered at $\lambda_0$ with radius $\varepsilon$ in $\mathbb{C}$ by $D(\lambda_0, \varepsilon)$ and

$$D^*(\lambda_0, \varepsilon) = D(\lambda_0, \varepsilon) \setminus \{\lambda_0\}.$$ The following Theorem establishes that if $T$ is a pseudo B-Fredholm operator, then $\lambda I - T$ is Fredholm in an open punctured neighborhood of $0$.

**Theorem 2.1.** Let $T \in B(X)$, a pseudo B-Fredholm operator, then there exists a constant $\varepsilon > 0$, such that $\lambda I - T$ is Fredholm for all $\lambda \in D^*(0, \varepsilon)$.

**Proof.** If $T$ is pseudo B-Fredholm, then there exists two closed $T$-invariant subspaces $X_1$ and $X_2$ such that $X = X_1 \oplus X_2$; $T_{|X_1}$ is Fredholm, $T_{|X_2}$ is quasi-nilpotent and $T = T_{|X_1} \oplus T_{|X_2}$.

If $X_1 = \{0\}$, $T$ is quasi nilpotent then $\lambda I - T$ is invertible for all $\lambda \neq 0$, hence $\lambda I - T$ is Fredholm for all $\lambda \neq 0$.

If $X_1 \neq \{0\}$ then $T_{|X_1}$ is Fredholm $\Rightarrow \exists \varepsilon > 0$ such that $(\lambda I - T)_{|X_1}$ is Fredholm for all $\lambda \in D(0, \varepsilon)$. As $T_{|X_2}$ is quasi-nilpotent $\Rightarrow \forall \lambda \neq 0$, $(\lambda I - T)_{|X_2}$ is invertible, then $(\lambda I - T)_{|X_2}$ is Fredholm for all $\lambda \in D^*(0, \varepsilon)$. Since $(\lambda I - T)_{|X_2}$ and $(\lambda I - T)_{|X_1}$ are Fredholm $\forall \lambda \in D^*(0, \varepsilon)$, we have $\lambda I - T$ is Fredholm $\forall \lambda \in D^*(0, \varepsilon)$.

Since $\sigma_W(T)$ is a closed set in $\mathbb{C}$, by the same argument, we can prove the following:
Theorem 2.2. Let $T \in \mathcal{B}(X)$, a pseudo B-Weyl operator, then there exists a constant $\varepsilon > 0$, such that $\lambda I - T$ is Weyl for all $\lambda \in D^*(0, \varepsilon)$.

As consequences of the Theorem 2.1 and Theorem 2.2, we have the following corollaries.

Corollary 2.1. Let $T \in \mathcal{B}(X)$, $\sigma_{pBF}(T)$ is a compact subset of $\mathbb{C}$. Moreover $\sigma_e(T) \setminus \sigma_{pBF}(T)$ consist of at most countably many isolated points.

Proof. We have $\sigma_{pBF}(T) \subset \sigma(T)$, by Theorem 2.1 $\mathbb{C} \setminus \sigma_{pBF}(T) = \rho_{pBF}(T)$ is open. Then $\sigma_{pBF}(T)$ is compact.

Furthermore, if $\lambda \in \sigma_e(T) \setminus \sigma_{pBF}(T)$, then $T - \lambda I$ is pseudo B-Fredholm. By Theorem 2.1, there exist $\varepsilon > 0$ such that, for all $\mu \in D^*(\lambda, \varepsilon)$, $T - \mu I$ is Fredholm. Hence $D^*(\lambda, \varepsilon) \subset \mathbb{C} \setminus \sigma_e(T)$, therefore $\lambda$ is an isolated point of $\sigma_e(T)$. It follows that $\sigma_e(T) \setminus \sigma_{pBF}(T)$ consists of at most countably many isolated points. \hfill $\square$

Corollary 2.2. Let $T \in \mathcal{B}(X)$, $\sigma_{pBW}(T)$ is a compact subset of $\mathbb{C}$. Moreover $\sigma_W(T) \setminus \sigma_{pBW}(T)$ consist of at most countably many isolated points.

Since $\sigma_{pBF}(T) \subset \sigma_{BF}(T) \subset \sigma_e(T)$ and $\sigma_{pBW}(T) \subset \sigma_{BW}(T) \subset \sigma_W(T)$, the following corollary hold.

Corollary 2.3. Let $T \in \mathcal{B}(X)$, $\sigma_{BW}(T) \setminus \sigma_{pBW}(T)$ and $\sigma_{BF}(T) \setminus \sigma_{pBF}(T)$ consist of at most countably many isolated points.

Lemma 2.1. [12] If $X = M \oplus N$, $T = T_1 \oplus T_2$, then $T$ is Kato if and only if $T_1$ and $T_2$ are Kato.

It is well known that if a bounded operator $T$ is a Fredholm operator, then there exists a $\gamma > 0$ for which $T - \lambda I$ is a Kato operator for all $\lambda \in D(0, \gamma)$ [8, Proposition 3.7.2]. In the following Theorem, we extend this result to the pseudo B-Fredholm operator.

Theorem 2.3. Let $T \in \mathcal{B}(X)$ a pseudo B-Fredholm operator. Then there exists a constant $\varepsilon > 0$ such that for all $\lambda \in D^*(0, \varepsilon)$, $\lambda I - T$ is a Kato operator.

Proof. If $T$ is pseudo B-Fredholm, then there exists two closed $T$-invariant subspaces $X_1, X_2 \subset X$ such that $X = X_1 \oplus X_2$; $T_{|X_1}$ is Fredholm, $T_{|X_2}$ is quasi-nilpotent and $T = T_{|X_1} \oplus T_{|X_2}$.

If $X_1 = \{0\}$, $T$ is quasi nilpotent, then for all $\lambda \neq 0$, $\lambda I - T$ is invertible, hence $\lambda I - T$ is Kato.

If $X_1 \neq \{0\}$ then $T_{|X_1}$ is Fredholm, by [8, Proposition 3.7.2] there exists $\varepsilon > 0$ such that $(\lambda I - T)_{|X_1}$ is a Kato operator for all $\lambda \in D^*(0, \varepsilon)$. 

As \( T_{iX_2} \) is quasi-nilpotent \( \Rightarrow \forall \lambda \neq 0 \ (\lambda I - T)_{iX_2} \) is invertible, then \((\lambda I - T)_{iX_2}\) is a Kato operator \( \forall \lambda \in D^*(0,\varepsilon) \). Since \((\lambda I - T)_{iX_2}\) and \((\lambda I - T)_{iX_1}\) are Kato operators \( \forall \lambda \in D^*(0,\varepsilon) \), by lemma 2.1, \( \lambda I - T \) is a Kato operator \( \forall \lambda \in D^*(0,\varepsilon) \).

\[\square\]

**Corollary 2.4.** Let \( T \in \mathcal{B}(X) \), then \( \sigma_K(T) \setminus \sigma_{pBF}(T) \) consist of at most countably many isolated points.

**Proof.** If \( \lambda \in \sigma_K(T) \setminus \sigma_{pBF}(T) \), then \( \lambda I - T \) is a pseudo B-Fredholm operator. According to Theorem 2.3, there exists \( \varepsilon > 0 \) such that \( \lambda I - T \) is a Kato operator for all \( \lambda \in D^*(0,\varepsilon) \). Therefore \( \lambda \) is an isolated point of \( \sigma_K(T) \). It follows that \( \sigma_K(T) \setminus \sigma_{pBF}(T) \) consists of at most countably many isolated points. \( \square \)

### 3. SVEP, Pseudo B-Fredholm and Pseudo B-Weyl Operators

The main result of this section is the following Theorem.

**Theorem 3.1.** Let \( T \in \mathcal{B}(X) \), \( \lambda_0 \in \sigma(T) \), \( T \) and \( T^* \) have the SVEP at \( \lambda_0 \). Then: \( \lambda_0 I - T \) is a pseudo B-Weyl operator if and only if \( \lambda_0 I - T \) is a pseudo B-Fredholm operator.

**Remark 1.** If \( \lambda_0 \notin \sigma(T) \), the result is clear.

To prove theorem, we need the following lemmas:

**Lemma 3.1.** Let \( T \in \mathcal{B}(X) \), suppose that \( T - \lambda_0 I \) is a pseudo B-Fredholm operator. Then the following statements are equivalent:

1. \( T \) and \( T^* \) have the SVEP at \( \lambda_0 \),
2. \( \sigma(T) \) does not cluster at \( \lambda_0 \).

**Proof.** Without loss of generality, we can assume that \( \lambda_0 = 0 \).

2) \( \Rightarrow \) 1) See [1].

1) \( \Rightarrow \) 2) Suppose that \( T \) is a pseudo B-Fredholm operator, then there exists two closed \( T \)-invariant subspaces \( X_1, X_2 \subset X \) such that \( X = X_1 \oplus X_2 \), \( T_{iX_1} \) is Fredholm, \( T_{iX_2} \) is quasi-nilpotent and \( T = T_{iX_1} \oplus T_{iX_2} \). Since \( T_{iX_1} \) is Fredholm, then \( T_{iX_1} \) is of Kato type, since \( T \) and \( T^* \) have the SVEP at \( \lambda_0 \) by [3, Theorem 2.2, Theorem 2.5] there exists a constant \( \varepsilon > 0 \) such that for all \( \lambda \in D^*(0,\varepsilon) \), \( (\lambda I - T)_{iX_1} \) is invertible. Since \( T_{iX_2} \) is quasi-nilpotent, \( (\lambda I - T)_{iX_2} \) is invertible.

\[\square\]
for all $\lambda \neq 0$. Hence $\lambda I - T$ is invertible for all $\lambda \in D^*(0, \varepsilon)$. Therefore $\sigma(T)$ does not cluster at $\lambda_0$. \hfill \Box

**Lemma 3.2.** Let $T \in \mathcal{B}(X)$, suppose that $T - \lambda_0 I$ is a pseudo $B$-Weyl operator. Then the following statements are equivalent:

1. $T$ or $T^*$ have the SVEP at $\lambda_0$,

2. $\sigma(T)$ does not cluster at $\lambda_0$.

**Proof.** Without loss of generality, we can assume that $\lambda_0 = 0$.

2) $\Rightarrow$ 1) [1].

1) $\Rightarrow$ 2) Let $T \in \mathcal{B}(X)$ such that $T$ or $T^*$ have the SVEP at 0, since $T$ is a pseudo $B$-Weyl operator, according to [16, corollary 2.8], $T$ is a generalized Drazin invertible, then there exists $X_1$, $X_2$ such that $X = X_1 \oplus X_2$, $T|_{X_1}$ is invertible and $T|_{X_2}$ is quasi-nilpotent. Hence there exists $\delta > 0$ such that, $(\lambda I - T)|_{X_1}$ is invertible for all $|\lambda| < \delta$. Since $T|_{X_2}$ is quasi nilpotent then $(\lambda I - T)|_{X_2}$ is invertible for all $\lambda \neq 0$. It follows that for all $\lambda \in D^*(0, \delta)$, $\lambda I - T$ is invertible, hence $D^*(0, \delta) \subset \rho(T)$, therefore $\sigma(T)$ does not cluster at 0. \hfill \Box

**Proof of Theorem 3.1.** Suppose that $\lambda_0 \in \sigma(T)$, $T$ and $T^*$ have the SVEP.

**Step 1:** $\lambda_0 I - T$ is a pseudo $B$-Fredholm operator if and only if $\lambda_0$ is an isolated point of $\sigma(T)$.

Assume that $\lambda_0 I - T$ is a pseudo $B$-Fredholm operator, since $T$ and $T^*$ have the SVEP at $\lambda_0$, then $\lambda_0$ is an isolated point of $\sigma(T)$ by lemma 3.1. Conversely, if $\lambda_0$ is an isolated point of $\sigma(T)$, then $X = H_0(T - \lambda_0 I) \oplus K(T - \lambda_0 I)$, [14, Theorem 4] $(T - \lambda_0 I)|_{H_0(T - \lambda_0 I)}$ is quasi nilpotent and $(T - \lambda_0 I)|_{K(T - \lambda_0 I)}$ is surjective, hence $(T - \lambda_0 I)|_{K(T - \lambda_0 I)}$ is Fredholm. Indeed, $\lambda_0$ is an isolated point, then $T$ has the SVEP at $\lambda_0$, hence $(T - \lambda_0 I)|_{K(T - \lambda_0 I)}$ has the SVEP at 0 and surjective, according to [1, corollary 2.24] $(T - \lambda_0)|_{K(T - \lambda_0 I)}$ is bijective.

**Step 2:** $\lambda_0 I - T$ is a pseudo $B$-Weyl operator if and only if $\lambda_0$ is an isolated point of $\sigma(T)$.

Suppose that $T - \lambda_0 I$ is a pseudo $B$-Weyl operator, since $T$ and $T^*$ have the SVEP at $\lambda_0$, by Lemma 3.2, $\sigma(T)$ does not cluster at $\lambda_0$. So $\lambda_0$ is an isolated point of $\sigma(T)$. The converse is similar to the converse of Step 1.

Now the result is clear. \hfill \Box

Since $T$ and $T^*$ have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum, the following corollary hold.
Corollary 3.1. Let $T \in \mathcal{B}(X)$, $\lambda_0 \in \partial(\sigma(T))$. Then: $\lambda_0 I - T$ is a pseudo $B$-Weyl operator if and only if $\lambda_0 I - T$ is a pseudo $B$-Fredholm operator.

As a straightforward consequence of the Theorem 3.1, we have the following corollary.

Corollary 3.2. Let $T \in \mathcal{B}(X)$, suppose that $T$ and $T^*$ have the SVEP. Then:

$$\sigma_{pBW}(T) = \sigma_{pBF}(T).$$

In particular, this holds if $T$ is decomposable.

Remark 2. We have $\sigma_{pBF}(T) \subset \sigma_{gD}(T)$, this inclusion is proper. Indeed:

Consider the operator $T$ defined in $l^2(\mathbb{N})$ by

$$T(x_1, x_2, ...) = (0, x_1, x_2, ...), \quad T^*(x_1, x_2, ...) = (x_2, x_3, ...).$$

Let $B = T \oplus T^*$. Then $\sigma_{gD}(B) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$ and we have $0 \notin \sigma_{pBF}(B)$.

This shows that the inclusion $\sigma_{pBF}(B) \subset \sigma_{gD}(B)$ is proper.

According to corollary 3.2 and [16, corollary 2.8], the following corollary hold.

Corollary 3.3. Let $T \in \mathcal{B}(X)$, suppose that $T$ and $T^*$ have the SVEP. Then:

$$\sigma_{gD}(T) = \sigma_{pBF}(T).$$

In particular, this holds if $T$ is decomposable.

Theorem 3.2. Let $T \in \mathcal{B}(X)$, for which $\sigma_e(T) = \partial(\sigma(T))$ and every $\lambda \in \partial(\sigma(T))$ is non-isolated in $\sigma(T)$. Then

$$\sigma_e(T) = \sigma_{BF}(T) = \sigma_{pBF}(T).$$

Proof. Let $\lambda \in \partial(\sigma(T))$ a non-isolated point of $\sigma(T)$ and suppose that $\lambda \notin \sigma_{pBF}(T)$, then $T - \lambda I$ is a pseudo $B$-Fredholm operator, since $\lambda \in \partial(\sigma(T))$, then $T$ and $T^*$ have the SVEP at $\lambda$. By lemma 3.1, $\sigma(T)$ does not cluster at $\lambda$. Hence, every non-isolated boundary point of $\sigma(T)$ belongs to $\sigma_{pBF}(T)$. Then

$$\sigma_e(T) = \partial(\sigma(T)) \subseteq \sigma_{pBF}(T) \subseteq \sigma_{BF}(T) \subseteq \sigma_e(T)$$

Example 1. Let $C_p$ the Cesaro operator on the classical Hardy space $H^p(\mathcal{D})$, where $\mathcal{D}$ the open unit disc of $\mathbb{C}$ and $1 \leq p < \infty$, is given by:
\[ C_p f(\lambda) := \frac{1}{\lambda} \int_0^\lambda \frac{f(\zeta)}{1 - \zeta} d\zeta \]

The spectrum of \( C_p \) is: \( \sigma(T) = \Gamma_p \) the closed disc centered at \( \frac{p}{2} \) with radius \( \frac{p}{2} \).

According to [8, Example 3.7.9], we have \( \sigma_e(C_p) = \partial \Gamma_p \), from Theorem 3.2

\[ \sigma_{pBF}(C_p) = \partial \Gamma_p \]

In addition, for arbitrary \( 1 < p < q < \infty \)

\[ \sigma_{pBF}(C_p) \cap \sigma_{pBF}(C_q) = \{0\} \]

**Example 2.** Let \( T \) is an unilateral weighted right shift on \( l^p(\mathbb{N}) \), \( 1 \leq p < \infty \), with weight sequence \( (w_n)_{n \in \mathbb{N}} \). If \( c(T) = \lim_{n \to \infty} \inf(w_1 ... w_n)^{1/n} = 0 \), then \( T \) and \( T^* \) have the SVEP, according to corollary 3.2, we have:

\[ \sigma_{pBW}(T) = \sigma_{pBF}(T) \]

By [1, corollary 3.118]: \( \sigma(T) = D(0, r(T)) \) where \( D(0, r(T)) \) is the closed disc centered at 0 with radius \( r(T) \), then \( iso(\sigma(T)) = \emptyset \). By Theorem 3.1, \( \sigma_{pBW}(T) = \sigma_{pBF}(T) = D(0, r(T)) \).

**References**


