International Journal of Pure and Applied Mathematics
Volume 108 No. 4 2016, 741-749
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v108i4.1
ijpam.eu

NEW TRAVELLING WAVE SOLUTIONS FOR NEW POTENTIAL NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper we apply the extended hyperbolic functions method, to solve the potential Padé-II equation and the potential Benjamin-Bona-Mahony (BBM) equation. Exact travelling wave solutions are obtained and expressed in terms of hyperbolic functions and trigonometric functions. The extended hyperbolic functions method, is the used method for solving other nonlinear evalution equations.

Key Words: extended hyperbolic functions method, exact solutions, potential Padé-II equation and the potential Benjamin-Bona-Mahony (BBM) equation

1. Introduction

In this paper we establish new travelling wave solutions to the potential Padé-II equation and the potential Benjamin-Bona-Mahony (BBM) equation given by

$$u_t(x,t) + u_x(x,t) + u_x^2(x,t) + \alpha u_{xxx}(x,t) + \beta u_{xxt}(x,t) = 0, \quad \alpha, \beta \in R \quad (1)$$

and

Received: May 2, 2016
Published: August 16, 2016

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$$u_t(x,t) + u_x(x,t) + \beta u_x^2(x,t) - \alpha u_{xxt}(x,t) = 0, \quad \alpha > 0, \beta \neq 0, \tag{2}$$

respectively.

As these above equations and their solutions are very important in the real world for many potential applications in physics and applied mathematics.

It is well known (see for example [1]) that the standard Padé-II equation has a form

$$u_t + u_x + uu_x - \frac{9}{10}u_{xxx} - \frac{19}{10}u_{xxt} = 0, u = u(x,t), \tag{3}$$

while the modified Padé-II equation has a form

$$u_t + u_x + u^2u_x - \frac{9}{10}u_{xxx} - \frac{19}{10}u_{xxt} = 0. \tag{4}$$

Also,it is well known [1] and [2] that standard Benjamin-Bona-Mahony (BBM) has a form

$$u_t + u_x + \beta uu_x - \alpha u_{xxt} = 0, \quad \beta > 0, \alpha \neq 0 \tag{5}$$

while the modified Benjamin-Bona-Mahony (mBBM) equation has a form

$$u_t + u_x + \beta u^2u_x - \alpha u_{xxt} = 0, \; u = u(x,t), \quad \beta > 0, \alpha \neq 0. \tag{6}$$

Finding exact solutions of nonlinear partial differential equations (PDE's) has become more and more attractive field in different branches of physics and applied mathematics subject due to the widespread of computer algebraic system (CAS), such as Maple and Mathematica.

CAS allows us to do tedious and lengthy manipulations. Moreover, CAS can help us find new exact solutions of nonlinear PDE's.

In order to get exact solutions directly, many powerful methods have been introduced such as the mapping method [3] and [4], the $\left(\frac{G'}{G}\right)$-expansion method [5], inverse scattering method [6] and [7], Hirota's bilinear method [8] and [9], the tanh method [10] and [11], the sine-cosine method [12] and [13], Backlund transformation method [14] and [15], the homogeneous balance [16] and [17], Darboux transformation [18], the Jacobi elliptic function expansion method [19], the first integral method [20], and the multiple simplest equation method [21].

The goal of this paper is to implement the extended hyperbolic functions method in [22] to obtain new exact travelling wave solutions of the potential Padé-II equation and the potential Benjamin-Bona-Mahony (BBM) equation.

2. Description of the Method

Given a nonlinear partial differential equation, for instance, in two variables, as follows

$$P(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \tag{7}$$

where P is in general a nonlinear function of its arguments, the subscripts denote the partial derivatives. Let $u(x,t) = u(\xi), \xi = \lambda\left(x - ct\right), c^2 = g\left(h + a\right),$ where λ is the wave number, c is the speed of the solitary wave, "a is the maximum amplitude above the water surface, h is the finite depth and g is the acceleration of gravity".

The solitary waves are therfore called gravity waves, then equation (7) reduces to a nonlinear ordinary differential equation (ODE)

$$Q(u, u', u'', ...) = 0. \tag{8}$$

We suppose that the solution of the ODE (8) is of the form

$$u(x,t) = u(\xi) = a_0 + \sum_{i=1}^{n} a_i \left(v(\xi)\right)^i + \sum_{j=1}^{n} b_j \left(v(\xi)\right)^{-j}. \tag{9}$$

Where the coefficients $a_0, a_i, b_j (i = 1, 2, ...n \; ; \; j = 1, 2, ...n)$ are constants to be determined and $v = v(\xi)$ satisfies a nonlinear ordinary differential equation of first order

$$v' = \frac{dv\left(\xi\right)}{d\xi} = a + bv^2\left(\xi\right), \quad a, b \neq 0 \in R. \tag{10}$$

The degree of the polynomial in equation (9) can be determined via balancing the highest order derivative terms and the nonlinear term in ODE. Substituting equation (9) into equation (8), using equation (10) repeatedly, and setting the coefficients of each order of v^i to zero, we obtain a set of nonlinear algebraic equations for $a_0, a_i, b_j (i = 1, 2, ...n \; ; \; j = 1, 2, ...n), a, b$ and c. With the aid of the computer program Maple we can solve the set of nonlinear algebraic equations and obtain all the constants $a_0, a_i, b_j (i = 1, 2, ...n \; ; \; j = 1, 2, ...n), a, b$ and c.

The solutions of equation (10), are given by:

1. $v(\xi) = \operatorname{sgn}(a)\sqrt{\frac{a}{b}}\tan\left(\sqrt{ab}\xi\right),\ ab > 0,$

2. $v(\xi) = -\operatorname{sgn}(a)\sqrt{\frac{a}{b}}\cot\left(\sqrt{ab}\xi\right), ab > 0,$

3. $v(\xi) = \operatorname{sgn}(a)\sqrt{-\frac{a}{b}}\tanh\left(\sqrt{-ab}\xi\right),\ ab < 0,$
4. $v(\xi) = \operatorname{sgn}(a)\sqrt{-\frac{a}{b}}\coth\left(\sqrt{-ab}\xi\right),\ ab < 0,$
5. $v(\xi) = -\frac{1}{b\xi}, a = 0, b > 0,$
6. $v(\xi) = a\xi,\ a \in R, b = 0.$

The multiple exact special solutions of nonlinear partial differential equation (7) are obtained by making use of equation (9) and general solutions of ODE (10).

3. Potential Padé-II Equation and Potential BBM Equation

3.1. Exact Solutions for Potential Padé-II Equation

In this section, we present our proposed equation, namely, a potential Padé-II equation as the form

$$u_t(x,t) + u_x(x,t) + u_x^2(x,t) + \alpha u_{xxx}(x,t) + \beta u_{xxt}(x,t) = 0, \tag{11}$$

where α and β are real numbers.

Now, we apply the spacial case of the mapping method (the extended hyperbolic function method), to solve our equation. Consequently we get the original solutions for our new equation, as the following

Substituting $u(x,t) = u(\xi)$, $\xi = \lambda(x - ct)$, in equation (11) we obtain

$$(1-c)\frac{d}{d\xi}u(\xi) + \lambda\left(\frac{d}{d\xi}u(\xi)\right)^2 + \lambda^2(\alpha - \beta c)\frac{d^3}{d\xi^3}u(\xi) = 0, \tag{12}$$

where $u(\xi), u'(\xi), u''(\xi) \to 0$ as $\xi \to \pm\infty$. Equation (12) is nonlinear ordinary differential equation.

Balancing the order of the nonlinear term $(u')^2$ with the highest derivative u''' gives $2(m+1) = m+3$, that gives $m = 1$. Thus, the solution of equation (12) has the form

$$u(\xi) = a_0 + a_1 v(\xi) + b_1 v^{-1}(\xi), \tag{13}$$

Substituting equation (13) in equation (12) and using equation (10), collecting the coefficients of each power of $v^i, 0 \le i \le 8$, setting each coefficient to zero, and solving the resulting system obtain the following sets of solutions

1. $a_0 = a_0, a_1 = 0, b_1 = 0, \lambda = \lambda,$

2. $a_0 = a_0, a_1 = 0, b_1 = \pm 3\sqrt{\frac{a}{b}(1-c)(\alpha - c\beta)}$, $\lambda = \pm\frac{1}{2}\sqrt{\frac{1-c}{ab(\alpha - c\beta)}}$.

Using equation (13), the solution of equation (10), and the above sets of solutions $[1-2]$, we get

$$u_1(x,t) = a_0 \qquad \text{(travelling solution)},$$

For $ab > 0$, we receive

$$u_2(x,t) = a_0 - 3\operatorname{sgn}(a)\sqrt{\frac{b}{a}}\sqrt{\frac{a}{b}(1-c)(\alpha - c\beta)}\tan\left(\frac{1}{2}\sqrt{ab}\sqrt{\frac{1-c}{ab(\alpha - c\beta)}}(x-ct)\right),$$

$$u_3(x,t) = a_0 + 3\operatorname{sgn}(a)\sqrt{\frac{b}{a}}\sqrt{\frac{a}{b}(1-c)(\alpha - c\beta)}\cot\left(\frac{1}{2}\sqrt{ab}\sqrt{\frac{1-c}{ab(\alpha - c\beta)}}(x-ct)\right),$$

For $ab < 0$, we obtain

$$u_4(x,t) = a_0 + 3\operatorname{sgn}(a)\sqrt{\frac{a}{b}(1-c)(\alpha - c\beta)}\sqrt{\frac{-b}{a}}\tanh\left(\frac{1}{2}\sqrt{-ab}\sqrt{\frac{1-c}{ab(\alpha - c\beta)}}(x-ct)\right),$$

$$u_5(x,t) = a_0 + 3\operatorname{sgn}(a)\sqrt{\frac{a}{b}(1-c)(\alpha - c\beta)}\sqrt{\frac{-b}{a}}\coth\left(\frac{1}{2}\sqrt{-ab}\sqrt{\frac{1-c}{ab(\alpha - c\beta)}}(x-ct)\right).$$

3.2. Exact Solutions for Potential BBM Equation

In this section, we present our proposed equation, namely, a potential BBM equation as the form

$$u_t(x,t) + u_x(x,t) + \beta u_x^2(x,t) - \alpha u_{xxt}(x,t) = 0, \alpha > 0, \beta \neq 0, \qquad (14)$$

in which $u(\xi), u'(\xi), u''(\xi) \to 0$ as $\xi \to \pm\infty$.

Now, we apply the extended hyperbolic function method, to solve our equation. Consequently we get the original solutions for our new equation, as the following Substituting $u(x,t) = u(\xi)$, $\xi = \lambda\,(x - ct)$, in equation (14) we get

$$(1-c)\,\frac{d}{d\xi}u\,(\xi) + \lambda\beta\left(\frac{d}{d\xi}u\,(\xi)\right)^2 + \lambda^2\,(\alpha c)\,\frac{d^3}{d\xi^3}u\,(\xi) = 0, \tag{15}$$

in which $u\,(\xi),\ u'\,(\xi),\ u''\,(\xi) \to 0$ as $\xi \to \pm\infty$.

Balancing the order of the nonlinear term $(u')^2$ with the highest derivative u''' gives $2\,(m+1) = m+3$ that gives $m = 1$. Thus, the solution of equation (15) has the form

$$u(\xi) = a_0 + a_1 v(\xi) + b_1 v^{-1}(\xi), \tag{16}$$

Substituting equation (16) in equation (15) and using equation (10), collecting the coefficients of each power of $v^i, 0 \le i \le 8$, setting each coefficient to zero, and solving the resulting system obtain the following sets of solutions

1. $a_0 = a_0, a_1 = 0, b_1 = 0, \lambda = \lambda,$
2. $a_0 = a_0, a_1 = \frac{-3\sqrt{\frac{(1-c)bc\alpha}{a}}}{2\beta}, b_1 = \frac{3\sqrt{\frac{(1-c)ac\alpha}{b}}}{2\beta},\ \lambda = \frac{1}{4}\sqrt{\frac{1-c}{abc\alpha}},$
3. $a_0 = a_0, a_1 = \frac{3\sqrt{\frac{(1-c)bc\alpha}{a}}}{2\beta}, b_1 = \frac{-3\sqrt{\frac{(1-c)ac\alpha}{b}}}{2\beta},\ \lambda = -\frac{1}{4}\sqrt{\frac{1-c}{abc\alpha}},$
4. $a_0 = a_0, a_1 = \pm\frac{3\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}, b_1 = 0,\ \lambda = \mp\frac{1}{2}\sqrt{\frac{1-c}{abc\alpha}},$
5. $a_0 = a_0, a_1 = 0, b_1 = \frac{\pm 3\sqrt{\frac{(1-c)ac\alpha}{b}}}{\beta},\ \lambda = \pm\frac{1}{2}\sqrt{\frac{1-c}{abc\alpha}}.$

Using equation (16), the solution of equation (10), and the above sets of solutions $[1-5]$, we get

$$u_1(x,t) = a_0 \qquad \text{(travelling solution)}.$$

For $ab > 0$, we get

$$u_2(x,t) = a_0 - \frac{3}{2}\mathrm{sgn}\,(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{\frac{a}{b}}\tan\left(\frac{1}{4}\sqrt{ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right)$$

$$+\frac{3}{2}\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{\frac{b}{a}}\cot\left(\frac{1}{4}\sqrt{ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right),$$

$$u_3(x,t)=a_0-3\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{\frac{a}{b}}\tan\left(\frac{1}{2}\sqrt{ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right),$$

$$u_4(x,t)=a_0-3\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{\frac{a}{b}}\cot\left(\frac{1}{2}\sqrt{ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right).$$

For $ab<0$, we obtain

$$u_5(x,t)=a_0+\frac{3}{2}\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{-\frac{a}{b}}\tanh\left(\frac{1}{4}\sqrt{-ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right)$$

$$+\frac{3}{2}\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{-\frac{b}{a}}\coth\left(\frac{1}{4}\sqrt{-ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right),$$

$$u_6(x,t)=a_0+3\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{-\frac{a}{b}}\tanh\left(\frac{1}{2}\sqrt{-ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right),$$

$$u_7(x,t)=a_0+3\operatorname{sgn}(a)\,\frac{\sqrt{\frac{(1-c)bc\alpha}{a}}}{\beta}\sqrt{-\frac{a}{b}}\coth\left(\frac{1}{2}\sqrt{-ab}\sqrt{\frac{1-c}{abc\alpha}}\,(x-ct)\right).$$

4. Conclusion

In this paper, we have analyzed the two new form of Padé-II equation and BBM equation. Therefore, we come out with the potential Padeé-II equation and potential BBM equation. Then, we give their solutions by using the extended hyperbolic function method.

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