NUMERICAL SOLUTION FOR VARIABLE-ORDER LINEAR CABLE EQUATIONS WITH SHIFTED SECOND KIND CHEBYSHEV POLYNOMIALS

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Abstract: In this paper, we apply second kind Chebyshev polynomials to solve variable-order linear cable equations. First, we convert second kind Chebyshev polynomials on the interval $[-1, 1]$ into $[0, R]$. Then we reduce variable-order linear cable equation to a set of algebraic equations by using second kind shifted Chebyshev polynomials and collocation method. To obtain an approximate solution of the linear cable equation, we should solve this algebraic system. The results demonstrate that the proposed method has high accuracy and effectiveness for solving the variable-order linear cable equations. The validity and effectiveness of the method are demonstrated by solving several numerical examples.

Key Words: variable-order linear cable equations, fractional PDE, Chebyshev polynomials, collocation method

1. Introduction

Fractional calculus is one of the important branches of applied mathematics which deals with derivatives and integrals of fractional order. Fractional calculus plays main role in Physics, Chemistry, Biology, Applied Sciences, and Engineering which has attracted the interest of many researchers in recent years (see for instance [1]).

Nowadays many mathematicians and scientists turn their attention to variable-order fractional, a new branch of calculus. In variable-order fractional calculus, the order of derivative and integral operator is a function. In fact,

fractional calculus can be considered as a part of variable-order fractional calculus; many researchers have devoted their efforts to work on the fractional order and the variable-order differential equations.

It is extremely hard to obtain an exact and analytical solution for the fractional differential and the variable-order differential equations, so many mathematicians use a lot of numerical methods to approximate solutions of these equations such as, The finite difference approximations [2], Homolocal method [3, 4], Adomian decomposition method [5, 6], Matrix operator method [7, 8, 9, 10], Homotopy method [11, 12], Collocation method, and also Maleki et. al. presented an adaptive pseudo-spectral method for solving a class of multivariate fractional boundary value problems [13].

However, until now, only few researchers have numerically analyzed the variable-order differential equations. The following are some works done on the variable-order differential equation: Coimbra proposed a stable approximate for solving the variable-order differential equation [14]. Line et al. [15] examined the stability and convergence of an approximation for the variable-order fractional equation. Chen et al. [16] analyzed some numerical methods with high accuracy for the variable-order electronic energy transfer equation. One of the fundamental equations is cable equation [17] which is widely used in modeling neuronal dynamics.

$$\frac{\partial u(x, t)}{\partial t} = D_t^{1-r_1} \frac{\partial^2 u(x, t)}{\partial x^2} - \mu D_t^{1-r_2} u(x, t) + f(x, t) \quad (1)$$

where $0 < r_1, r_2 < 1$ and $\mu > 0$ is a constant.

In this paper, we consider the variable-order linear cable equation in the form:

$$\frac{\partial u(x, t)}{\partial t} = D_t^{1-r_1(x, t)} \frac{\partial^2 u(x, t)}{\partial x^2} - \mu D_t^{1-r_2(x, t)} u(x, t) + f(x, t) \quad (2)$$

with the initial and the boundary conditions:

$$\begin{aligned} u(x, 0) &= g(x), & 0 \leq x \leq X, \\ u(0, t) &= \varphi(t), & 0 \leq t \leq T, \end{aligned}$$

where $0 < r_1(x, t), r_2(x, t) < 1$ and $\mu > 0$ is a constant. Here, $D_t^{1-r(x, t)} g(x)$ is Caputo fractional partial derivatives of order $1 - r(x, t)$.

In recent years, different kind of polynomials including: Taylor series [18, 19], Legendre polynomials [20, 21, 22], and Bernstein polynomials [22] have received considerable attention to obtain numerical solution of the differential equations, integral equations, differential integral equations, and fractional differential equations. Chebyshev polynomials are normal and orthogonal that

they are widely used for solving fractional differential equations. In this paper, we intend to use Chebyshev polynomials of the second kind for solving variable-order linear fractional equation. The rest of the paper is organized as follows. In Section 2, we introduce some necessary definitions. In Section 3, we present an analytical solution of the proposed method. In Section 4, several numerical examples are given to show the efficiency of the method. The last section is devoted to our concluding remarks.

2. Preliminaries

Definition 1. First kind of the Riemann-Liouville fractional integral of order $\alpha(t)$ is defined, as follows:

$$I_{a^+}^{\alpha(t)} u(t) = \frac{1}{\Gamma(\alpha(t))} \int_{a^+}^t (t - \tau)^{\alpha(t)-1} u(\tau) d\tau, \quad t > 0, \quad \operatorname{Re}(\alpha(t)) > 0, \quad (3)$$

Definition 2. First kind of the Riemann-Liouville fractional derivative of order $\alpha(t)$ is defined by:

$$D_{a^+}^{\alpha(t)} u(t) = \frac{1}{m - \Gamma(\alpha(t))} \frac{d^m}{dt^m} \int_a^t \frac{u(\tau)}{(t - \tau)^{\alpha(t)-m+1}} d\tau, \quad m - 1 \leq \alpha(t) < m, \quad (4)$$

Definition 3. First kind of the Caputo fractional derivative of order $\alpha(x, t)$ is defined by:

$$D_t^{\alpha(x,t)} u(x, t) = \frac{1}{1 - \Gamma(\alpha(x, t))} \int_0^t \frac{\frac{\partial u(x, \tau)}{\partial \tau}}{(t - \tau)^{\alpha(x,t)}} d\tau, \quad (5)$$

2.1. Chebyshev Polynomials of the Second Kind

We show Chebyshev polynomials of the second kind of degree n in the form of $u_n(x)$ and define them by:

$$\begin{cases} u_0(x) = 1, & u_1(x) = 2x, \\ u_{n+1}(x) = 2xu_n(x) - u_{n-1}(x), & n \geq 2, \end{cases} \quad (6)$$

2.2. Shifted Chebyshev Polynomials of the Second Kind

$u_n(x)$ is Chebyshev polynomials of the second kind of degree n on the interval $[-1, 1]$. Transferring these polynomials on the interval $[-1, 1]$ to $[0, R]$ leads to

obtain the following shifted Chebyshev polynomials of the second kind:

$$u_n^R(x) = u_n\left(\frac{2x-R}{R}\right) \quad (7)$$

Now, these polynomials are orthogonal with respect to the weight function $w^R(x) = \sqrt{1 - \frac{(x-R)^2}{R^2}}$ and we have:

$$\langle u_m^R(x), u_n^R(x) \rangle = \int_0^R u_m^R(x) u_n^R(x) w^R(x) dx = \frac{R\pi\delta_{mn}}{4}.$$

2.3. Approximate Solution of Functions of Two Variables using the Shifted Chebyshev

Let $f(x, t)$ be a function of two variables on the area $(x, t) \in [0, X] \times [0, T]$, In this case, we introduce an approximate solution of the function on this area, as follows:

$$f(x, t) \simeq \sum_{i=0}^M \sum_{j=0}^M c_{ij} u_i^R(x) u_j^T(t)$$

Where

$$c_{ij} = \frac{4}{X\pi} \frac{4}{T\pi} \int_0^T \int_0^R f(x, t) u_i^R(x) u_j^T(t) w^R(x) w^T(t) dx dt, \quad i, j = 0, 1, \dots, M.$$

3. Method of Solution

In this section, we present an approximate solution of the Eq. (2):

$$u(x, t) \simeq \tilde{u}(x, t) = \sum_{i=0}^M \sum_{j=0}^M a_{ij} u_i^X(x) u_j^T(t), \quad (8)$$

Substituting Eq. (8) into Eq. (2), we have:

$$\begin{cases} \frac{\partial \tilde{u}(x, t)}{\partial t} = D_t^{1-r_1(x, t)} \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2} - \mu D_t^{1-r_2(x, t)} \tilde{u}(x, t) + f(x, t), & (x, t) \in [0, X] \times [0, T], \\ \tilde{u}(x, 0) = g(x), & 0 \leq x \leq X, \\ \tilde{u}(0, t) = \varphi(t), & 0 \leq t \leq T, \end{cases} \quad (9)$$

therefore

$$\left\{ \begin{array}{l} \sum_{i=0}^M \sum_{j=0}^M a_{ij} u_i^X(x) \frac{d}{dt} u_j^T(t) = \sum_{i=0}^M \sum_{j=0}^M a_{ij} \frac{d^2}{dx^2} u_i^X(x) D_t^{1-r_1(x,t)} u_j^T(t) \\ \quad - \mu \sum_{i=0}^M \sum_{j=0}^M a_{ij} u_i^X(x) D_t^{1-r_2(x,t)} u_j^T(t) + f(x, t), \\ \sum_{i=0}^M \sum_{j=0}^M a_{ij} u_i^X(x) u_j^T(0) = g(x), \\ \sum_{i=0}^M \sum_{j=0}^M a_{ij} u_i^X(0) u_j^T(t) = \varphi(t), \end{array} \right. \quad (10)$$

In order to find the unknown coefficients a_{ij} and $i, j = 0, 1, \dots, M$. we satisfy the collocation points $(\theta_p, \eta_q) \in [0, X] \times [0, T]$ such that $p = 0, 1, \dots, M$ and $q = 0, 1, \dots, M$ into Eq. (10) where θ_p are the root of u_{M+1}^X and η_q are the root of u_{M+1}^T . In this case, Eq. (10) generates a set of $(M+1) \times (M+1)$ algebraic equations for $(M+1) \times (M+1)$ unknown coefficient a_{ij} , which can be solved for unknown coefficients with one of the existing methods. Substituting these coefficients into Eq. (8) leads to an approximate solution of Eq. (2).

4. Numerical Example

In this section, several examples are carried out to show the effectiveness of the proposed collocation method based on Chebyshev polynomials of the second kind. Then the obtained results are compared with the presented results in [22]. In order to obtain the high accuracy results at this step, we have performed the computation in MAPLE 16 with about 25 digits operations.

Example 1. We start with the following variable-order linear cable equation:

$$\left\{ \begin{array}{l} \frac{\partial u(x, t)}{\partial t} = D_t^{1-r_1(x,t)} \frac{\partial^2 u(x, t)}{\partial x^2} - 2D_t^{1-r_2(x,t)} u(x, t) + f(x, t), \\ u(x, 0) = x^3, \quad u(0, t) = t^2, \end{array} \right.$$

where $(x, t) \in [0, 2] \times [0, 3]$,

$$r_1(x, t) = \frac{1 + \sin(xt)}{3}, \quad r_2(x, t) = \frac{1 + \cos(xt)}{3},$$

Figure 1: The absolute error to Example 1.

$$f(x, t) = x + 2t + \frac{12t^{\frac{1+\cos(xt)}{3}}(6t + 4x + x \cos(xt))}{\Gamma\left(\frac{1+\cos(xt)}{3}\right)(9 + 10 \cos(xt) + \cos(2xt))}.$$

The exact solution is $u(x, t) = x^3 + t^2 + xt$. As seen in Fig. 1, plots the absolute error function $|u(x, t) - \tilde{u}(x, t)|$ at $(x, t) \in [0, 2] \times [0, 3]$ for $M = 3$.

Example 2. Consider the variable-order linear cable equation:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = D_t^{1-r_1(x, t)} \frac{\partial^2 u(x, t)}{\partial x^2} - 2D_t^{1-r_2(x, t)} u(x, t) + f(x, t), \\ u(x, 0) = 0, \quad u(0, t) = 0, \end{cases}$$

where $(x, t) \in [0, 2] \times [0, 3]$,

$$r_1(x, t) = \frac{x+t}{3}, \quad r_2(x, t) = \frac{1+x+t}{3}, \quad f(x, t) = x + \frac{6t^{\frac{1+x+t}{3}}x}{\Gamma\left(\frac{1+x+t}{3}\right)(1+x+t)}.$$

Figure 2: The absolute error to Example 2 and $M = 2$.

The exact solution is $u(x, t) = xt$. Fig. 2, plots the absolute error for $M = 2$. In addition, in Table 1, the absolute errors for $M = 4$, $t = 0.9$ and different values of x are compared with the presented results in [22]. As we can see, the results of the proposed method based on Chebyshev polynomials are more accurate than the presented results in [22] based on Bernstein polynomials.

Table 1: The absolute errors for $M = 4$, $t = 0.9$.

x	$M = 4$ [22]	Present Method
0.0	6.1062E-16	9.260E-21
0.4	1.5404E-15	4.080E-20
0.7	1.9984E-15	8.235E-20

5. Conclusion

In this paper, we convert the second kind Chebyshev polynomials on the interval $[-1, 1]$ into $[0, R]$. Then we generate a set of $(M + 1) \times (M + 1)$ algebraic equations for $(M + 1) \times (M + 1)$ unknown and present an approximate solution for solving the variable-order linear cable equation. Truly, the use of the Chebyshev polynomials leads to achieve high accuracy and low error corresponding with similar methods. Numerical examples demonstrate that the presented collocation method via the second kind Chebyshev polynomials on the area $[0, X] \times [0, T]$ is so appropriate for solving the fractional equations.

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