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REFINED HYERS-ULAM STABILITY OF MODULE LEFT (m, n)-DERIVATIONS IN METRIC SPACES

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Abstract: In this paper, we consider the Hyers–Ulam stability of Cauchy type functional equations of module left (m, n)-derivations and generalized module left (m, n)-derivations in complete metric spaces.

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Key Words: Hyers–Ulam stability, module left (m, n)-derivations, generalized module left (m, n)-derivations

1. Introduction

The Hyers–Ulam stability problem was originated by S. M. Ulam [24] in 1940. In regard to a group homomorphism, S. M. Ulam asked the question saying how likely to an automorphism a function must behave in order to guarantee the existence of an automorphism near such functions.

Ulam's question was partially solved by D. H. Hyers [13] in the case of approximately additive functions and when the groups in the question are Banach spaces. D.G. Bourgin [5] and T. Aoki [2] treated this problem for approximate additive mappings controlled by unbounded functions. In [22], Th. M. Rassias provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. Finally, P. Găvruta [11] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions. During the last three

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decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings [1, 7, 8, 16, 21].

Let us define some basic definitions of derivations. Assume that A be an algebra over the real or complex field and M a left A-module. An additive mapping $\delta : A \to M$ is said to be module left derivation if $\delta(xy) = x \cdot \delta(y) + y \cdot \delta(x)$ holds for all $x, y \in A$. Here, $\cdot$ denotes the module multiplication on M. Let $m, n \geq 0$ with $m+n \neq 0$ be some fixed integers. Then an additive mapping $\delta : A \to M$ is called a module left (m, n)-derivation if

$$(m+n)\delta(xy) = 2mx \cdot \delta(y) + 2ny \cdot \delta(x)$$

for all $x, y \in A$. Furthermore, an additive mapping $g : A \to M$ is called a generalized module left derivation if there exists a module left derivation $\delta : A \to M$ such that $g(xy) = x \cdot g(y) + y \cdot \delta(x)$ for all $x, y \in A$. Also we define a generalized module left (m, n)-derivation is an additive mapping $g : A \to M$ for which there exists a module left (m, n)-derivation $\delta : A \to M$ such that

$$(m+n)g(xy) = 2mx \cdot g(y) + 2ny \cdot \delta(x)$$

for all $x, y \in A$. Obviously, if $m = n$, then every module left (m, n)-derivation and generalized module left (m, n)-derivation is a module left derivation and a generalized module left derivation respectively.

Many authors have produced the stability results of derivations [15, 17, 20, 23]. In particular, A. Fošner investigated the stability of module left (m, n)-derivations [9] and generalized module left (m, n)-derivations [10] by using a crucial stability result over metric spaces in [6]. Now, we introduce a stability result of generalized module left (m, n)-derivations [10] :

Theorem 1. *Let A be a normed algebra, M a Banach left A-module, and $F : A^2 \to [0, \infty)$ a function such that $F(2x, y) = \eta_1 F(x, y)$ and $F(x, 2y) = \eta_2 F(x, y)$ for some nonnegative scalars η_1, η_2 with $\eta_1\eta_2 < 1$. Suppose that $g : A \to M$ is a mapping for which there exists a mapping $d : A \to M$ such that*

$$\|g(\lambda x + y) - \lambda g(x) - g(y)\| \leq F(x, y),$$
$$\|d(\lambda x + y) - \lambda d(x) - d(y)\| \leq F(x, y),$$

and

$$\|(m+n)g(xy) - 2mx \cdot g(y) - 2ny \cdot d(x)\| \leq F(x, y),$$

$$\|(m+n)d(xy) - 2mx \cdot d(y) - 2ny \cdot d(x)\| \leq F(x,y),$$

for all $x, y \in A$ *and* $\lambda \in \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. *Then there exists a unique linear generalized module left* (m,n)*-derivation* $G : A \to M$ *such that*

$$\|g(x) - G(x)\| \leq \frac{F(x,x)}{1 - \eta_1\eta_2}$$

for all $x \in A$.

This paper aims to obtain the refined Hyers-Ulam stability of Cauchy functional inequalities associated with module left (m,n)-derivations and generalized module left (m,n)-derivations from algebras to complete metric left A-modules, respectively. As results, we obtain refined stability results of A. Fošner results [9, 10]. Furthermore, we investigate superstability of such module left (m,n)-derivations.

2. Hyers-Ulam Stability of Module Left (m,n)-Derivations

In this section, we assume that A is an algebra and M is a complete metric left A-module with metric d if A and M are not mentioned in theorem. We say that an additive mapping $f : A \to M$ is $\mathbb{C}$-linear (or just linear) if $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and all scalars $\lambda \in \mathbb{C}$. In the following, Λ denotes the set of all complex units, *i.e.,*

$$\Lambda = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

For a given mapping $f : A \to M$, Park [14] obtained the following result.

Lemma 2. *If* f *is additive and* $f(\lambda x) = \lambda f(x)$ *for all* $x \in A$ *and all* $\lambda \in \Lambda$, *then* f *is* $\mathbb{C}$*-linear.*

Now, we present a theorem which is a generalized stability of module left (m,n)-derivations from algebras to complete metric left A-modules.

Theorem 3. *Let* $F : A^2 \to [0,\infty)$ *be a function such that* $F(2x,y) \leq \eta_1 F(x,y)$ *and* $F(x,2y) \leq \eta_2 F(x,y)$ *for all* $x, y \in A$ *and the metric* d *satisfy* $d(\frac{a}{2}, \frac{b}{2}) \leq \xi d(a,b)$ *for all* $a, b \in M$ *and for some scalar* $\xi, \eta_1, \eta_2 \geq 0$ *with* $\xi\eta_1\eta_2, \xi^2\eta_1\eta_2 < 1$ *and either* $\eta_1 < 1$ *or* $\eta_2 < 1$. *Assume that* $f : A \to M$ *is a mapping such that*

$$d(f(\lambda x + \lambda y), \lambda f(x) + \lambda f(y)) \leq F(x,y) \tag{1}$$

and

$$d((m+n)f(xy), 2mx \cdot f(y) + 2ny \cdot f(x)) \le F(x,y) \tag{2}$$

for all $x, y \in A$ *and* $\lambda \in \Lambda$. *Then there exists a unique linear module left* (m,n)*-derivation* $h : A \to M$ *such that*

$$d(f(x), h(x)) \le \frac{\xi F(x,x)}{1 - \xi\eta_1\eta_2} \tag{3}$$

for all $x \in A$.

Proof. Let substitute x for y in (1) and $\lambda = 1$. Then one get the following inequality

$$d(f(2x), 2f(x)) \le F(x,x)$$

for all $x \in A$. Using the above inequality, for every $x \in A$ and $k \in \mathbb{N}$,

$$\begin{aligned} d(f(x), \frac{f(2^k x)}{2^k}) &\le \sum_{j=0}^{k-1} d(\frac{f(2^j x)}{2^j}, \frac{f(2^{j+1}x)}{2^{j+1}}) \\ &\le \sum_{j=0}^{k-1} \xi^{j+1} F(2^j x, 2^j x) \le \sum_{j=0}^{k-1} (\xi\eta_1\eta_2)^j \xi F(x,x). \end{aligned}$$

By the similar proof of Corollary 3.2 in [6], we conclude that there exists a unique additive mappping $h : A \to M$ such that (3) holds for all $x \in A$. Next, replacing y (or x) by 0 in (1), we get

$$d(f(\lambda x), \lambda f(x)) \le F(x,0) \le \eta_2^k F(x,0)$$
$$\big(d(f(\lambda y), \lambda f(y)) \le F(0,y) \le \eta_1^k F(0,y)\big)$$

for all $x \in A$ $(y \in A)$ and k $\in \mathbb{N}$. Consequently, $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and $\lambda \in \Lambda$. By Lemma 2, f is linear.

We would like to show that h is a module left (m,n)-derivation. Since h is additive, $h(x) = 2^{-k} h(2^k x)$. Therefore, according to (3), we have

$$d(2^{-k} f(2^k x), h(x)) \le \xi^k d(f(2^k x), h(2^k x)) \le (\xi\eta_1\eta_2)^k \frac{\xi F(x,x)}{1 - \xi\eta_1\eta_2}$$

for all $x \in A$ and $k \in \mathbb{N}$. Hence,

$$\lim_{k\to\infty} 2^{-k} f(2^k x) = h(x)$$

for all $x \in A$ and $k \in \mathbb{N}$. We can also get

$$d(2^{-2k} f(2^{2k}xy), h(xy)) \leq \frac{\xi^{2k}\xi F(2^{2k}xy, 2^{2k}xy)}{1-\xi\eta_1\eta_2} \leq \frac{(\xi\eta_1\eta_2)^{2k}\xi F(xy, xy)}{1-\xi\eta_1\eta_2}$$

for all $x, y \in A$ and $k \in \mathbb{N}$. Thus,

$$\lim_{k\to\infty} 2^{-2k} f(2^{2k}xy) = h(xy)$$

for all $x, y \in A$ and $k \in \mathbb{N}$. According to inequality (2), we have

$$\begin{aligned} &d((m+n)2^{-2k} f(2^{2k}xy), 2m2^{-k}x \cdot f(2^k y) + 2n2^{-k}y \cdot f(2^k x)) \\ &\leq \xi^{2k} d((m+n)f(2^{2k}xy), 2m2^k x \cdot f(2^k y) + 2n2^k y \cdot f(2^k x)) \\ &\leq \xi^{2k} F(2^k x, 2^k y) \leq (\xi^2\eta_1\eta_2)^k F(x, y) \end{aligned} \tag{4}$$

for all $x, y \in A$ and $k \in \mathbb{N}$. By taking $k \to \infty$ in (4), we have

$$(m+n)h(xy) = 2mx \cdot h(y) + 2ny \cdot h(x).$$

for all $x, y \in A$. The proof is completed. □

Theorem 4. *Let $F : A^2 \to [0, \infty)$ be a function such that $F(\frac{x}{2}, y) \leq \eta_1 F(x, y)$ and $F(x, \frac{y}{2}) \leq \eta_2 F(x, y)$ for all $x, y \in A$ and the metric d satisfy $d(2a, 2b) \leq \xi d(a, b)$ for all $a, b \in M$ and for some scalar $\xi, \eta_1, \eta_2 \geq 0$ with $\xi\eta_1\eta_2, \xi^2\eta_1\eta_2 < 1$ and either $\eta_1 < 1$ or $\eta_2 < 1$. Assume that $f, g : A \to M$ are mappings such that*

$$\begin{aligned} d(f(\lambda x + \lambda y), \lambda f(x) + \lambda f(y)) &\leq F(x, y), \\ d(g(\lambda x + \lambda y), \lambda g(x) + \lambda g(y)) &\leq F(x, y), \\ d((m+n)f(xy), 2mx \cdot f(y) + 2ny \cdot g(x)) &\leq F(x, y), \\ d((m+n)g(xy), 2mx \cdot g(y) + 2ny \cdot g(x)) &\leq F(x, y) \end{aligned} \tag{5}$$

for all $x, y \in A$. Then there exists a unique linear generalized module left (m, n)-derivation $h : A \to M$ such that

$$d(f(x), h(x)) \leq \frac{\xi F(x, x)}{1-\xi\eta_1\eta_2} \tag{6}$$

for all $x \in A$.

Proof. According to the same proof of Theorem 3, there exists a linear mapping $h : A \to M$ and a module left (m, n)-derivation $G : A \to M$ such that (6) holds and

$$d(g(x), G(x)) \leq \frac{\xi F(x,x)}{1 - \xi\eta_1\eta_2} \tag{7}$$

for all $x \in A$. By (7), we get

$$d(2^{-k}g(2^k x), G(x)) \leq \xi^k d(g(2^k x), G(2^k x)) \leq (\xi\eta_1\eta_2)^k \frac{\xi F(x,x)}{1 - \xi\eta_1\eta_2}$$

for all $x \in A$ and $k \in \mathbb{N}$. Therefore

$$\lim_{k\to\infty} 2^{-k}g(2^k x) = G(x)$$

for all $x \in A$. Using (5), we have

$$\begin{aligned} &d((m+n)2^{-2k}f((2^k x)(2^k y)), 2m2^{-k}x \cdot f(2^k y) + 2n2^{-k}y \cdot g(2^k x)) \\ &\leq \xi^{2k}F(2^k x, 2^k y) \leq (\xi^2\eta_1\eta_2)^k F(x,y). \end{aligned}$$

for all $x, y \in A$ and $k \in \mathbb{N}$. If we take $k \to \infty$ above inequality, then we conclude that $(m+n)h(xy) = 2mx \cdot h(y) + 2ny \cdot G(x)$. So the proof is completed. □

The following corollary is a stability of generalized module left (m, n)-derivations in normed spaces which is a improved result of Theorem 1.

Corollary 5. *Let A be a normed algebra, M be a Banach left module, and $F : A^2 \to [0, \infty)$ be a function such that $F(2x, y) \leq \eta_1 F(x, y)$ and $F(x, 2y) \leq \eta_2 F(x, y)$ for all $x, y \in A$ and for some scalar $\eta_1, \eta_2 \geq 0$ with $\eta_1\eta_2 < 2$ and either $\eta_1 < 1$ or $\eta_2 < 1$. Assume that $f, g : A \to M$ are mappings such that*

$$\begin{aligned} \|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| &\leq F(x,y), \\ \|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| &\leq F(x,y), \\ \|(m+n)f(xy) - 2mx \cdot f(y) - 2ny \cdot g(x)\| &\leq F(x,y), \\ \|(m+n)g(xy) - 2mx \cdot g(y) - 2ny \cdot g(x)\| &\leq F(x,y) \end{aligned}$$

for all $x, y \in A$ and $\lambda \in \Lambda$. Then there exists a unique linear generalized module left (m, n)-derivation $h : A \to M$ such that

$$\|f(x) - h(x)\| \leq \frac{F(x,x)}{2 - \eta_1\eta_2}$$

for all $x \in A$.

Theorem 6. *Let $F : A^2 \to [0, \infty)$ be a function such that $F(2x, y) \leq \eta_1 F(x, y)$ and $F(x, 2y) \leq \eta_2 F(x, y)$ for all $x, y \in A$ and the metric d satisfy $d(\frac{a}{2}, \frac{b}{2}) \leq \xi d(a, b)$ for all $a, b \in M$ for some scalar $\xi, \eta_1, \eta_2 \geq 0$ with $\xi\eta_1\eta_2, \xi^2\eta_1\eta_2 < 1$ and either $\eta_1 < 1$ or $\eta_2 < 1$. Assume that $f : A \to M$ is a mapping satisfying (1) and (2) for all $x, y \in A$. Then there exists a unique linear module left (m, n)-derivation $h : A \to M$ such that*

$$d(f(x), h(x)) \leq \frac{\eta_1\eta_2 F(x, x)}{1 - \xi\eta_1\eta_2} \tag{8}$$

for all $x \in A$.

Proof. Recall the following inequality

$$d(f(2x), 2f(x)) \leq F(x, x)$$

for all $x \in A$. We easily find the following inequality

$$d(f(x), 2f(\frac{x}{2})) \leq F(\frac{x}{2}, \frac{x}{2}) \leq \eta_1\eta_2 F(x, x)$$

for all $x \in A$. Using above inequality,

$$d(f(x), 2^n f(\frac{x}{2^n})) \leq \sum_{k=1}^{n} (\xi\eta_1\eta_2)^k F(x, x)$$

for all $x \in A$. By the same pattern of the proof of Theorem 3, we get (8) for all $x \in A$ and h is a linear module left (m, n)-derivation. $\square$

Theorem 7. *Let $F : A^2 \to [0, \infty)$ be a function such that $F(2x, y) \leq \eta_1 F(x, y)$ and $F(x, 2y) \leq \eta_2 F(x, y)$ for all $x, y \in A$ and the metric d satisfy $d(\frac{a}{2}, \frac{b}{2}) \leq \xi d(a, b)$ for all $a, b \in M$ and for some scalar $\xi, \eta_1, \eta_2 \geq 0$ with $\xi\eta_1\eta_2, \xi^2\eta_1\eta_2 < 1$ and either $\eta_1 < 1$ or $\eta_2 < 1$. Assume that $f, g : A \to M$ are mappings such that*

$$\begin{aligned} d(f(\lambda x + \lambda y), \lambda f(x) + \lambda f(y)) &\leq F(x, y), \\ d(g(\lambda x + \lambda y), \lambda g(x) + \lambda g(y)) &\leq F(x, y), \\ d((m + n)f(xy), 2mx \cdot f(y) + 2ny \cdot g(x)) &\leq F(x, y), \\ d((m + n)g(xy), 2mx \cdot g(y) + 2ny \cdot g(x)) &\leq F(x, y) \end{aligned}$$

for all $x, y \in A$. Then there exists a unique linear generalized module left (m, n)-derivation $h : A \to M$ such that

$$d(f(x), h(x)) \leq \frac{\eta_1\eta_2 F(x, x)}{1 - \xi\eta_1\eta_2}$$

for all $x \in A$.

Corollary 8. *Let A be a normed algebra, M be a Banach left module and $F : A^2 \to [0,\infty)$ be a function such that $F(\frac{x}{2}, y) \le \eta_1 F(x,y)$ and $F(x, \frac{y}{2}) \le \eta_2 F(x,y)$ for all $x, y \in A$ and for some scalar $\eta_1, \eta_2 \ge 0$ with $4\eta_1\eta_2 < 1$ and either $\eta < 1$ or $\eta_2 < 1$. Assume that $f, g : A \to M$ are mappings such that*

$$\begin{aligned} \|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| &\le F(x,y), \\ \|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| &\le F(x,y), \\ \|(m+n)f(xy) - 2mx \cdot f(y) - 2ny \cdot g(x)\| &\le F(x,y), \\ \|(m+n)g(xy) - 2mx \cdot g(y) - 2ny \cdot g(x)\| &\le F(x,y) \end{aligned}$$

for all $x, y \in A$ and $\lambda \in \Lambda$. Then there exists a unique linear generalized module left (m,n)-derivation $h : A \to M$ such that

$$\|f(x) - h(x)\| \le \frac{\eta_1\eta_2 F(x,x)}{1 - 2\eta_1\eta_2}$$

for all $x \in A$.

3. Superstability of Modulef Left (m,n)-Derivations

Let us introduce the concept of superstability. If a function f satisfying the equation ε approximately must actually be a solution of it, then we say that f is ε-superstable. The notion of superstability has appeared in connection with the investigation of stability of the exponential equation $f(x+y) = f(x)f(y)$. The first result for the superstability of this equation was proved by D.G. Bourgin [4]. Later, this problem was renewed and investigated by J. Baker, J. Lawrence, and J. Zorzitto [3]. Recently various functional equation's superstability has been investigated by many authors [12, 18, 19].

In this section, we will consider the superstability of module left (m,n)-derivaion and generalized module left (m,n)-derivation in metric spaces. Note that a translation invariant metric d is a metric satisfying $d(x,y) = d(x+a, y+a)$ for all $x, y, a \in A$ and if A has a unit element e such that $e \cdot x = x$ for all $x \in M$, then a left A-module M is called unitary.

Theorem 9. *Let A be an algebra with a unit e, M a unitary complete metric left A-module with translation invariant metric d, and $F : A^2 \to [0,\infty)$ a function such that $F(2x, y) \le \eta_1 F(x,y)$ and $F(x, 2y) \le \eta_2 F(x,y)$ and $d(\frac{a}{2}, \frac{b}{2}) = \xi d(a,b)$ and $d((m+n)a, (m+n)b) \le (m+n)d(a,b)$ for all $x, y \in A$ and $a, b \in M$ and for some scalar $\xi, \eta_1, \eta_2 \ge 0$ with $\xi\eta_1\eta_2, \xi^2\eta_1\eta_2, \xi\eta_1 < 1$ and either $\eta_1 < 1$*

or $\eta_2 < 1$. Assume that $f : A \to M$ is a mapping satisfying (1) and (2) for all $x, y \in A$. Then f is a linear module left (m,n)-derivation.

Proof. First, we will show that $f(2^l x) = 2^l f(x)$ for all $x \in X$ and $l \in \mathbb{N}$. Let $l \in \mathbb{N}$ be a fixed integer. By Theorem 3, we can check that there exists a module left (m,n)-derivation h such that (3) is true. Since h is additive, $h(2^l x) = 2^l h(x)$ for all $x \in A$. Also we have

$$\begin{aligned}
&d(h((m+n)(2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot f(2^k e))\\
&\leq d(2^l(m+n)h((2^k e)x), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot f(2^k e))\\
&\leq \xi^{-l} d((m+n)h(2^k ex), (m+n)f(2^k ex))\\
&\quad +\xi^{-l} d((m+n)f(2^k ex), 2m2^k e \cdot f(x) + 2nx \cdot f(2^k e))\\
&\leq \xi^{-l} d((m+n)h(2^k ex), (m+n)f(2^k ex))\\
&\quad +\xi^{-l} d((m+n)f(2^k ex), 2m2^k e \cdot f(x) + 2nx \cdot f(2^k e))\\
&\leq \xi^{-l}(m+n)\frac{\xi F(2^k ex, 2^k ex)}{1-\xi\eta_1\eta_2} + \xi^{-l} F(2^k e, x)\\
&\leq \xi^{-l}(m+n)\frac{\xi(\eta_1\eta_2)^k F(x,x)}{1-\xi\eta_1\eta_2} + \xi^{-l}\eta_1^k F(e,x)
\end{aligned}$$

for all $x \in A$ and $k \in \mathbb{N}$. Using above inequality and (3), we obtain

$$\begin{aligned}
&d((m+n)f((2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x f(2^k e))\\
&\leq d((m+n)f((2^k e)(2^l x), (m+n)h((2^k e)(2^l x))\\
&\quad +d((m+n)h((2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot f(2^k x)\\
&\leq (m+n)\frac{\xi F((2^k e)(2^l x), (2^k e)(2^l x))}{1-\xi\eta_1\eta_2} + (m+n)\xi^{-l}\frac{\xi(\eta_1\eta_2)^k F(x,x)}{1-\xi\eta_1\eta_2}\\
&\quad +\xi^{-l}\eta_1^k F(e,x)\\
&\leq (m+n)\frac{\xi(\eta_1\eta_2)^k F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2} + (m+n)\xi^{-l}\frac{\xi(\eta_1\eta_2)^k F(x,x)}{1-\xi\eta_1\eta_2}\\
&\quad +\xi^{-l}\eta_1^k F(e,x)
\end{aligned}$$

for all $x \in A$ and $k \in \mathbb{N}$. This yields

$$\begin{aligned}
&d(2m2^k f(2^l x), 2m2^k 2^l f(x)) = d(2m2^k e \cdot f(2^l x), 2m2^k e \cdot 2^l f(x))\\
&\leq d(2m2^k e \cdot f(2^l x) + 2n2^l x \cdot f(2^k e), (m+n)f((2^k e)(2^l x)))\\
&\quad +d((m+n)f((2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot f(2^k e))\\
&\leq F(2^k e, 2^l x) + (m+n)\frac{(\xi\eta_1\eta_2)^k \xi F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2}
\end{aligned}$$

$$
\begin{aligned}
&+(m+n)\xi^{-l}\frac{(\xi\eta_1\eta_2)^k\xi F(x,x)}{1-\xi\eta_1\eta_2}+\xi^{-l}\eta_1^kF(e,x)\\
&\leq \eta_1^kF(e,2^lx)+(m+n)\frac{(\xi\eta_1\eta_2)^k\xi F(2^lx,2^lx)}{1-\xi\eta_1\eta_2}\\
&+(m+n)\xi^{-l}\frac{(\xi\eta_1\eta_2)^k\xi F(x,x)}{1-\xi\eta_1\eta_2}+\xi^{-l}\eta_1^kF(e,x)
\end{aligned}
$$

for all $x \in A$ and $k \in \mathbb{N}$. It follows that

$$
\begin{aligned}
d(mf(2^lx), m2^lf(x)) \quad &\leq \quad \xi^{k+1}(\eta_1^kF(e,2^lx)+(m+n)\frac{(\xi\eta_1\eta_2)^k\xi F(2^lx,2^lx)}{1-\xi\eta}\\
&+(m+n)\xi^{-l}\frac{(\xi\eta_1\eta_2)^k\xi F(x,x)}{1-\xi\eta_1\eta_2}+\xi^{-l}\eta_1^kF(e,x))
\end{aligned}
$$

for all $x \in A$ and $k \in \mathbb{N}$. As letting $k \to \infty$, we can conclude that $f(2^lx) = 2^lf(x)$.

Next we claim that $f = h$. We obtain the following inequality

$$
\begin{aligned}
d(f(x), h(x)) \quad &= \quad d(2^{-l}f(2^lx), 2^{-l}h(2^lx))\\
&\leq \quad \xi^{l+1}\frac{F(2^lx,2^lx)}{1-\xi\eta_1\eta_2} = (\xi\eta_1\eta_2)^l\xi\frac{F(x,x)}{1-\xi\eta_1\eta_2}
\end{aligned}
$$

for all $x \in A$ and $l \in \mathbb{N}$. Thus, if we take $l \to \infty$, we can conclude that $f = h$. In other words, f is a module left (m,n)-derivation on A. □

Theorem 10. *Let A be an algebra with a unit e, M a unitary complete metric left A-module with translation invariant metric d, and $F : A^2 \to [0,\infty)$ a function such that $F(2x,y) \leq \eta_1F(x,y)$ and $F(x,2y) \leq \eta_2F(x,y)$ and $d(\frac{a}{2},\frac{b}{2}) = \xi d(a,b)$ and $d((m+n)a,(m+n)b) \leq (m+n)d(a,b)$ for all $x,y \in A$ and $a,b \in M$ and for some scalar $\xi,\eta_1,\eta_2 \geq 0$ with $\xi\eta_1\eta_2, \xi^2\eta_1\eta_2, \xi\eta_1 < 1$ and either $\eta_1 < 1$ or $\eta_2 < 1$. Assume that $f,g : A \to M$ are mappings such that*

$$
\begin{aligned}
d(f(\lambda x+\lambda y)-\lambda f(x)-\lambda f(y)) &\leq F(x,y),\\
d(g(\lambda x+\lambda y)-\lambda g(x)-\lambda g(y)) &\leq F(x,y),\\
d((m+n)f(xy), 2mx\cdot f(y)+2ny\cdot g(x)) &\leq F(x,y),\\
d((m+n)g(xy), 2mx\cdot g(y)+2ny\cdot g(x)) &\leq F(x,y)
\end{aligned}
$$

for all $x,y \in A$. Then f is a linear generalized module left (m,n)-derivation.

Proof. By Theorem 9, g is a linear module left (m,n)-derivation. We would like to prove that $f(2^lx) = 2^lf(x)$ for all $x \in A$ and $l \in \mathbb{N}$. Let l be a fixed integer. Suppose that $h : A \to M$ is a module left (m,n)-derivation from the proof of Theorem 3. Recall that h is additive and therefore $h(2^l x) = 2^l h(x)$ for all $x \in A$. Moreover h satisfies (6). By using (8), we have

$$\begin{aligned}
&d((m+n)h((2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot g(2^k e))\\
&\le d((m+n)2^l h(2^k ex), (m+n)2^l f(2^k ex))\\
&\quad + d((m+n)2^l f(2^k ex), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot g(2^k e))\\
&\le \xi^{-l}(m+n)\frac{\xi F(2^k ex, 2^k ex)}{1-\xi\eta_1\eta_2} + \xi^{-l} F(2^k e, x)\\
&\le \xi^{-l}(m+n)(\eta_1\eta_2)^k \frac{F(x,x)}{1-\xi\eta_1\eta_2} + \xi^{-l}\eta_1^k F(e,x).
\end{aligned}$$

for all $x \in A$ and $k \in \mathbb{N}$. Thus,

$$\begin{aligned}
&d((m+n)f((2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot g(2^k e))\\
&\le d((m+n)f((2^k e)(2^l x)), (m+n)h((2^k e)(2^l x)))\\
&\quad + d((m+n)h((2^k e)(2^l x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot g(2^k e))\\
&\le (m+n)\xi\frac{F(2^k 2^l x, 2^k 2^l x)}{1-\xi\eta_1\eta_2} + \xi^{-l}(m+n)(\eta_1\eta_2)^k \frac{F(x,x)}{1-\xi\eta_1\eta_2} + \xi^{-l}\eta_1^k F(e,x).\\
&\le (m+n)\xi(\eta_1\eta_2)^k\frac{F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2} + \xi^{-l}(m+n)(\eta_1\eta_2)^k \frac{F(x,x)}{1-\xi\eta_1\eta_2} + \xi^{-l}\eta_1^k F(e,x).
\end{aligned}$$

for all $x \in A$ and $k \in \mathbb{N}$. This yields that

$$\begin{aligned}
&d(2m2^k f(2^l x), 2m2^k 2^l f(x)) = d(2m2^k e \cdot f(2^l x), 2m2^k e \cdot 2^l f(x))\\
&\le d(2m2^k e \cdot f(2^l x) + 2n2^l x \cdot g(2^k e), (m+n)f((2^k e)(2^l x)))\\
&\quad + d((m+n)f((2^k e)(2^k x)), 2m2^l 2^k e \cdot f(x) + 2n2^l x \cdot g(2^k e))\\
&\le F(2^k e, 2^l x) + (m+n)\xi(\eta_1\eta_2)^k \frac{F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2}\\
&\quad + \xi^{-l}(m+n)(\eta_1\eta_2)^k \frac{F(x,x)}{1-\xi\eta_1\eta_2} + \xi^{-l}\eta_1^k F(e,x)\\
&\le \eta_1^k F(e, 2^l x) + (m+n)\xi(\eta_1\eta_2)^k \frac{F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2}\\
&\quad + \xi^{-l}(m+n)(\eta_1\eta_2)^k \frac{F(x,x)}{1-\xi\eta_1\eta_2} + \xi^{-l}\eta_1^k F(e,x)
\end{aligned}$$

for all $x \in A$ and $k \in \mathbb{N}$. Therefore,

$$d(2mg(2^l x), 2m2^l g(x)) \quad \le \quad \xi^k(\eta_1^k F(e, 2^l x) + (m+n)\xi(\eta_1\eta_2)^k \frac{F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2}$$

$$+\xi^{-l}(m+n)(\eta_1\eta_2)^k\frac{F(x,x)}{1-\xi\eta_1\eta_2}+\xi^{-l}\eta_1^k F(e,x))$$

for all $x \in A$ and $k \in \mathbb{N}$. Letting $k \to \infty$, we get $g(2^l x) = 2^l g(x)$ for all $x \in A$.

Finally, we would remain to prove that $f = h$. Using (6), we obtain

$$\begin{aligned} d(f(x), h(x)) &= d(2^{-l}f(2^l x), 2^{-l}h(2^l x)) \\ &\leq \xi^l \frac{\xi F(2^l x, 2^l x)}{1-\xi\eta_1\eta_2} = (\xi\eta_1\eta_2)^l \frac{\xi F(x,x)}{1-\xi\eta_1\eta_2} \end{aligned}$$

for all $x \in A$ and $l \in \mathbb{N}$. As $l \to \infty$, we conclude that $f = h$. Therefore, f is a generalized module left (m,n)-derivation on A. This completes the proof. □

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