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# A NOTE ON WAVELET PACKETS

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**Abstract:** The construction of wavelet packets was studied by Chui et. al. in (SIAM J. Math. Anal. 24 (3) 1993, 712-738). In this paper, for any integer dilation factor $p \geq 2$, we construct a orthogonal $p$-wavelet packets based on the Walsh-Fourier transform.

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**Key Words:** $p$-wavelet packets, walsh-Fourier transform, multiresolution analysis

## 1. Introduction

The theory of wavelet packet analysis is an important generalization of wavelet analysis. Shen [7] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. The orthogonal multi wavelet packets [1]. Farkov [4] constructed orthogonal compactly supported $p$-wavelets on $\mathbb{R}_+$. Shah et al. provided a method for constructing wavelet packet on $p$-adic field in [6]. Recently, Chen and Chang [2] investigated the construction of a class of orthogonal vector-valued wavelet packets. Xiao-Feng et al. [5] gave the construction and characterization of vector-valued multivariate wavelet packets associated with dilation matrix by means of time-frequency analysis method, matrix theory and operator theory. In this work, we consider Walsh-Fourier transform and we shall investigate construction of $p$-wavelet packets with and

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integer dilation factor $p \geq 2$ and construct orthonormal wavelet packet bases of space $L^2(\mathbb{R}_+)$ based generalized Walsh functions.

## 2. Preliminaries and Notations

Analogue of all the definitions and properties in the following can also be found in [3, 4]. Denote by $[x]$ the integer part of $x$. For $x$ in $\mathbb{R}_+$ and any positive integer $j$, we set

$$x_j = [p^j x] \pmod p\,, \quad x_{-j} = [p^{1-j} x] \pmod p. \tag{2.1}$$

Consider on $\mathbb{R}_+$ the addition defined as follows: for $x$ and $y$ if $z = x \oplus y$ then

$$z = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with

$$\zeta_j = x_j + y_j \pmod p \quad (j \in \mathbb{Z})$$

where $\zeta_j \in \{0, 1, \cdots, p-1\}$ and $x_j$, $y_j$ are calculated by (2.1). As usual, the equality $z = x \ominus y$ means that $z \oplus y = x$.
For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & x \in [0, 1/p) \\ \varepsilon_p^l, & x \in [lp^{-1}, (l+1)p^{-1}) \quad (l = 1, \cdots, p-1) \end{cases}$$

where $\varepsilon_p = exp(2\pi i/p)$. The extension of the function $r_0$ to $\mathbb{R}_+$ is denoted by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}_+$. Then the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}_+\}$ are defined by

$$w_0(x) \equiv 1, \quad w_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^{\mu_j}$$

where

$$m = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, ..., p-1\}, \ \ \mu_k \neq 0.$$

For $x, w \in \mathbb{R}_+$, we set

$$\chi(x,w) = exp(\frac{2\pi i}{p}\sum_{j=1}^{\infty}(x_j w_{-j} + x_{-j} w_j)). \tag{2.2}$$

where $x_j, w_j$ are given by (2.1). The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}_+} f(x)\overline{\chi(x,\xi)}dx,$$

where $\chi(x,\xi)$ is given by (**??**). The properties of the Walsh-Fourier transform are similar to the Fourier transform. Particular, if $f \in L^2(\mathbb{R}_+)$, then $\hat{f} \in L^2(\mathbb{R}_+)$ and

$$\| \hat{f} \|_{L^2(\mathbb{R}_+)} = \| f \|_{L^2(\mathbb{R}_+)} .$$

Also, the inverse formula is the form

$$f(x) = \int_{\mathbb{R}_+} \hat{f}(x)\overline{\chi(x,\xi)}d\xi,$$

We recall that the definition of multiresolution $p$-analysis in $L^2(\mathbb{R}_+)$.

**Definition 2.1.** A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}_+)$ is called a vector-valued multiresolution $p$-analysis of $L^2(\mathbb{R}_+)$, if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;

(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}_+)$;

(iii) $f(\cdot) \in V_j$ if and only if $f(p\cdot) \in V_{j+1}, \forall j \in \mathbb{Z}$;

(iv) There exists a function $\varphi \in V_0$, called the scaling function, such that $\{\varphi_k(x) := \varphi(\cdot \ominus k) : \ k \in \mathbb{Z}_+\}$ constitute a orthogonal basis for $V_0$.

Since $\varphi \in V_{-1} \subset V_0$, there exist a sequence $\{P_k : k \in \mathbb{Z}_+\}$ with $\sum k \in \mathbb{Z}_+ |P_k|^2 < \infty$ such that

$$p^{-1}\varphi(xp^{-1}) = \sum_{k\in\mathbb{Z}_+} P_k \varphi(x \ominus k). \tag{2.3}$$

## 3. Orthogonal $p$-Wavelet Packets

Let $W_0$ be the orthogonal complement of $V_0$ to $V_1$, i.e., $V_1 = V_0 \oplus W_0$. If there exists $\{\psi_1, \cdots \psi_{p-1}\}$ such that $\{\psi_l(\cdot \ominus k) : 1 \le l \le p, k \in \mathbb{Z}_+\}$ forms an orthonormal basis for $W_0$ . We call $\{\psi_1, \cdots \psi_{p-1}\}$ an orthogonal p-wavelet associated with $\varphi(x)$.

From $p^{-1}\psi_l(\cdot/p) \in W_{-1} \subset V_0$, there exists a sequence $\{Q_k : k \in \mathbb{Z}_+\}$ with $\sum_{k\in\mathbb{Z}_+} |Q_k|^2 < \infty$ such that

$$p^{-1}\psi_l(xp^{-1}) = \sum_{k\in\mathbb{Z}_+} Q_k^l \psi(x \ominus k). \tag{3.1}$$

set

$$P(\xi) = \sum_{k\in\mathbb{Z}_+} P_k \overline{\chi(k,\xi)} \quad \text{and} \quad Q_l(\xi) = \sum_{k\in\mathbb{Z}_+} Q_k \overline{\chi(k,\xi)}. \tag{3.2}$$

By taking Walsh-Fourier transform, we get

$$\hat{\varphi}(\xi p) = P(\xi)\hat{\varphi}(\xi) \quad \text{and} \quad \hat{\psi}_l(\xi p) = Q_l(\xi)\hat{\varphi}(\xi) \tag{3.3}$$

$P(\xi)$ and $Q(\xi)$ are refinement mask and wavelet mask, respectively.

**Lemma 3.1.** *for $\varphi \in L^2(\mathbb{R}_+)$, $\{\varphi(\cdot \ominus k) : k \in \mathbb{Z}_+\}$ are orthonormal in $L^2(\mathbb{R}_+)$ if and only if*

$$\sum_{l\in\mathbb{Z}_+} |\hat{\varphi}(\xi + l)|^2 = 1 \ \ a.e\xi \in \mathbb{R}_+$$

*Proof.* We have

$$\begin{aligned} \int_{\mathbb{R}_+} \varphi(x)\overline{\varphi(x \ominus k)}dx &= \int_{\mathbb{R}_+} |\hat{\varphi}(\xi)|^2 \overline{\chi(k,\xi)}d\xi \\ &= \sum_{l=0}^{\infty} \int_l^{l+1} |\hat{\varphi}(\xi)|^2 \overline{\chi(k,\xi)}d\xi \\ &= \int_0^1 (\sum_{l=0}^{\infty} |\hat{\varphi}(\xi + l)|^2) \overline{\chi(k,\xi)}d\xi. \end{aligned}$$

Therefor, $\{\varphi(\cdot \ominus k) : k \in \mathbb{Z}_+\}$ are orthonormal if and only if $\sum_{l\in\mathbb{Z}_+} |\hat{\varphi}(\xi + l)|^2 = 1 \ \ a.e\xi \in \mathbb{R}_+$ □

Now let $P_k^i$ be a high-pass filter sequence associated to $\psi_i(x)$, $i = 1, \cdots p-1$. Denote $U_0(x) = \varphi(x)$, and $U_i(x) = \psi_i(x)$, $i = 1, \cdots p-1$. Then the two scaling equation can be written as

$$U_0(x) = \sum_{k \in \mathbb{Z}_+} P_k^0 U_0(px \ominus k) \text{ and } U_i(x) = \sum_{k \in \mathbb{Z}_+} P_k^i U_0(px \ominus k) \;\; i = 1, \cdots p-1$$

orthogonal p-wavelet packets with respect to $\varphi(x)$ are defined by

$$U_{pl+i}(x) = \sum_{k \in \mathbb{Z}_+} P_k^i U_l(px \ominus k). \tag{3.4}$$

where $i = 0, \cdots p-1$, $l = 0, 1, \cdots$. Note that (3.4) defines $U_n$ for all $n \geq 0$. Taking Walsh-Fourier transform in both side of (3.4), we get

$$(U_{pl+i})^{\wedge}(\xi) = Q_i(p^{-1}\xi)\hat{\omega}_l(p^{-1}\xi), \;\; i = 0, \cdots p-1. \tag{3.5}$$

The functions will be called the basic p-wavelet packets associated with multiresolution p-analysis. For $n = 0, 1, \cdots$, in order to describe the Fourier transform of $U_{pl+i}$, we need write $n$ in the form of $n = \sum_{j=0}^{k} \epsilon_j p^j$, $\epsilon_j \in \{0, 1, \cdots p-1\}$, $\epsilon_k \neq 0$, $k \in \mathbb{Z}_+$.

**Theorem 3.2.** *Let $\{U_n : n \geq 0\}$ be the basic p-wavelet packets constructed above and the unique expansion of $n$ be as the above. Then*

$$\hat{U}_n(\xi) = \prod_{j=0}^{k} Q_{\epsilon_j}(p^{-j}\xi)\hat{\varphi}(p^{-k}\xi) \tag{3.6}$$

*Proof.* For arbitrary non-negative integer $n$, we show it by induction. If the length of $n$ is equal 1, clearly (3.6) hold. Assume that it true for all n of length $k$. Then

$$\hat{U}_{\epsilon+pn} = Q_\epsilon(p^{-1\xi})\hat{U}_n(p^{-1\xi}) \tag{3.7}$$

$$= \prod_{j=0}^{k+1} Q_{\epsilon_j}(p^{-j}\xi)\hat{\varphi}(p^{-(k+1)}\xi). \tag{3.8}$$

□

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