

**VISCOSITY APPROXIMATION METHODS OF  
NONEXPANSIVE MAPPINGS IN HILBERT  
SPACES AND APPLICATIONS**

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**Abstract:** In this paper, we introduce a new viscosity approximation method of nonexpansive mappings in Hilbert spaces. The strong convergence theorems of the rules are proved under certain assumptions imposed on the sequence of parameters. Moreover, we give applications of the purposed viscosity method.

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## 1. Introduction

In this paper, we shall take  $H$  as a real Hilbert space,  $\langle \cdot, \cdot \rangle$  as the inner product,  $\| \cdot \|$  as the induced norm, and  $C$  as a nonempty closed subset of  $H$ .

**Definition 1.1.** Let  $T : H \rightarrow H$  be a mapping. Then  $T$  is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

**Definition 1.2.** A mapping  $f : H \rightarrow H$  is called a *contraction* if for all  $x, y \in H$  and  $\theta \in [0, 1)$

$$\|f(x) - f(y)\| \leq \theta \|x - y\|.$$

**Definition 1.3.**  $P_C : H \rightarrow C$  is called a *metric projection* if for every  $x \in H$  there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

**Theorem 1.4.** [2] (The demiclosedness principle) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$  and  $T : C \rightarrow C$  such that  $x_n \rightarrow x^* \in C$  and  $(I - T)x_n \rightarrow 0$ . Then  $x^* = Tx^*$ . Here  $\rightarrow$  and  $\rightharpoonup$  denote strong and weak convergence, respectively.*

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

**Theorem 1.5.** [9] *Assume that  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence with

- (1)  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ,
- (2)  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$  or  $\sum_{n=0}^{\infty} |\delta_n| < \infty$ .

Then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

The following strong convergence theorem, which is also called the *viscosity approximation method* for nonexpansive mappings in real Hilbert spaces is given by Moudafi [6] in 2000.

**Theorem 1.6.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T) := \{x \in H : T(x) = x\}$  is nonempty. Let  $f$  be a contraction of  $C$  into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n), n \geq 0,$$

where the sequence  $\{\epsilon_n\} \in (0, 1)$  satisfies:

- (1)  $\lim_{n \rightarrow \infty} \epsilon_n = 0,$
- (2)  $\sum_{n=0}^{\infty} \epsilon_n = \infty,$
- (3)  $\lim_{n \rightarrow \infty} \left| \frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n} \right| = 0.$

Then  $\{x_n\}$  converges strongly to a fixed point  $x^*$  of the mapping  $T$ , which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in F(T).$$

In 2015, Xu et al. [9] applied the viscosity method on the midpoint rule for nonexpansive mappings and they give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T \left( \frac{x_n + x_{n+1}}{2} \right), \quad \forall n \geq 0.$$

This, using contraction, regularizes the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of  $T$ . Ke and Ma [5], motivated and inspired by the idea of Xu et al. [9], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T (s_n x_n + (1 - s_n) x_{n+1})$$

and

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T (s_n x_n + (1 - s_n) x_{n+1}).$$

Our contribution in this direction is the following new viscosity method:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right). \end{cases} \tag{1.1}$$

### 2. The Main Result

**Theorem 2.1.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T : C \rightarrow C$  be a nonexpansive mapping with  $F(T) \neq \emptyset$  and  $f : C \rightarrow C$  be a contraction with coefficient  $\theta \in [0, 1)$ . Pick any  $x_0 \in C$ , let  $\{x_n\}$  be a sequence generated by the viscosity method (1.1), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:*

- (i)  $\alpha_n + \beta_n + \gamma_n = 1$ ,
- (ii)  $\lim_{n \rightarrow \infty} \alpha_n = 0 = \lim_{n \rightarrow \infty} \beta_n$  and  $\lim_{n \rightarrow \infty} \gamma_n = 1$ ,
- (iii)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,
- (iv)  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,
- (v)  $\lim_{n \rightarrow \infty} \|x_n - T(x_n)\| = 0$ .

Then  $\{x_n\}$  converges strongly to a fixed point  $x^*$  of the mapping  $T$ , which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in F(T).$$

In other words,  $x^*$  is the unique fixed point of the contraction  $P_{F(T)}f$ , that is,  $P_{F(T)}f(x^*) = x^*$ .

*Proof.* The proof is divided into four steps:

*Step 1.* ( $\{x_n\}$  is bounded)

Take  $p \in F(T)$  arbitrarily, we have

$$\begin{aligned} & \|x_{n+1} - p\| \\ &= \|T(y_n) - p\| \\ &= \left\| T \left( \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) \right) - p \right\| \\ &\leq \left\| \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) - p \right\| \\ &= \left\| \alpha_n x_n - \alpha_n p + \beta_n f(x_n) - \beta_n p + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) + (\alpha_n + \beta_n - 1)p \right\| \\ &\leq \alpha_n \|x_n - p\| + \beta_n \|f(x_n) - p\| + \gamma_n \left\| T \left( \frac{x_n + x_{n+1}}{2} \right) - p \right\| \\ &\leq \alpha_n \|x_n - p\| + \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + \gamma_n \left\| \frac{x_n + x_{n+1}}{2} - p \right\| \\ &\leq \alpha_n \|x_n - p\| + \theta \beta \|x_n - p\| + \beta \|f(p) - p\| \\ &\quad + \gamma_n \left[ \frac{1}{2} \|x_n - p\| + \frac{1}{2} \|x_{n+1} - p\| \right] \end{aligned}$$

$$= \left( \alpha_n + \theta\beta_n + \frac{\gamma_n}{2} \right) \|x_n - p\| + \frac{\gamma_n}{2} \|x_{n+1} - p\| + \beta_n \|f(p) - p\|.$$

It follows that

$$\begin{aligned} & \left( 1 - \frac{\gamma_n}{2} \right) \|x_{n+1} - p\| \\ & \leq \left( \alpha_n + \theta\beta_n + \frac{\gamma_n}{2} \right) \|x_n - p\| + \beta_n \|f(p) - p\|. \end{aligned} \tag{2.1}$$

Since  $\gamma_n \in (0, 1)$ ,  $1 - \frac{\gamma_n}{2} \geq 0$ . Moreover, by equation (2.1) and  $\alpha_n + \beta_n + \gamma_n = 1$ , we get

$$\begin{aligned} & \|x_{n+1} - p\| \\ & \leq \frac{\alpha_n + \theta\beta_n + \frac{\gamma_n}{2}}{1 - \frac{\gamma_n}{2}} \|x_n - p\| + \frac{\beta_n}{1 - \frac{\gamma_n}{2}} \|f(p) - p\| \\ & \leq \left[ 1 - \frac{\beta_n(1 - \theta)}{1 - \frac{\gamma_n}{2}} \right] \|x_n - p\| + \frac{\beta_n(1 - \theta)}{1 - \frac{\gamma_n}{2}} \left( \frac{1}{1 - \theta} \|f(p) - p\| \right). \end{aligned}$$

Thus, we have

$$\|x_{n+1} - p\| \leq \max \left\{ \|x_n - p\|, \frac{1}{1 - \theta} \|f(p) - p\| \right\}.$$

Now by induction, we obtain

$$\|x_{n+1} - p\| \leq \max \left\{ \|x_0 - p\|, \frac{1}{1 - \theta} \|f(p) - p\| \right\}.$$

Hence, we concluded that  $\{x_n\}$  is bounded. It follows immediately that  $\{f(x_n)\}$  and  $\{T(\frac{x_n+x_{n+1}}{2})\}$  are bounded.

*Step 2.* ( $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ )

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ & = \|T(y_n) - x_n\| \\ & = \left\| T \left( \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) \right) - x_n \right\| \\ & \leq \left\| \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) - x_n \right\| \\ & = \left\| \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) - \left[ \alpha_{n-1} x_{n-1} + \beta_{n-1} f(x_{n-1}) \right. \right. \\ & \quad \left. \left. + \gamma_{n-1} T \left( \frac{x_{n-1} + x_n}{2} \right) \right] \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \beta_n(f(x_n) - f(x_{n-1})) \right. \\
&\quad \left. + (\beta_n - \beta_{n-1})f(x_{n-1}) + \gamma_n \left[ T\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right) \right] \right. \\
&\quad \left. + (\gamma_n - \gamma_{n-1})T\left(\frac{x_{n-1} + x_n}{2}\right) \right\| \\
&= \left\| \alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \beta_n(f(x_n) - f(x_{n-1})) \right. \\
&\quad \left. + (\beta_n - \beta_{n-1})f(x_{n-1}) + \gamma_n \left[ T\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right) \right] \right. \\
&\quad \left. - [(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})] T\left(\frac{x_{n-1} + x_n}{2}\right) \right\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left\| x_{n-1} - T\left(\frac{x_{n-1} + x_n}{2}\right) \right\| \\
&\quad + \beta_n \|f(x_n) - f(x_{n-1})\| + |\beta_n - \beta_{n-1}| \left\| f(x_{n-1}) - T\left(\frac{x_n + x_{n-1}}{2}\right) \right\| \\
&\quad + \gamma_n \left\| T\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right) \right\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) M + \theta \beta_n \|x_n - x_{n-1}\| \\
&\quad + \frac{\gamma_n}{2} \left\| \frac{x_{n+1} + x_n}{2} - \frac{x_n - x_{n-1}}{2} \right\| \\
&= \alpha_n \|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) M + \theta \beta_n \|x_n - x_{n-1}\| \\
&\quad + \frac{\gamma_n}{2} \|x_{n+1} - x_n\| + \frac{\gamma_n}{2} \|x_n - x_{n-1}\|,
\end{aligned}$$

where  $M > 0$  is a constant such that

$$M \geq \max \left\{ \sup_{n \geq 0} \left\| x_{n-1} - T\left(\frac{x_{n-1} + x_n}{2}\right) \right\|, \sup_{n \geq 0} \left\| f(x_{n-1}) - T\left(\frac{x_n + x_{n-1}}{2}\right) \right\| \right\}.$$

It gives

$$\begin{aligned}
&\left(1 - \frac{\gamma_n}{2}\right) \|x_{n+1} - x_n\| \\
&\leq \left(\alpha_n + \beta_n + \frac{\gamma_n}{2}\right) \|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| - |\beta_n - \beta_{n-1}|)M,
\end{aligned}$$

that is,

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ & \leq \left( \frac{\alpha_n + \beta_n + \frac{\gamma_n}{2}}{1 - \frac{\gamma_n}{2}} \right) \|x_n - x_{n-1}\| + \frac{M}{1 - \frac{\gamma_n}{2}} (|\alpha_n - \alpha_{n-1}| - |\beta_n - \beta_{n-1}|) \\ & = \left[ 1 - \frac{\beta_n(1 - \theta)}{1 - \frac{\gamma_n}{2}} \right] \|x_n - x_{n-1}\| + \frac{M}{1 - \frac{\gamma_n}{2}} (|\alpha_n - \alpha_{n-1}| - |\beta_n - \beta_{n-1}|). \end{aligned}$$

Note that for  $\gamma_n \in (0, 1)$ ,  $1 - \frac{\gamma_n}{2} \leq 1$  and  $\frac{\beta_n(1-\theta)}{1-\frac{\gamma_n}{2}} \geq \beta_n(1 - \theta)$ . Hence

$$\begin{aligned} \|x_{n+1} - x_n\| & \leq [1 - \beta_n(1 - \theta)] \|x_n - x_{n-1}\| \\ & \quad + \frac{M}{1 - \frac{\gamma_n}{2}} (|\alpha_n - \alpha_{n-1}| - |\beta_n - \beta_{n-1}|). \end{aligned}$$

Since  $\sum_{n=0}^\infty \beta_n = \infty$ ,  $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ , and  $\sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty$ , by Theorem 1.5, we have  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

STEP 3.  $(\limsup_{n \rightarrow \infty} \langle x^* - f(x^*), x^* - x_n \rangle \leq 0$ , where  $x^* = P_{F(T)}f(x^*)$ )

Indeed, we take a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  which converges weakly to a fixed point  $p$  of  $T$ . Without loss of generality, we may assume that  $\{x_{n_i}\} \rightharpoonup p$ . From  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$  and Theorem 1.4, we have  $p = Tp$ . This together with the property of the metric projection implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle x^* - f(x^*), x^* - x_n \rangle & = \limsup_{n \rightarrow \infty} \langle x^* - f(x^*), x^* - x_{n_i} \rangle \\ & = \langle x^* - f(x^*), x^* - p \rangle \\ & \leq 0. \end{aligned}$$

Step 4.  $(x_n \rightarrow x^*$  as  $n \rightarrow \infty)$

Take  $x^* \in F(T)$  the unique fixed point of the contraction  $P_{F(T)}f$ . Consider

$$\begin{aligned} & \|x_{n+1} - x_n\|^2 \\ & = \|T(y_n) - x_n\|^2 \\ & = \left\| T \left( \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) \right) - x_n \right\|^2 \\ & \leq \left\| \alpha_n x_n + \beta_n f(x_n) + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) - x_n \right\|^2 \\ & = \left\| \alpha_n x_n - \alpha_n x^* + \beta_n f(x_n) - \beta_n x^* + \gamma_n T \left( \frac{x_n + x_{n+1}}{2} \right) \right. \\ & \quad \left. + (\alpha_n + \beta_n - 1)x_n \right\|^2 \end{aligned}$$

$$\begin{aligned}
&= \alpha_n^2 \|x_n - x^*\|^2 + \beta_n^2 \|f(x_n) - x^*\|^2 + \gamma_n^2 \left\| T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\|^2 \\
&\quad + 2\alpha_n\beta_n \langle x_n - x^*, f(x_n) - x^* \rangle + 2\alpha_n\gamma_n \left\langle x_n - x^*, T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\rangle \\
&\quad + 2\beta_n\gamma_n \langle x_n - x^*, f(x_n) - x^* \rangle \\
&\leq \alpha_n^2 \|x_n - x^*\|^2 + \beta_n^2 \|f(x_n) - x^*\|^2 + \gamma_n^2 \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\|^2 \\
&\quad + 2\alpha_n\beta_n \langle x_n - x^*, f(x_n) - x^* \rangle \\
&\quad + 2\alpha_n\gamma_n \|x_n - x^*\| \left\| T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\| \\
&\quad + 2\beta_n\gamma_n \left\langle f(x_n) - f(x^*), T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\rangle \\
&\quad + 2\beta_n\gamma_n \left\langle f(x^*) - x^*, T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\rangle \\
&\leq \alpha_n^2 \|x_n - x^*\|^2 + \gamma_n^2 \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\|^2 \\
&\quad + 2\alpha_n\gamma_n \|x_n - x^*\| \left\| T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\| \\
&\quad + 2\beta_n\gamma_n \|f(x_n) - f(x^*)\| \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\| + K_n \\
&\leq \alpha_n^2 \|x_n - x^*\|^2 + \gamma_n^2 \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\|^2 \\
&\quad + 2\alpha_n\gamma_n \|x_n - x^*\| \left\| T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\| \\
&\quad + 2\theta\beta_n\gamma_n \|x_n - x^*\| \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\| + K_n \\
&= \alpha_n^2 \|x_n - x^*\|^2 + \gamma_n^2 \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\|^2 \\
&\quad + 2\gamma_n(\alpha_n + \theta\beta_n) \|x_n - x^*\| \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\| + K_n,
\end{aligned}$$

where

$$\begin{aligned}
K_n &= \beta_n^2 \|f(x_n) - x^*\|^2 + 2\alpha_n\beta_n \langle x_n - x^*, f(x_n) - x^* \rangle \\
&\quad + 2\beta_n\gamma_n \left\langle f(x^*) - x^*, T\left(\frac{x_{n+1} + x_n}{2}\right) - x^* \right\rangle.
\end{aligned}$$



It becomes

$$\begin{aligned} & \gamma_n^2 \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\|^2 + 2\gamma_n(\alpha_n + \theta\beta_n)\|x_n - x^*\| \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\| \\ & \quad + K_n + \alpha_n^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ & \geq 0. \end{aligned}$$

Solving this quadratic inequality for  $\left\| \frac{x_{n+1} + x_n}{2} - x^* \right\|$ , we get

$$\begin{aligned} & \left\| \frac{x_{n+1} + x_n}{2} - x^* \right\| \\ & \geq \frac{1}{2\gamma_n^2} [-2\gamma_n(\alpha_n + \theta\beta_n)\|x_n - x^*\| \\ & \quad + \sqrt{4\gamma_n^2(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - 4\gamma_n^2(K_n + \alpha_n^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2)}] \\ & \quad \times \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right) \|x_n - x^*\| + \frac{\gamma_n}{2}\|x_{n+1} - x^*\| \\ & \geq \sqrt{(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - K_n - \alpha_n^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2} \\ & \quad \times (\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - K_n - \alpha_n^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \\ & \leq \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right)^2 \|x_n - x^*\|^2 \\ & \quad + \frac{\gamma_n^2}{4}\|x_{n+1} - x^*\|^2 + 2\left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right) \frac{\gamma_n}{2} \|x_n - x^*\| \|x_{n+1} - x^*\| \\ & \leq \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right)^2 \|x_n - x^*\|^2 + \frac{\gamma_n^2}{4}\|x_{n+1} - x^*\|^2 \\ & \quad + \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right) \frac{\gamma_n}{2} [\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2], \end{aligned}$$

which reduces to

$$\begin{aligned} & \left( 1 - \frac{\gamma_n^2}{4} - \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right) \frac{\gamma_n}{2} \right) \|x_{n+1} - x^*\|^2 \\ & \leq \left[ \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n \right) \left( \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n + \frac{\gamma_n}{2} \right) + \alpha_n^2 - (\alpha_n + \theta\beta_n)^2 \right] \\ & \quad \times \|x_n - x^*\|^2 + K_n, \end{aligned}$$

that is,

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ & \leq \frac{[(\frac{\gamma_n}{2} + \alpha_n + \theta\beta_n)(\gamma_n + \alpha_n + \theta\beta_n) + \alpha_n^2 - (\alpha_n + \theta\beta_n)^2] \|x_n - x^*\|^2 + K_n}{1 - \frac{\gamma_n}{2}(\frac{\gamma_n}{2} + \frac{\gamma_n}{2} + \alpha_n + \theta\beta_n)} \\ & \leq \frac{[(\frac{\gamma_n}{2} + \alpha_n + \theta\beta_n)(1 - \beta_n + \theta\beta_n) + \alpha_n^2 - (\alpha_n + \theta\beta_n)^2] \|x_n - x^*\|^2 + K_n}{1 - \frac{\gamma_n}{2}(\gamma_n + \alpha_n + \theta\beta_n)} \\ & \leq \frac{[(\frac{\gamma_n}{2} + \alpha_n + \theta\beta_n)(1 + (\theta - 1)\beta_n) - 2\alpha_n\theta\beta_n - \theta^2\beta_n^2] \|x_n - x^*\|^2}{1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)} \\ & \quad + \frac{K_n}{1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)}. \end{aligned}$$

Let

$$\begin{aligned} y_n &= \frac{1}{\beta_n} \left[ 1 - \frac{(\frac{\gamma_n}{2} + \alpha_n + \theta\beta_n)(1 + (\theta - 1)\beta_n) - 2\alpha_n\theta\beta_n - \theta^2\beta_n^2}{1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)} \right] \\ &= \frac{2 + 2\theta\alpha_n - \beta_n}{1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)}. \end{aligned}$$

Then  $\lim_{n \rightarrow \infty} y_n = 4$ . If  $0 < \rho < 4$ , then there exists an integer  $N$  such that  $y_n > \rho, \forall n \geq N$ . Hence we have

$$\frac{(\frac{\gamma_n}{2} + \alpha_n + \theta\beta_n)(1 + (\theta - 1)\beta_n) - 2\alpha_n\theta\beta_n - \theta^2\beta_n^2}{1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)} \leq 1 - \rho\beta_n, \quad \forall n \geq N.$$

It turns out that

$$\|x_{n+1} - x^*\|^2 \leq (1 - \rho\beta_n)\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)}. \tag{2.2}$$

By  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$  and  $\lim_{n \rightarrow \infty} \gamma_n = 1$  we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{K_n}{\rho\beta_n[1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)]} \\ &= \limsup_{n \rightarrow \infty} \left( \frac{\beta_n\|f(x_n) - x^*\|^2 + 2\alpha_n\langle x_n - x^*, f(x_n) - x^* \rangle}{\rho[1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)]} \right. \\ & \quad \left. + \frac{2\gamma_n\langle f(x^*) - x^*, T(\frac{x_{n+1} + x_n}{2}) - x^* \rangle}{\rho[1 - \frac{\gamma_n}{2}(1 + (\theta - 1)\beta_n)]} \right) \\ & \leq 0. \end{aligned} \tag{2.3}$$

From equations (2.2), (2.3), and Theorem 1.4 we have  $\lim_{n \rightarrow \infty} \|x_{n+1} - x^*\|^2 = 0$ , which implies that  $x_n \rightarrow x^*$ . This completes the proof. □

### 3. Applications

#### 3.1. A More General System of Variational Inequalities

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$  and  $\{A_i\}_{i=1}^N : C \rightarrow H$  be a family of mappings. In [1], Cai and Bu considered the problem of finding  $(x_1^*, x_2^*, \dots, x_N^*) \in C \times C \times \dots \times C$  such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \dots\dots\dots, \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{3.1}$$

The equation (3.1) can be written as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N)x_N^*, x - x_1^* \rangle \geq 0, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1})x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \dots\dots\dots, \\ \langle x_3^* - (I - \lambda_2 A_2)x_2^*, x - x_3^* \rangle \geq 0, \\ \langle x_2^* - (I - \lambda_1 A_1)x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases}$$

which is a more general system of variational inequalities in Hilbert spaces, where  $\lambda_i > 0$  for all  $i \in \{1, 2, 3, \dots, N\}$ . We also have following lemmas.

**Lemma 3.1.** [1] *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . For  $i \in \{1, 2, 3, \dots, N\}$ , let  $A_i : C \rightarrow H$  be  $\delta_i$ -inverse-strongly monotone for some positive real number  $\delta_i$ , namely,*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C.$$

Let  $G : C \rightarrow C$  be a mapping defined by

$$\begin{aligned} G(x) &= P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ &P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \end{aligned} \tag{3.2}$$

If  $0 < \lambda_i \leq 2\delta_i$  for all  $i \in \{1, 2, \dots, N\}$ , then  $G$  is nonexpansive.

**Lemma 3.2.** [4] *Let  $C$  be a nonempty closed convex subset of a real Hilbert spaces  $H$ . Let  $A_i : C \rightarrow H$  be a nonlinear mapping, where  $i \in$*

$\{1, 2, 3, \dots, N\}$ . For given  $x_i^* \in C$ ,  $i \in \{1, 2, 3, \dots, N\}$ ,  $(x_1^*, x_2^*, x_3^*, \dots, x_N^*)$  is a solution of the problem (3.1) if and only if

$$\begin{aligned} x_1^* &= P_C(I - \lambda_N A_N)x_N^*, \\ x_i^* &= P_C(I - \lambda_{i-1} A_{i-1})x_{i-1}^*, \quad i = 2, 3, 4, \dots, N, \end{aligned} \tag{3.3}$$

that is,

$$\begin{aligned} x_1^* &= P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ &\quad P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*, \quad \forall x \in C. \end{aligned}$$

From Lemma 3.2, we know that  $x_1^* = G(x_1^*)$ , that is,  $x_1^*$  is a fixed point of the mapping  $G$ , where  $G$  is defined by (3.2). Moreover, if we find the fixed point  $x_1^*$ , it is easy to get the other points by (3.3). Applying Theorem 2.1 we get the result

**Theorem 3.3.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . For  $i \in \{1, 2, 3, \dots, N\}$ , let  $A_i : C \rightarrow H$  be  $\delta_i$ -inverse-strongly monotone for some positive real number  $\delta_i$  with  $F(G) \neq \emptyset$ , where  $G : C \rightarrow C$  is defined by*

$$\begin{aligned} G(x) &= P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ &\quad P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \end{aligned}$$

Let  $f : C \rightarrow C$  be a contraction with coefficient  $\theta \in [0, 1)$ . Pick any  $x_0 \in C$ , let  $\{x_n\}$  be a sequence generated by

$$\begin{cases} x_{n+1} = G(y_n), \\ y_n = \alpha_n x_n + \beta_n f(x_n) + \gamma_n G\left(\frac{x_n + x_{n+1}}{2}\right), \end{cases}$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying the conditions (i)-(iv) and

(vi)  $\lim_{n \rightarrow \infty} \|x_n - G(x_n)\| = 0$ .

Then  $\{x_n\}$  converges strongly to a fixed point  $x^*$  of the nonexpansive mapping  $G$  which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in F(G).$$

In other words,  $x^*$  is the unique fixed point of the contraction  $P_{F(G)}f$ , that is,  $P_{F(G)}f(x^*) = x^*$ .

### 3.2. The Constrained Convex Minimization Problem

Now, we consider the following constrained convex minimization problem:

$$\min_{x \in C} \phi(x), \tag{3.4}$$

where  $\phi : C \rightarrow R$  is a real valued convex function and assumes that the problem (3.4) is consistent. Let  $\Omega$  denote its solution set. For the minimization problem (3.4), if  $\phi$  is (Fréchet) differentiable, then we have the following lemma.

**Lemma 3.4.** (Optimality Condition) [7] *A necessary condition of optimality for a point  $x^* \in C$  to be a solution of the minimization problem (3.4) is that  $x^*$  solves the variational inequality*

$$\langle \nabla\phi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.5}$$

Equivalently,  $x^* \in C$  solves the fixed point equation

$$x^* = P_C (x^* - \lambda \nabla\phi(x^*))$$

for every constant  $\lambda > 0$ . If, in a addition  $\phi$  is convex, then the optimality condition (3.5) is also sufficient.

It is well known that the mapping  $P_C(I - \lambda A)$  is nonexpansive when the mapping  $A$  is  $\delta$ -inverse-strongly monotone and  $0 < \lambda < 2\delta$ . We therefore have the following result.

**Theorem 3.5.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . For the minimization problem (3.4), assume that  $\phi$  is (Fréchet) differentiable and the gradient  $\nabla\phi$  is a  $\delta$ -inverse-strongly monotone mapping for some positive real number  $\delta$ . Let  $f : C \rightarrow C$  be a contraction with coefficient  $\theta \in [0, 1)$ . Pick any  $x_0 \in C$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{cases} x_{n+1} = P_C(I - \lambda \nabla\phi)(y_n), \\ y_n = \alpha_n x_n + \beta_n f(x_n) + \gamma_n P_C(I - \lambda \nabla\phi) \left( \frac{x_n + x_{n+1}}{2} \right), \end{cases}$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying the conditions (i)-(iv) and

$$(vii) \lim_{n \rightarrow \infty} \|x_n - P_C(I - \lambda \nabla\phi)(x_n)\| = 0.$$

Then  $\{x_n\}$  converges strongly to a solution  $x^*$  of the minimization problem (3.4), which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in \Omega.$$

In other words,  $x^*$  is the unique fixed point of the contraction  $P_\Omega f$ , that is,  $P_\Omega f(x^*) = x^*$ .

### 3.3. The $K$ -Mapping

Kangtunyakarn and Suantai [3] in 2009 gave  $K$ -mappings generated by  $T_1, T_2, T_3, \dots, T_N$  and  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$  as follows.

**Definition 3.6.** [3] Let  $C$  be a nonempty convex subset of a real Banach space. Let  $\{T_i\}_{i=1}^N$  be a family of mappings of  $C$  into itself and let  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$  be real numbers such that  $0 \leq \lambda_i \leq 1$  for every  $i = 1, 2, 3, \dots, N$ . We define a mapping  $K : C \rightarrow C$  as follows:

$$\begin{cases} U_1 = \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\ \dots\dots\dots, \\ U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\ U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a  $K$ -mapping generated by  $T_1, T_2, T_3, \dots, T_N$  and  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ .

In 2014, Suwannaut and Kangtunyakarn [8] established the following result for  $K$ -mappings generated by  $T_1, T_2, T_3, \dots, T_N$  and  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ .

**Lemma 3.7.** [8] Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . For  $i = 1, 2, 3, \dots, N$ , let  $\{T_i\}_{i=1}^N$  be a finite family of  $K_i$ -strictly pseudo-contractive mappings of  $C$  into itself with  $K_i \leq \omega_i$  and  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ , namely there exist constants  $K_i \in [0, 1)$  such that

$$\|T_i x - T_i y\|^2 \leq \|x - y\|^2 + K_i \|(I - T_i)x - (I - T_i)y\|^2, \quad \forall x, y \in C.$$

Let  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$  be real numbers with  $0 < \lambda_i < \omega_i, \forall i = 1, 2, 3, \dots, N$  and  $\omega_1 + \omega_2 < 1$ . Let  $K$  be the  $K$ -mapping generated by  $T_1, T_2, T_3, \dots, T_N$  and  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ . Then the following properties hold:

- (a)  $F(K) = \bigcap_{i=1}^N F(T_i)$ .
- (b)  $K$  is a nonexpansive mapping.

On the bases of above lemma, we have the following result.

**Theorem 3.8.** Let  $C$  be a nonempty closed convex subset of the real Hilbert space  $H$ . For  $i = 1, 2, 3, \dots, N$ , let  $\{T_i\}_{i=1}^N$  be a finite family of  $K_i$ -strictly pseudo-contractive mappings of  $C$  into itself with  $K_i \leq \omega_i$  and  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Let  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$  be real numbers with  $0 < \lambda_i < \omega_i, \forall i = 1, 2, 3, \dots, N$  and  $\omega_1 + \omega_2 < 1$ . Let  $K$  be the  $K$ -mapping generated by  $T_1, T_2, T_3, \dots, T_N$  and

$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ . Let  $f : C \rightarrow C$  be a contraction with coefficient  $\theta \in [0, 1)$ . Pick any  $x_0 \in C$ , let  $\{x_n\}$  be sequence generated by

$$\begin{cases} x_{n+1} = K(y_n), \\ y_n = \alpha_n x_n + \beta_n f(x_n) + \gamma_n K\left(\frac{x_n + x_{n+1}}{2}\right), \end{cases}$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying the conditions (i)-(iv) and

$$(viii) \lim_{n \rightarrow \infty} \|x_n - K(x_n)\| = 0.$$

Then  $\{x_n\}$  converges strongly to a fixed point  $x^*$  of the mappings  $\{T_i\}_{i=1}^N$ , which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle, \quad \forall y \in F(K) = \bigcap_{i=1}^N F(T_i).$$

In other words,  $x^*$  is the unique fixed point of the contraction  $P_{\bigcap_{i=1}^N F(T_i)} f$ , that is,  $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$ .

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### References

- [1] G. Cai, S.Q. Bu, Hybrid algorithm for generalized mixed equilibrium problems and variational inequality problems and fixed point problems, *Comput. Math. Appl.*, **62** (2011), 4772-4782. doi: 10.1016/j.camwa.2011.10.068
- [2] K. Goebel, W. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990).
- [3] A. Kangtunyakarn, S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings, *Nonlinear Anal.*, **71** (2009), 4448-4460. doi: 10.1016/j.na.2009.03.003
- [4] Y.F. Ke, C.F. Ma, A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem, *Fixed point Theory Appl.*, **126** (2013), 21 pages. doi: 10.1186/1687-1812-2013-126
- [5] Y.F. Ke, C.F. Ma, The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces, *Fixed point Theory Appl.*, **190** (2015), 21 pages. doi: 10.1186/s13663-015-0439-6

- [6] A. Moudafi, Viscosity approximation methods for fixed points problems, *J. Math. Anal. Appl.*, **241** (2000), 46-55. **doi:** 10.1006/jmaa.1999.6615
- [7] M. Su, H.K. Xu, Remarks on gradient-projection algorithm, *J. Nonlinear Anal. Optim.*, **1** (2010), 35-43.
- [8] S. Suwannaut, A. Kangtunyakarn, Strong convergence theorem for the modified generalized equilibrium problem and fixed point problem of strictly pseudo-contractive mappings, *Fixed Point Theory Appl.*, **86** (2014), 31 pages. **doi:** 10.1186/1687-1812-2014-86
- [9] H.K. Xu, M.A. Alghamdi, N. Shahzad., The viscosity technique for the implicit mid point rule of nonexpansive mappings in Hilbert spaces, *Fixed point Theory Appl.*, **41** (2015), 12 pages. **doi:** 10.1186/s13663-015-0282-9