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$\{T, S\}$ SPLITTINGS OF RECTANGULAR MATRICES REVISITED

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Abstract: In this short note, $\{T, S\}$ splittings of rectangular matrices are considered and derived certain convergence and comparison results.

These results involve outer inverses (or $\{2\}$-inverses) with prescribed range and null space, of matrices emerge from the $\{T, S\}$ splittings.

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Key Words: $\{2\}$-inverse, $\{T, S\}$ splitting, convergence theorems, comparison theorems

1. Introduction

Consider a linear system

$$Ax = b, \tag{1}$$

where A is a nonsingular real matrix. In order to solve such a system, iterative methods of the form

$$x^{i+1} = Hx^i + c \tag{2}$$

are often employed. The iterative scheme (2) is obtained by decomposing A into the form $A = U - V$, where U, V are real square matrices and U is nonsingular, and then setting $H = U^{-1}V$ and $c = U^{-1}b$. It is well known that the iterative scheme (2) converges to the solution of $Ax = b$ (irrespective of the choice of initial vector x°) if and only if the spectral radius $\rho(H)$ (i.e. the maximum of

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the moduli of the eigenvalues of H) of an *iteration matrix* H is less than one. Standard iterative methods like the Guass-Jacobi, Guass-Seidal and successive over-relaxation methods arise from different choices of U and V. For more details one could refer to the books (see [3], [10]). A splitting $A = U - V$ is called a *regular splitting* if U is invertible, $U^{-1} \geq 0$ and $V \geq 0$, where the comparison is entrywise and 0 is the null matrix. Varga (see [10]) and several others introduced the notion of regular splitting and obtained the conditions under which $\rho(U^{-1}V) < 1$ for any regular splitting $A = U - V$.

Berman and Plemmons (see [2]) extended the concept of splittings to rectangular matrices by introducing a matrix decomposition namely proper splitting and derived the conditions under which $\rho(U^{\dagger}V) < 1$ where $U^{\dagger}$ denotes the Moore-Penrose inverse (see [1] for the definition) of U, for any proper splitting $A = U - V$ of a real rectangular matrix A. Analogous to the invertible case, they have shown (see [2, Corollary 1]) that for any proper splitting $A = U - V$ the iterative scheme $x^{i+1} = Hx^i + c$, where $H = U^{\dagger}V$ and $c = U^{\dagger}b$, converges to $x = A^{\dagger}b$ for any initial vector x° if and only if $\rho(U^{\dagger}V) < 1$.

If the matrix A has two decompositions or splittings then the comparison of the spectral radius of the corresponding iteration matrices, is an important problem in analyzing the iterative schemes of type (2). The comparison of asymptotic rates of convergence of the iterative matrix induced by two splittings of a given matrix has been studied by many authors; for example Varga (see [10]), Elsner, Song, to name a few. Elsner (see [5]) considered weak regular splittings and multisplittings, proved comparison results. Song (see [9]) studied a comparison theorem for a nonnegative splittings and then applied it to study different basic iterative methods.

Recently, Mishra and Sivakumar (see [8]) considered a subclass of proper splitting of matrices and derived few comparison theorems. Also, in another paper Mishra (see [7]) proposed the extension of the nonnegative splitting for rectangular matrices called proper nonnegative splitting and established different convergence and comparison theorems. Jena et al. (see [6]) obtained several convergence and comparison theorems for proper regular splittings and proper weak regular splittings of rectangular matrices.

The main purpose of this article is to prove certain convergence theorems and comparison theorems for different iterative schemes arising out of a decomposition (called $\{T, S\}$ splitting) introduced by Djordjević and Stanimirović (see [4]). There the authors used $\{T, S\}$ splitting to discuss properties and representations of generalized inverses as well as iterative methods for computing various solutions of singular linear system $Ax = b$. The main results of this paper generalize some of the results of Mishra (see [7]) and Jena et al. (see

[6]) for $\{T, S\}$ splittings of rectangular matrices. In Section 2, notations and preliminary results are introduced and in Section 3 main results are proved.

2. Notations, Definitions and Preliminaries

Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^{m\times n}$ denote the set of all real matrices of order $m \times n$. For $A \in \mathbb{R}^{m\times n}$ the matrix $X \in \mathbb{R}^{n\times m}$ satisfying the equation $XAX = X$ is called $\{2\}$-inverse (or outer inverse) of A. It always exists and is denoted by $A^{(2)}$. Let T and S be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively then for $A \in \mathbb{R}^{m\times n}$, the matrix $A^{(2)}_{T,S} \in \mathbb{R}^{n\times m}$ denotes $\{2\}$-inverse with range T and null space S. It is well known that $\{2\}$-inverses have wide range of applications, for example, in the iterative schemes for solving the nonlinear differential equations and in the applications to statistics. We refer the reader to books (see [1], [3]) for details of these results.

The following theorem guarantees the existence and uniqueness of $A^{(2)}_{T,S}$ for a given matrix $A \in \mathbb{R}^{m\times n}$.

Theorem 1. *(Theorem 14, [1]) Let $A \in \mathbb{R}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{R}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{R}^m$ of dimension $m - s$. Then $A^{(2)}_{T,S}$ exists and unique if and only if $AT \oplus S = \mathbb{R}^m$, where $AT \oplus S$ denotes the direct sum of subspaces AT and S of $\mathbb{R}^m$.*

Note that well known generalized inverses namely the Moore-Penrose inverse, the weighted Moore-Penrose inverse, the Drazin inverse, the Group inverse, the Bott-Duffin inverse and the generalized Bott-Duffin inverse are $A^{(2)}_{T,S}$ generalized inverses for suitable choice of T and S. For definitions and details of these results, we refer the reader to (see [1]).

We now turn our attention to the notion of $\{T, S\}$ splitting. As we mentioned earlier, this notion was first introduced by Djordjević and Stanimirović (see [4]). Its definition is the following.

Definition 2. (Definition 2.1, [4]) Let $A \in \mathbb{R}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{R}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{R}^m$ of dimension $m - s$. Then the splitting $A = U - V$ is called the $\{T, S\}$ splitting of A if $UT \oplus S = \mathbb{R}^m$.

Let $R(A)$ and $N(A)$ denote the range space, and null space of $A \in \mathbb{R}^{m\times n}$, respectively. The index of a square matrix A is denoted by $ind(A)$ and defined as the smallest positive integer k such that $rank(A^{k+1}) = rank(A^k)$. Let $A \in \mathbb{R}^{n\times n}$ with $k = ind(A)$. Then the splitting $A = U - V$ is called an index splitting of

A if $R(U) = R(A^k)$ and $N(U) = N(A^k)$. Note that $\{T, S\}$ splitting reduces to index splitting if $m = n$, $T = R(U) = R(A^k)$ and $S = N(U) = N(A^k)$, where $k \geq ind(A)$. On the other hand, $\{T, S\}$ splitting reduces to proper splitting if $T = R(U^t) = R(A^t)$ and $S = N(U^t) = N(A^t)$, where A^t denotes the transpose of A. We refer the reader to (see [3]) for the detailed study of these splittings.

Next, we present a result which characterize the generalized inverse $A^{(2)}_{T,S}$ and that will be used in proving main results of this article.

Theorem 3. *(Theorem 2.1, [4]) Let $A \in \mathbb{R}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{R}^n$ of dimension $s \leq r$ and let S be a subspace of $\mathbb{R}^m$ of dimension $m - s$, such that $AT \oplus S = \mathbb{R}^m$. Assume that $A = U - V$ is a $\{T, S\}$ splitting of A and $dim(T) \leq rank(U)$. Then the generalized inverse $A^{(2)}_{T,S}$ satisfies the following conditions:*
(i) $U^{(2)}_{T,S} - A^{(2)}_{T,S} = -U^{(2)}_{T,S} V A^{(2)}_{T,S} = -A^{(2)}_{T,S} V U^{(2)}_{T,S}$,
(ii) $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} = U^{(2)}_{T,S} (I - V U^{(2)}_{T,S})^{-1}$ *and*
(iii) $U^{(2)}_{T,S} = (I + A^{(2)}_{T,S} V)^{-1} A^{(2)}_{T,S} = A^{(2)}_{T,S} (I + V A^{(2)}_{T,S})^{-1}$.

As we mentioned in the introduction, the spectral radius of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$, where λ_1, λ_2,...,λ_n are the eigenvalues of A. We say that a real matrix A is nonnegative, if it is entrywise nonnegative, and we write this as $A \geq 0$. Then same notation and nomenclature are also used for vectors. If A and B are real matrices, we write $B \geq A$ if $B - A \geq 0$.

If $A = U - V$ is a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, then the following result gives equivalent conditions for $\rho(U^{(2)}_{T,S} V) < 1$.

Theorem 4. *(Theorem 2.2, [4]) Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, such that the conditions of Theorem 3 are satisfied. Further, let $U^{(2)}_{T,S} \geq 0$ and $U^{(2)}_{T,S} V \geq 0$. Then the following statements are equivalent:*
(i) $A^{(2)}_{T,S} \geq 0$.
(ii) $A^{(2)}_{T,S} V \geq 0$.
(iii) $\rho(U^{(2)}_{T,S} V) = \dfrac{\rho(A^{(2)}_{T,S} V)}{1 + \rho(A^{(2)}_{T,S} V)} < 1$.

We now collect some results connecting nonnegativity of a matrix and its spectral radius.

Theorem 5. *(Theorem 3.16, [10]) Let $B \in \mathbb{R}^{n\times n}$ and $B \geq 0$. Then $\rho(B) < 1$ if and only if $(I - B)^{-1}$ exists and $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \geq 0$.*

The next theorem is a part of the Perron-Frobenius theorem.

Theorem 6. *(Theorem 2.20, [10]) Let A be a real square nonnegative matrix. Then we have the following.*
(i) A has a nonnegative real eigenvalue equal to the spectral radius.
(ii) There exists a nonnegative real eigenvector for its spectral radius.

Another result which relates the spectral radius of two nonnegative matrices is given below.

Theorem 7. *(Theorem 2.21, [10]) If $A \geq B \geq 0$, then $\rho(A) \geq \rho(B)$.*

We conclude this section with a result that has motivated us to prove the main results of this article.

Theorem 8. *(Corollary 2.3, [4]) Let $A = U - V$ be a $\{T,S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, such that the conditions of Theorem 3 are satisfied. Suppose that $x \in T$, then:*
(i) The vector $A^{(2)}_{T,S}b$ is the unique solution of the system $x = U^{(2)}_{T,S}Vx + U^{(2)}_{T,S}b$ for any $b \in \mathbb{R}^n$.
(ii) The iteration $x^{i+1} = U^{(2)}_{T,S}Vx^i + U^{(2)}_{T,S}b$, $b \in \mathbb{R}^n$, converges to $A^{(2)}_{T,S}b$ for every $x^\circ \in \mathbb{R}^n$ if and only if $\rho(U^{(2)}_{T,S}V) < 1$.

3. Main Results

We begin this section with a result that gives a relation between eigenvalues of certain matrices arises from the $\{T,S\}$ splitting of a rectangular matrix.

Lemma 3.1. *Let $A = U - V$ be a $\{T,S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, such that the conditions of Theorem 3 are satisfied. Let μ_i, $1 \leq i \leq p$ and λ_j, $1 \leq j \leq p$ be the eigenvalues of the matrices $U^{(2)}_{T,S}V$ (or $VU^{(2)}_{T,S}$) and $A^{(2)}_{T,S}V$ (or $VA^{(2)}_{T,S}$), respectively. Then for every j, we have $1+\lambda_j \neq 0$. Also, for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ and for every j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.*

Proof. Let λ be an eigenvalue of the matrix $U^{(2)}_{T,S}V$ and x be the corresponding eigenvector. By Theorem 3, $U^{(2)}_{T,S} = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S}$. This implies that $(I + A^{(2)}_{T,S}V)U^{(2)}_{T,S}Vx = A^{(2)}_{T,S}Vx$. Then $(I + A^{(2)}_{T,S}V)\lambda x = A^{(2)}_{T,S}Vx$. Thus $A^{(2)}_{T,S}Vx = \frac{\lambda}{1-\lambda}x$ $(\lambda \neq 1)$. This shows that the matrices $U^{(2)}_{T,S}V$ and $A^{(2)}_{T,S}V$ have the same eigenvectors.

Now, let y be an eigenvector corresponding to the eigenvalue μ_i of the matrix $U^{(2)}_{T,S}V$. Then, y is also an eigenvector of the matrix $A^{(2)}_{T,S}V$ corresponding to some eigenvalue λ_j. By Theorem 3, $\mu_i y = U^{(2)}_{T,S}Vy = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S}Vy = \frac{\lambda_j}{1+\lambda_j}y$. This implies that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$. Similarly, one can show that for each j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$. $\square$

Note that if $T = R(U^t) = R(A^t)$ and $S = N(U^t) = N(A^t)$ in Lemma 3.1 then it reduces to Lemma 2.6 in [8].

We now present a convergent result for the $\{T, S\}$ splitting of a rectangular matrix. This result generalizes the Lemma 3.4 in [7].

Theorem 9. *Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 3 are satisfied. Suppose that $A^{(2)}_{T,S}U \geq 0$ and $U^{(2)}_{T,S}V \geq 0$. Then* $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}U) - 1}{\rho(A^{(2)}_{T,S}U)} < 1.$

Proof. We have $U^{(2)}_{T,S}V \geq 0$. So, by the Perron-Frobenius theorem there exists a nonnegative vector $0 \neq x$ such that $U^{(2)}_{T,S}Vx = \rho(U^{(2)}_{T,S}V)x$. Hence $x \in R(U^{(2)}_{T,S}) = T = R(A^{(2)}_{T,S})$ so that $U^{(2)}_{T,S}Ux = x$. Also, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ by Theorem 3. So, $A^{(2)}_{T,S}U = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}U$. Then $A^{(2)}_{T,S}Ux = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}Ux = (I - U^{(2)}_{T,S}V)^{-1}x = \frac{1}{1-\rho(U^{(2)}_{T,S}V)}x$. Since $A^{(2)}_{T,S}U \geq 0$, it follows that $\frac{1}{1-\rho(U^{(2)}_{T,S}V)}$ is a nonnegative eigenvalue of $A^{(2)}_{T,S}U$. Hence $0 \leq \frac{1}{1-\rho(U^{(2)}_{T,S}V)} \leq \rho(A^{(2)}_{T,S}U)$. This implies that $\rho(U^{(2)}_{T,S}V) \leq \frac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)}$.

Again, the condition $A^{(2)}_{T,S}U \geq 0$ implies existence of a nonnegative vector $0 \neq y$ such that $A^{(2)}_{T,S}Uy = \rho(A^{(2)}_{T,S}U)y$. Then $y \in R(A^{(2)}_{T,S}) = R(U^{(2)}_{T,S})$. Therefore $\rho(A^{(2)}_{T,S}U)y = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}Uy = (I - U^{(2)}_{T,S}V)^{-1}y$. So, we have $(I - U^{(2)}_{T,S}V)^{-1}y = \rho(A^{(2)}_{T,S}U)y$. This implies that $\frac{1}{\rho(A^{(2)}_{T,S}U)}y = y - U^{(2)}_{T,S}Vy$ i.e., $U^{(2)}_{T,S}Vy = \frac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)}y$. Then $\rho(U^{(2)}_{T,S}V) \geq \frac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)}$. Now, from the earlier part of the proof it follows that $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}U) - 1}{\rho(A^{(2)}_{T,S}U)} < 1.$ $\square$

Next, we obtain another convergence result which generalizes the Lemma 3.5 in [7], for the $\{T, S\}$ splitting.

Theorem 10. *Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 3 are satisfied. Suppose that $U^{(2)}_{T,S}V \geq 0$. Then $A^{(2)}_{T,S}V \geq 0$ if and only if $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} < 1$.*

Proof. We first assume that $A^{(2)}_{T,S}V \geq 0$. Let λ and μ be the eigenvalues of $A^{(2)}_{T,S}V$ and $U^{(2)}_{T,S}V$, respectively. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, $\lambda \geq 0$. Then f is a strictly increasing function. We have $\mu = \frac{\lambda}{1+\lambda}$ by Lemma 3.1. So, μ attains its maximum when λ is maximum. However, λ is maximum when $\lambda = \rho(A^{(2)}_{T,S}V)$. As a result, the maximum value of μ is $\rho(U^{(2)}_{T,S}V)$. Hence $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} < 1$.
Conversely, assume that $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} < 1$. Since $A = U - V$ is a $\{T, S\}$ splitting, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ by Theorem 3. The condition $\rho(U^{(2)}_{T,S}V) < 1$ implies that $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k$. Therefore $A^{(2)}_{T,S}V = \sum_{k=1}^{\infty}(U^{(2)}_{T,S}V)^k \geq 0$. □

The following result provides some more properties of $\{T, S\}$ splitting in addition to the properties of $\{T, S\}$ splitting discussed in Section 2.

Theorem 11. *Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 3 are satisfied. Suppose that $U^{(2)}_{T,S}V \geq 0$ and $A^{(2)}_{T,S}V \geq 0$. Then*
(i) $(I - U^{(2)}_{T,S}V)^{-1} \geq 0$.
(ii) $(I - U^{(2)}_{T,S}V)^{-1} \geq I$.
(iii) $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$.

Proof. *(i)* By Theorem 10, we have $\rho(U^{(2)}_{T,S}V) < 1$. Then by Theorem 5, $(I - U^{(2)}_{T,S}V)^{-1}$ exists and $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k \geq 0$ since $U^{(2)}_{T,S}V \geq 0$.
(ii) $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k = I + \sum_{k=1}^{\infty}(U^{(2)}_{T,S}V)^k \geq I$ since $U^{(2)}_{T,S}V \geq 0$.
(iii) From Theorem 3, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$. So $(I - U^{(2)}_{T,S}V)A^{(2)}_{T,S} =$

$U^{(2)}_{T,S}$. Upon post multiplying by V, we have $A^{(2)}_{T,S}V - U^{(2)}_{T,S}V = U^{(2)}_{T,S}VA^{(2)}_{T,S}V \geq 0$ since $U^{(2)}_{T,S}V \geq 0$ and $A^{(2)}_{T,S}V \geq 0$. Hence $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$. □

Now we prove one more convergent result for $\{T, S\}$ splitting which is a generalization of the Theorem 3.9 in [7].

Theorem 12. *Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 3 are satisfied. Suppose that $U^{(2)}_{T,S}V \geq 0$, $\rho(U^{(2)}_{T,S}V) < 1$ and $A^{(2)}_{T,S} \geq 0$, then*
(i) $A^{(2)}_{T,S} \geq U^{(2)}_{T,S}$.
(ii) $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$.
(iii) $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1 + \rho(A^{(2)}_{T,S}V)}$.

Proof. *(i)* By Theorem 3, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ so that $(I - U^{(2)}_{T,S}V)A^{(2)}_{T,S} = U^{(2)}_{T,S}$. Therefore $A^{(2)}_{T,S} - U^{(2)}_{T,S} = U^{(2)}_{T,S}VA^{(2)}_{T,S} \geq 0$. i.e., $A^{(2)}_{T,S} \geq U^{(2)}_{T,S}$.
(ii) $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ by Theorem 3. The condition $\rho(U^{(2)}_{T,S}V) < 1$ implies that $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k$. Therefore $A^{(2)}_{T,S}V - U^{(2)}_{T,S}V = \sum_{k=2}^{\infty}(U^{(2)}_{T,S}V)^k \geq 0$. i.e., $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$.
(iii) From *(ii)* we have $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V \geq 0$. Hence $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1 + \rho(A^{(2)}_{T,S}V)}$ by Theorem 10. □

We next obtain comparison theorems for $\{T, S\}$ splittings.

Theorem 13. *Let $A \in \mathbb{R}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{R}^n$ of dimension $s \leq r$ and let S be a subspace of $\mathbb{R}^m$ of dimension $m - s$, such that $AT \oplus S = \mathbb{R}^m$. Let $A = U - V = P - Q$ be two $\{T, S\}$ splittings of A such that $dim(T) \leq min\{rank(U), rank(P)\}$ and $U^{(2)}_{T,S} \geq 0$, $P^{(2)}_{T,S} \geq 0$, $V \geq 0$, $Q \geq 0$. If $A^{(2)}_{T,S} \geq 0$ and $Q \geq V$, then $1 > \rho(P^{(2)}_{T,S}Q) \geq \rho(U^{(2)}_{T,S}V)$.*

Proof. By Theorem 10, we have $\rho(U^{(2)}_{T,S}V) < 1$ and $\rho(P^{(2)}_{T,S}Q) < 1$. Also, $A^{(2)}_{T,S} \geq 0$ and $Q \geq V \geq 0$. Then $A^{(2)}_{T,S}Q \geq A^{(2)}_{T,S}V \geq 0$ and Theorem 7 yields $\rho(A^{(2)}_{T,S}Q) \geq \rho(A^{(2)}_{T,S}V)$. Let λ_1 and λ_2 be the eigenvalues of $A^{(2)}_{T,S}V$ and $A^{(2)}_{T,S}Q$, respectively. Then $1 > \rho(P_{T,S}^{(2)}Q) = \dfrac{\rho(A_{T,S}^{(2)}Q)}{1+\rho(A_{T,S}^{(2)}Q)} \geq \dfrac{\rho(A_{T,S}^{(2)}V)}{1+\rho(A_{T,S}^{(2)}V)} = \rho(U_{T,S}^{(2)}V)$ since $\frac{\lambda}{1+\lambda}$ is a strictly increasing function. □

Note that if $T = R(U^t) = R(P^t) = R(A^t)$ and $S = N(U^t) = N(P^t) = N(A^t)$ then the above result reduces to Theorem 3.2 in [6]. We illustrate Theorem 13 with The following example .

Example 3.1. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Set $U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Also, let $T = \{(x, x, 0)^t : x \in \mathbb{R}\}$ and $S = \{(0, y)^t : y \in \mathbb{R}\}$, then T is a subspace of $\mathbb{R}^3$ with dimension 1 and S is a subspace of $\mathbb{R}^2$ with dimension 1. Further $AT \oplus S = \mathbb{R}^2$, $UT \oplus S = \mathbb{R}^2$ and $PT \oplus S = \mathbb{R}^2$. So $A = U - V = P - Q$ are two $\{T, S\}$ splittings of A and $Q \geq V \geq 0$. Now $A_{T,S}^{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$, $U_{T,S}^{(2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$, $P_{T,S}^{(2)} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$, $U_{T,S}^{(2)}V = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P_{T,S}^{(2)}Q = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, $\rho(U_{T,S}^{(2)}V) = \frac{1}{2}$ and $\rho(P_{T,S}^{(2)}Q) = \frac{2}{3}$. Hence $1 > \rho(P_{T,S}^{(2)}Q) \geq \rho(U_{T,S}^{(2)}V)$.

Another comparison theorem for $\{T, S\}$ splitting is proved, next. This generalizes Theorem 3.3 in [6].

Theorem 14. *Let $A = U - V = P - Q$ be two $\{T, S\}$ splittings of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 13 are satisfied. If $A_{T,S}^{(2)} \geq 0$ and $U_{T,S}^{(2)} \geq P_{T,S}^{(2)}$, then $1 > \rho(P_{T,S}^{(2)}Q) \geq \rho(U_{T,S}^{(2)}V)$.*

Proof. By Theorem 4, we have $\rho(U_{T,S}^{(2)}V) < 1$ and $\rho(P_{T,S}^{(2)}Q) < 1$. Also $\rho(U_{T,S}^{(2)}V)$ and $\rho(P_{T,S}^{(2)}Q)$ are strictly monotone increasing functions of $\rho(A_{T,S}^{(2)}V)$ and $\rho(A_{T,S}^{(2)}Q)$, respectively. Therefore, it is enough to show that $\rho(A_{T,S}^{(2)}Q) \geq \rho(A_{T,S}^{(2)}V)$. By the hypothesis, we have $A = U - V = P - Q$ two $\{T, S\}$ splittings satisfying the conditions of Theorem 13 and $A_{T,S}^{(2)} \geq 0$. So, $I + A_{T,S}^{(2)}V$ and $I + QA_{T,S}^{(2)}$ are both invertible and nonnegative. Now $U_{T,S}^{(2)} \geq P_{T,S}^{(2)}$ implies $A_{T,S}^{(2)}(I + QA_{T,S}^{(2)}) \geq (I + A_{T,S}^{(2)}V)A_{T,S}^{(2)}$ i.e., $A_{T,S}^{(2)}QA_{T,S}^{(2)} \geq A_{T,S}^{(2)}VA_{T,S}^{(2)}$. Then post multiplying by Q, and again by V, we have $(A_{T,S}^{(2)}Q)^2 \geq A_{T,S}^{(2)}VA_{T,S}^{(2)}Q$ and

$A^{(2)}_{T,S}QA^{(2)}_{T,S}V \geq (A^{(2)}_{T,S}V)^2$. Therefore, by Theorem 7, we have

$$\rho^2(A^2_{T,S}Q) \geq \rho(A^{(2)}_{T,S}VA^{(2)}_{T,S}Q) = \rho(A^{(2)}_{T,S}QA^{(2)}_{T,S}V) \geq \rho^2(A^{(2)}_{T,S}V).$$

Hence $\rho(A^{(2)}_{T,S}Q) \geq \rho(A^{(2)}_{T,S}V)$. □

The following example illustrates Theorem 14.

Example 3.2. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Set $U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \geq 0$, $P = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \geq 0$. Also, let $T = \{(x,0,0)^t : x \in \mathbb{R}\}$ and $S = \{(0,y)^t : y \in \mathbb{R}\}$, then T is a subspace of $\mathbb{R}^3$ with dimension 1 and S is a subspace of $\mathbb{R}^2$ with dimension 1. Further $AT \oplus S = \mathbb{R}^2$, $UT \oplus S = \mathbb{R}^2$ and $PT \oplus S = \mathbb{R}^2$. So $A = U - V = P - Q$ are two $\{T,S\}$ splittings of A.
Now $A^{(2)}_{T,S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$, $U^{(2)}_{T,S} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $P^{(2)}_{T,S} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $U^{(2)}_{T,S}V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P^{(2)}_{T,S}Q = \frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So $U^{(2)}_{T,S} \geq P^{(2)}_{T,S} \geq 0$ and $\rho(U^{(2)}_{T,S}V) = \frac{1}{2}$, $\rho(P^{(2)}_{T,S}Q) = \frac{2}{3}$.
Hence $1 > \rho(P^{(2)}_{T,S}Q) \geq \rho(U^{(2)}_{T,S}V)$.

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