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COMMON COUPLED FIXED POINTS FOR FOUR MAPS USING α-ADMISSIBLE FUNCTIONS IN COMPLEX VALUED b- METRIC SPACES

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Abstract: In this paper, we obtain a unique common coupled fixed point theorem for four self maps using α-admissible function in complex valued b-metric spaces. Also we give an example to illustrate our main theorem.

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1. Introduction and Preliminaries

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and

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uniqeness of solutions to various mathematical models.

Azam et al. [1] introduced the notion of complex valued metric space which is a generalization of the classical metric space and obtained sufficent conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive condition.Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer [3, 4, 6, 8, 11, 12, 14, 16, 20, 21, 28]. Recently some authors,for example[7, 13, 17, 22, 24, 25, 26], obtained coupled and common coupled fixed point point theorems for a pair of mappings in complex valued metric spaces.

In this paper,we prove a unique common fixed point theorem for two pairs of mappings satisfying a contractive condition of rational type in the frame work of complex valued b-metric spaces using α-admissible function. The proved result generalizes and extends some of the results of [13, 22, 24, 25].

To begin with, we recall some basic definitions, notations and results.

Throughout this paper $\mathcal{R}$, $\mathcal{R}^+$, $\mathcal{N}$ and $\mathcal{C}$ denote the set of all real numbers, non-negative real numbers, positive integers and complex numbers respectively.First we refer the following preliminaries.

Let $z_1, z_2 \in \mathcal{C}$. Define a partial order $\precsim$ on $\mathcal{C}$ follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2), Im(z_1) \le Im(z_2)$.

Thus $z_1 \precsim z_2$ if one of the following holds:

$(1) Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

$(2) Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

$(3) Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

$(4) Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Clearly $z_1 \precsim z_2 \Rightarrow |z_1| \le |z_2|$.

Definition 1. ([1]) Let X be a non empty set. A function $d : X \times X \to \mathcal{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.

The pair $(X, d)$is called a complex valued metric space.

Now, we briefly review the notation about complex valued b-metric spaces.

Definition 2. ([9]) Let X be a non empty set and $s \geq 1$. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b- metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \precsim s\left[d(x, z) + d(z, y)\right].$

The pair $(X, d)$is called a complex valued b- metric space.

Remark 1. Let (X, d) be a complex valued b-metric space. Then

(i)$|d(x, y)|$ *or* $|d(u, v)| < |1 + d(x, y) + d(u, v)|, \forall x, y, u, v \in X.$

(ii) If $x \neq y$ then $|d(x, y)| > 0$.

(iii) For $0 \leq k < 1$ and $z, w \in \mathcal{C}$, if $|z| \leq k|w|$ and $|w| \leq k|z|$ then $z = w = 0$.

Definition 3. ([9]) Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathcal{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathcal{C}$ such that $B(x, r) \cap (X - A) \neq \phi$.

(iii) A subset $B \subseteq X$ is called open whenever each point of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B is in B.

(v) The family $F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}$ is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topologyτ_c is Hausdorff.

Let $\{x_n\}$ be a sequence in X and $x \in X$.If for every $c \in \mathcal{C}$ with $0 \preceq c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0, d(x_n, x) \prec c$,then $\{x_n\}$ is said to be convergent to x and x is the limit point of $\{x_n\}$.We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathcal{C}$ with $0 \prec c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$, where $m \in \mathcal{N}$,then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complex valued b-metric space. We require the follwing lemmas.

Lemma 1.([9]) Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.([9]) Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is a Cauchy sequence if and only if

$|d(x_n, x_{n+m})| \to 0$ as $n, m \to \infty$.

One can easily prove the following lemma

Lemma 3. Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y respectively. Then

(i) $\frac{1}{s}\,|d(x,z)| \leq \lim\limits_{n\to\infty} |d(x_n, z)| \leq s\ \ |d(x,z)|$ for all $z \in X$,

(ii) $\frac{1}{s^2}\,|d(x,y)| \leq \lim\limits_{n\to\infty} |d(x_n, y_n)| \leq s^2\,|d(x,y)|$.

Recently Bhaskar and Lakshmikantham [23] introduced the concept of coupled fixed point and discussed some problems of the uniqueness of a coupled fixed point and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems. Later Lakshmikantham and Ciric [27] proved some coupled coincidence and coupled common fixed point results in partially ordered metric spaces.

Definition 4. Let X be a non empty set and $F : X \times X \to X$ and $S : X \to X$.

(i)([27]) An element $(x, y) \in X \times X$ is called a common coupled fixed point of mappings F and S if $x = Sx = F(x, y)$ and $y = Sy = F(y, x)$.

(ii)([10]) The pair (F, S) is called w-compatible if $S(F(x, y)) = F(Sx, Sy)$ and $S(F(y, x)) = F(Sy, Sx)$, whenever $Sx = F(x, y)$ and $Sy = F(y, x)$.

Now we extend the definition of compatible maps introduced by Jungck [5] in metric spaces to complex valued b-metric spaces to the maps $F : X \times X \to X$ and $S : X \to X$ as follows.

Definition 5. Let (X, d) be a complex valued b-metric space and $F : X \times X \to X$ and $S : X \to X$. Then the pair (F, f) is said to be compatible if

(i) $\lim\limits_{n\to\infty} |d(S(F(x_n, y_n)), F(Sx_n, Sy_n))| = 0$ and

(ii) $\lim\limits_{n\to\infty} |d(S(F(y_n, x_n)), F(Sy_n, Sx_n))| = 0$

whenever there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $F(x_n, y_n) \to t, Sx_n \to t$ and $F(y_n, x_n) \to t', Sy_n \to t'$ for some $t, t' \in X$.

Samet et al. [2] introduced the notion of α- admissible mappings as follows

Definition 6. ([2]) Let X be a non empty set, $T : X \to X$ and $\alpha : X \times X \to \mathcal{R}^+$ be mappings. Then T is called α- admissible if for all $x, y \in X$, we have $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

In the sequel,Kaushik et al. [18]introduced the following α-admissible mapping concept which is a generalization of the concept introduced by Mursaleen et al. [15].

Definition 7. ([18])Let X be a nonempty set and $\alpha : X^2 \times X^2 \to \mathcal{R}^+$ be a function.Let $F : X \times X \to X$ and $S : X \to X$ be mappings. Then F and S are said to be α-admissible if

$$\alpha((Sx, Sy), (Su, Sv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1,$$

for all $x, y, u, v \in X$.

If $S = I$(Identity map), then the above definition is the concept of Mursaleen et al. [15].

In this paper, we give an extension of the above definition for four maps.

Definition 8. Let X be a nonempty set and $\alpha : X^2 \times X^2 \to \mathcal{R}^+$.Let $F, G : X \times X \to X$ and $S, T : X \to X$ be mappings.Then we say that the pair (S, T) is α-admissible with respect to the pair (F, G) if

(i) $\alpha((Sx, Sy), (Tu, Tv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (G(u, v), G(v, u))) \geq 1$,

(ii) $\alpha((Tx, Ty), (Su, Sv)) \geq 1 \Rightarrow \alpha((G(x, y), G(y, x)), (F(u, v), F(v, u))) \geq 1$, for all $x, y, u, v \in X$.

Combining all the above notions,now we prove our main result.

2. Main Result

Theorem 9. *Let (X, d) be a complete complex valued b-metric space with $s \geq 1$.Let $\alpha : X^2 \times X^2 \to \mathcal{R}^+$ be a function and $F, G : X \times X \to X$ and $S, T : X \to X$ be mappings satisfying the following:*

(9.1) $F(X \times X) \subseteq T(X)$, $G(X \times X) \subseteq S(X)$,

(9.2)

$$\begin{aligned}\alpha((Sx, Sy),(Tu, Tv))\ s^4\ d(F(x, y), G(u, v)) &\precsim a_1\ d(Sx, Tu)\\ &+ a_2\ d(Sy, Tv) + a_3\ d(Sx, F(x, y))\end{aligned}$$

$$+ a_4\ d(Sy, F(y,x)) + a_5\ d(Tu, G(u,v)) + a_6\ d(Tv, G(v,u))$$
$$+ a_7 \frac{d(Sx, F(x,y))\ d(Tu, G(u,v))}{1 + s^4 d(Sx, Tu) + s^4 d(Sy, Tv)}$$
$$+ a_8 \frac{d(Tu, F(x,y))\ d(Sx, G(u,v))}{1 + s^4 d(Sx, Tu) + s^4 d(Sy, Tv)},$$

for all $x, y, u, v \in X$, where a_i,$i = 1, 2, \ldots.., 8$ are non-negative real numbers such that $\sum_{i=1}^{8} a_i < \frac{1}{s}$,

(9.3) *(a)* $\alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1$,

(b) $\alpha((Sy_1, Sx_1), (F(y_1, x_1), F(x_1, y_1))) \geq 1$,

(c) $\alpha((F(x_1, y_1), F(y_1, x_1)), (Sx_1, Sy_1)) \geq 1$ *and*

(d) $\alpha((F(y_1, x_1), F(x_1, y_1)), (Sy_1, Sx_1)) \geq 1$ *for some* $x_1, y_1 \in X$,

(9.4) *the pair (S, T) is α-admissible with respect to the pair (F, G),*

(9.5)(a) *the pair (F, S) is compatible,S is continuous and the pair (G, T) is w-compatible and if there exist sequences $\{z_n\}, \{w_n\}$ in X such that*

$$\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1,$$
$$\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1,$$
$$\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1,$$
$$\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1,$$

with $z_n \to z$ and $w_n \to w$ imply

(i) $\alpha((Sz_{2n}, Sw_{2n}), (z_{2n+1}, w_{2n+1})) \geq 1, \alpha((Sw_{2n}, Sz_{2n}), (w_{2n+1}, z_{2n+1})) \geq 1$,

(ii) $\alpha((z, w), (z_{2n+1}, w_{2n+1})) \geq 1, \alpha((w, z), (w_{2n+1}, z_{2n+1})) \geq 1$,

(iii) $\alpha(z, w), (z, w)) \geq 1, \alpha(w, z), (w, z)) \geq 1$,

(iv) $\alpha((z, w), (Tz, Tw)) \geq 1, \alpha((w, z), (Tw, Tz)) \geq 1$ *for all* $n \in \mathcal{N}$.
(or)

(9.5)(b) *the pair (G, T) is compatible,T is continuous and the pair (F, S) is w-compatible and if there exist sequences $\{z_n\}, \{w_n\}$ in X such that*

$$\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1,$$
$$\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1,$$
$$\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1,$$

$$\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1,$$

with $z_n \to z$ *and* $w_n \to w$ *imply*

(i) $\alpha((z_{2n}, w_{2n}), (Tz_{2n+1}, Tw_{2n+1})) \geq 1, \alpha((w_{2n}, z_{2n}), (Tw_{2n+1}, Tz_{2n+1})) \geq 1,$

(ii) $\alpha(z_{2n}, w_{2n}), (z, w)) \geq 1, \alpha(w_{2n}, z_{2n}), (w, z)) \geq 1,$

(iii) $\alpha((z, w), (z, w)) \geq 1, \alpha((w, z), (w, z)) \geq 1,$

(iv) $\alpha((Sz, Sw), (z, w)) \geq 1, \alpha((Sw, Sz), (w, z)) \geq 1$ *for all* $n \in \mathcal{N}$.
Then F, G, S *and* T *have a common coupled fixed point.*

(9.6) *Further if we assume that* $\alpha((z, w), (z', w')) \geq 1$ *and* $\alpha((w, z), (w', z')) \geq 1$ *whenever* (z, w) *and* (z', w') *are common coupled fixed points of* F, G, S *and* T *then* F, G, S *and* T *have a unique common coupled fixed point in* $X \times X$.

Proof. Let x_1 and y_1 be in X satisfying (9.3). Now define the sequences $\{z_n\}$ and $\{w_n\}$ from (9.1) as follows:

$$\begin{aligned} z_{2n+1} =& F(x_{2n+1}, y_{2n+1}) = Tx_{2n+2}, \\ z_{2n+2} =& G(x_{2n+2}, y_{2n+2}) = Sx_{2n+3}, \\ w_{2n+1} =& F(y_{2n+1}, x_{2n+1}) = Ty_{2n+2}, \\ w_{2n+2} =& G(y_{2n+2}, x_{2n+2}) = Sy_{2n+3}, \text{ for } n = 0, 1, 2, 3, \cdots \end{aligned}$$

From $(9.3)(a)$, we have

$$\begin{aligned} &\alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1 \\ &\Rightarrow \alpha((Sx_1, Sy_1), (Tx_2, Ty_2)) \geq 1, \text{ from definition of } \{z_n\} \textit{ and } \{w_n\} \\ &\Rightarrow \alpha((F(x_1, y_1), F(y_1, x_1)), (G(x_2, y_2), G(y_2, x_2))) \geq 1, \text{ from(9.4)} \\ &\Rightarrow \alpha((z_1, w_1), (z_2, w_2)) \geq 1 \text{ from definition of } \{z_n\} \textit{ and } \{w_n\} \\ &\Rightarrow \alpha((Tx_2, Ty_2), (Sx_3, Sy_3)) \geq 1, \text{ from definition of } \{z_n\} \textit{ and } \{w_n\} \\ &\Rightarrow \alpha((G(x_2, y_2), G(y_2, x_2)), (F(x_3, y_3), F(y_3, x_3))) \geq 1, \text{ from(9.4)} \\ &\Rightarrow \alpha((z_2, w_2), (z_3, w_3)) \geq 1. \end{aligned}$$

Continuing in this way, we have

$$\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1, \quad \forall\ n. \tag{1}$$

Similarly from $(9.3)(c), (9.3)(b)$ and $(9.3)(d)$ we can obtain

$$\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1. \tag{2}$$

$$\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1. \tag{3}$$

$$\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1. \tag{4}$$

Let $R_n = \max\{d(z_n, z_{n+1}), d(w_n, w_{n+1})\}$.

Case (i) Suppose $R_{2m} = 0$ for some m. Then $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$. Now

$$\alpha((Sx_{2m+1}, Sy_{2m+1}),(Tx_{2m+2}, Ty_{2m+2})) = \alpha((z_{2m}, w_{2m}), (z_{2m+1}, w_{2m+1})) \geq 1,$$

from (1),

$$\begin{aligned} d(z_{2m+1}, z_{2m+2}) =& d(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2})) \\ \precsim & \alpha((Sx_{2m+1}, Sy_{2m+1}), (Tx_{2m+2}, Ty_{2m+2})) \\ & s^4 d(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2})) \\ \precsim & a_1 d(z_{2m}, z_{2m+1}) + a_2 d(w_{2m}, w_{2m+1}) + a_3 d(z_{2m}, z_{2m+1}) \\ & + a_4 d(w_{2m}, w_{2m+1}) + a_5 d(z_{2m+1}, z_{2m+2}) \\ & + a_6 d(w_{2m+1}, w_{2m+2}) \\ & + a_7 \frac{d(z_{2m}, z_{2m+1}) d(z_{2m+1}, z_{2m+2})}{1 + s^4 d(z_{2m}, z_{2m+1}) + s^4 d(w_{2m}, w_{2m+1})} \\ & + a_8 \frac{d(z_{2m+1}, z_{2m+1}) d(z_{2m}, z_{2m+2})}{1 + s^4 d(z_{2m}, z_{2m+1}) + s^4 d(w_{2m}, w_{2m+1})} \\ =& a_5 d(z_{2m+1}, z_{2m+2}) + a_6 d(w_{2m+1}, w_{2m+2}). \end{aligned}$$

Thus $|d(z_{2m+1}, z_{2m+2})| \leq (a_5 + a_6)\,|R_{2m+1}|$.

Similarly using (3), we have $|d(w_{2m+1}, w_{2m+2})| \leq (a_5 + a_6)\,|R_{2m+1}|$.

Thus $|R_{2m+1}| \leq (a_5 + a_6)\,|R_{2m+1}|$ which in turn yields that $z_{2m+1} = z_{2m+2}$ and $w_{2m+1} = w_{2m+2}$.

Continuing in this way, we get $z_{2m} = z_{2m+1} = z_{2m+2} = \cdots$ and $w_{2m} = w_{2m+1} = w_{2m+2} = \cdots$.

Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in (X, d).

Case (ii): Assume that $R_n \neq 0$ for all n.

As in Case (i),using Remark 1(i), we have

$$|d(z_{2n+1}, z_{2n+2}| \leq (a_1 + a_2 + a_3 + a_4)\,|R_{2n}| + (a_5 + a_6 + a_7)\,|R_{2n+1}|$$

and

$$|d(w_{2n+1}, w_{2n+2}| \leq (a_1 + a_2 + a_3 + a_4)\,|R_{2n}| + (a_5 + a_6 + a_7)\,|R_{2n+1}|.$$

Thus $|R_{2n+1}| \leq k\,|R_{2n}|$,where $k_1 = \frac{a_1+a_2+a_3+a_4}{1-a_5-a_6-a_7}$.

Similarly using (2) and (4), we have $|R_{2n+2}| \leq k_2\,|R_{2n+1}|$, where $k_2 = \frac{a_1+a_2+a_5+a_6}{1-a_3-a_4-a_7}$. Let $k = \max\{k_1, k_2\}$. Thus

$$\begin{aligned} |R_n| \leq &\ k\,|R_{n-1}|, \quad n = 2, 3, ... \\ \leq &\ k^{n-1}\,|R_1| \end{aligned} \tag{5}$$

For $m > n$,using (5) we have

$$\begin{aligned} |d(z_n, z_m)| &\leq s|d(z_n, z_{n+1})| + s^2|d(z_{n+1}, z_{n+2})| + ... + s^{m-n-1}|d(z_{m-1}, z_m)| \\ &\leq \left[sk^{n-1} + s^2k^n + ... + s^{m-n-1}k^{m-2}\right]\ |R_1| \\ &\leq \left[(sk)^{n-1} + (sk)^n + ... + (sk)^{m-2}\right]\ |R_1| \\ &\leq \frac{(sk)^{n-1}}{1-sk}\ |R_1| \to 0\ as\ n, m \to \infty, since\ sk < 1. \end{aligned}$$

Thus $\{z_n\}$ is a Cauchy sequence in (X, d). Similarly we can show that $\{w_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete,there exist $z, w \in X$ such that $\lim\limits_{n\to\infty} |d(z_n, z)| = 0$ and $\lim\limits_{n\to\infty} |d(w_n, w)| = 0$ from Lemma 1. Thus

$$\begin{aligned} z = \lim_{n\to\infty} F(x_{2n+1}, y_{2n+1}) &= \lim_{n\to\infty} G(x_{2n+2}, y_{2n+2}) \\ &= \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1} \end{aligned} \tag{6}$$

and

$$\begin{aligned} w = \lim_{n\to\infty} F(y_{2n+1}, x_{2n+1}) &= \lim_{n\to\infty} G(y_{2n+2}, x_{2n+2}) \\ &= \lim_{n\to\infty} Ty_{2n} = \lim_{n\to\infty} Sy_{2n+1} \end{aligned} \tag{7}$$

Suppose $(9.5)(a)$ holds.

Since S is continuous at z and w we have

$$SSx_{2n+1} \to Sz, SSy_{2n+1} \to Sw, S(F(x_{2n+1}, y_{2n+1})) \to Sz$$

and

$$S(F(y_{2n+1}, x_{2n+1})) \to Sw.$$

Since(F, S) is compatible and by Lemma 1, we have

$$\begin{aligned} |d(F(Sx_{2n+1}, Sy_{2n+1}), Sz)| \leq &s\,|d(S(F(x_{2n+1}, y_{2n+1})), F(Sx_{2n+1}, Sy_{2n+1}))| \\ &+ s\ \ |d(S(F(x_{2n+1}, y_{2n+1})), Sz)| \to 0. \end{aligned}$$

Hence $F(Sx_{2n+1}, Sy_{2n+1}) \to Sz$.

Similarly,we get $F(Sy_{2n+1}, Sx_{2n+1}) \to Sw$. Now from (9.5)$(a)(i)$ we have

$$\alpha((SSx_{2n+1}, SSy_{2n+1}), (Tx_{2n+2}, Ty_{2n+2})) \\ = \alpha((Sz_{2n}, Sw_{2n}), (z_{2n+1}, w_{2n+1})) \geq 1. \quad (8)$$

Consider

$$\begin{aligned} & 1 + s^2 d(Sz, z) + s^2 d(Sw, w) \\ & \precsim 1 + s^2[sd(Sz, SSx_{2n+1}) + s^2 d(SSx_{2n+1}, Tx_{2n+2}) + s^2 d(Tx_{2n+2}, z)] \\ & \qquad + s^2[sd(Sw, SSy_{2n+1}) + s^2 d(SSy_{2n+1}, Ty_{2n+2}) + s^2 d(Ty_{2n+2}, z)] \\ & = [1 + s^4 d(SSx_{2n+1}, Tx_{2n+2}) + s^4 d(SSy_{2n+1}, Ty_{2n+2})] \\ & \quad + s^3 d(Sz, SSx_{2n+1}) + s^4 d(Tx_{2n+2}, z) + s^3 d(Sw, SSy_{2n+1}) + s^4 d(Ty_{2n+2}, w). \end{aligned}$$

$$\begin{aligned} & \left|1 + s^2 d(Sz, z) + s^2 d(Sw, w)\right| \\ & \leq \left|1 + s^4 d(SSx_{2n+1}, Tx_{2n+2}) + s^4 d(SSy_{2n+1}, Ty_{2n+2})\right| + s^3 \left|d(Sz, SSx_{2n+1})\right| \\ & + s^4 \left|d(Tx_{2n+2}, z)\right| + s^3 \left|d(Sw, SSy_{2n+1})\right| + s^4 \left|d(Ty_{2n+2}, w)\right|. \end{aligned}$$

Letting $n \to \infty$ and using Lemma 1, we get

$$\left|1 + s^2 d(Sz, z) + s^2 d(Sw, w)\right| \\ \leq \lim_{n\to\infty} \left|1 + s^4 d(SSx_{2n+1}, Tx_{2n+2}) + s^4 d(SSy_{2n+1}, Ty_{2n+2})\right|. \quad (9)$$

Now we consider

$$\begin{aligned} s^2 \left|d(Sz, z)\right| = & s^4 \frac{1}{s^2} \left|d(Sz, z)\right| \\ \leq & s^4 \lim_{n\to\infty} \left|d(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2}))\right| \\ \leq & \lim_{n\to\infty} \alpha((SSx_{2n+1}, SSy_{2n+1}), (Tx_{2n+2}, Ty_{2n+2}))s^4 \\ & \left|d(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2}))\right| \\ \leq & \lim_{n\to\infty} [a_1 \left|d(SSx_{2n+1}, Tx_{2n+2})\right| + a_2 \left|d(SSy_{2n+1}, Ty_{2n+2})\right| \\ & + a_3 \left|d(SSx_{2n+1}, F(Sx_{2n+1}, Sy_{2n+1}))\right| \\ & + a_4 \left|d(SSy_{2n+1}, F(Sy_{2n+1}, Sx_{2n+1}))\right| \\ & + a_5 \left|d(Tx_{2n+2}, G(x_{2n+2}, y_{2n+2}))\right| + a_6 \left|d(Ty_{2n+2}, G(y_{2n+2}, x_{2n+2}))\right| \\ & + a_7 \frac{\left|d(SSx_{2n+1},\ F(Sx_{2n+1}, Sy_{2n+1}))\right| \ \left|d(Tx_{2n+2},\ G(x_{2n+2}, y_{2n+2}))\right|}{\left|1 + s^4 d(SSx_{2n+1}, Tx_{2n+2}) + s^4 d(SSy_{2n+1}, Ty_{2n+2})\right|} \\ & + a_8 \frac{\left|d(Tx_{2n+2}, F(Sx_{2n+1}, Sy_{2n+1}))\right| \ \left|d(SSx_{2n+1}, G(x_{2n+2}, y_{2n+2}))\right|}{\left|1 + s^4 d(SSx_{2n+1}, Tx_{2n+2}) + s^4 d(SSy_{2n+1}, Ty_{2n+2})\right|}\Bigg] \\ \leq & a_1 s^2 \left|d(Sz, z)\right| + a_2 s^2 \left|d(Sw, w)\right| + a_8 \frac{s^2 \left|d(Sz, z)\right| \ s^2 \left|d(Sz, z)\right|}{\left|1 + s^2 d(Sz, z) + s^2 d(Sw, w)\right|}, \end{aligned}$$

$$\begin{aligned}&\text{from Lemma 3 and (9)}\\ &<(a_1+a_8)s^2\,|d(Sz,z)|+a_2s^2\,|d(Sw,w)|\,,\\ &\text{from Remark 1(i).}\end{aligned}$$

Thus $|d(z,Sz)| < \frac{a_2}{1-a_1-a_8}\,|d(w,Sw)|$.

Similarly using $(9.5)(a)(i)$, we have

$$|d(w,Sw)| < \frac{a_2}{1-a_1-a_8}\,|d(z,Sz)|\,.$$

From Remark 1(iii), we have $Sz=z$ and $Sw=w$.

From $(9.5)(a)(ii)$, we have
$\alpha((Sz,Sw),(Tx_{2n+2},Ty_{2n+2}))=\alpha((z,w),(z_{2n+1},w_{2n+1}))\geq 1$.

From Lemma 3:

$$\begin{aligned}&\tfrac{1}{s}\,|d(F(z,w),z)|\\ &\leq \lim_{n\to\infty}|d(F(z,w),G(x_{2n+2},y_{2n+2}))|\\ &\leq \lim_{n\to\infty}\alpha((Sz,Sw),(Tx_{2n+2},Ty_{2n+2}))\ s^4\,|d(F(z,w),G(x_{2n+2},y_{2n+2}))|\\ &\leq \lim_{n\to\infty}\left[\begin{array}{c} a_1\,|d(z,z_{2n+1})|+a_2\,|d(w,w_{2n+1})|+a_3\,|d(z,F(z,w))|\\ a_4\,|d(w,F(w,z))|+a_5\,|d(z_{2n+1},z_{2n+2})|+a_6\,|d(w_{2n+1},w_{2n+2})|\\ +a_7\frac{|d(z,F(z,w))|\ |d(z_{2n+1},z_{2n+2})|}{|1+d(z,z_{2n+1})+d(w,w_{2n+1})|}+a_8\frac{|d(z_{2n+1},F(z,w))|\ |d(z,z_{2n+2})|}{|1+d(z,z_{2n+1})+d(w,w_{2n+1})|}\end{array}\right]\\ &\leq a_3\,|d(z,F(z,w))|+a_4\,|d(w,F(w,z))|\,,\ \text{by Lemmas 1 and 3.}\end{aligned}$$

Thus $|d(z,F(z,w))| \leq \frac{a_4 s}{1-a_3 s}\,|d(w,F(w,z))|$.

Similarly using $(9.5)(a)(ii)$,we have
$|d(w,F(w,z))| \leq \frac{a_4 s}{1-a_3 s}\,|d(z,F(z,w))|$. From Remark 1(iii), we have $z=F(z,w)$ and $F(w,z)=w$. Thus $Sz=z=F(z,w), Sw=w=F(w,z)$. Since $z=F(z,w)\in F(X\times X)\subseteq T(X)$ and $w=F(w,z)\in F(X\times X)\subseteq T(X)$,there exist u and v in X such that $z=Tu$ and $w=Tv$. From $(9.5)(a)(iii)$,we have $\alpha((Sz,Sw),(Tu,Tv))=\alpha((z,w),(z,w))\geq 1$. Now from (9.2), we have

$$\begin{aligned}d(z,G(u,v)) =&d(F(z,w),G(u,v))\\ \precsim&\alpha((Sz,Sw),(Tu,Tv))s^4d(F(z,w),G(u,v))\\ \precsim&a_1d(z,z)+a_2d(w,w)+a_3d(z,z)+a_4d(w,w)+a_5d(z,G(u,v))\\ &+a_6d(w,G(v,u))+a_7\frac{d(z,z)\ d(z,G(u,v))}{1+s^4d(z,z)+s^4d(w,w)}\\ &+a_8\frac{d(z,z)\ d(z,G(u,v))}{1+s^4d(z,z)+s^4d(w,w)}\\ =&a_5d(z,G(u,v))+a_6d(w,G(v,u)).\end{aligned}$$

Thus $|d(z,G(u,v)| \leq \frac{a_6}{1-a_5} |d(w,G(v,u))|$.

Similarly using (9.5)$(a)(iii)$,we have

$$|d(w,G(v,u)| \leq \frac{a_6}{1-a_5} |d(z,G(u,v))|.$$

From Remark 1(iii), we have $z = G(u,v)$ and $w = G(v,u)$.

Since the pair(G,T) is w-compatible, we have $Tz = G(z,w)$ and $w = G(w,z)$. From (9.5)$(a)(iv)$,we have
$\alpha((Sz,Sw),(Tz,Tw)) = \alpha((z,w),(Tz,Tw)) \geq 1$. From (9.2), we have

$$\begin{aligned}
|d(z,Tz)| =& d(F(z,w),G(z,w)) \\
\leq& \alpha((Sz,Sw),(Tz,Tw))s^4 |d(F(z,w),G(z,w))| \\
\leq& a_1 |d(z,Tz)| + a_2 |d(w,Tw)| + a_3 |d(z,z)| + a_4 |d(w,w)| \\
&+ a_5 |d(Tz,Tz)| + a_6 |d(Tw,Tw)| + a_7 \frac{|d(z,z)| \ |d(Tz,G(z,w))|}{|1+s^4 d(z,Tz)+s^4 d(w,Tw)|} \\
&+ a_8 \frac{|d(z,Tz)| \ |d(z,Tz)|}{|1+s^4 d(z,Tz)+s^4 d(w,Tw)|} \\
\leq& a_1 |d(z,Tz)| + a_2 |d(w,Tw)| + a_8 \ s^4 \frac{|d(z,Tz)| \ |d(z,Tz)|}{|1+s^4 d(z,Tz)+s^4 d(w,Tw)|} \\
\leq& a_1 |d(z,Tz)| + a_2 |d(w,Tw)| + a_8 |d(z,Tz)|, fromRemark\ 1(i).
\end{aligned}$$

Thus $|d(z,Tz)| < \frac{a_2}{1-a_1-a_8} |d(w,Tw)|$.

Similarly using (9.5)$(a)(iv)$, we have

$$|d(w,Tw)| < \frac{a_2}{1-a_1-a_8} |d(z,Tz)|.$$

Thus from Remark 1(iii), we have $Tz = z$ and $Tw = w$. Thus (z,w) is a common coupled fixed point of F, G, S and T.

Uniqueness of common coupled fixed point of F, G, S and T follows easily from (9.6) and (9.2). □

Now we give an example to illustrate Theorem 9.

Example 1. Let $X = [0,2]$ and $d(x,y) = i\,|x-y|^2,\ \forall\ x,y \in X$.

Define $F,G : X \times X \to X$ by $F(x,y) = \frac{x^2+y^2}{24}$ and $G(x,y) = \frac{x^2+y^2}{36}$ and $S,T : X \to X$ by $Sx = \frac{x^2}{2}$ and $Tx = \frac{x^2}{3}$.

Define $\alpha : X^2 \times X^2 \to \mathcal{R}^+$ by

$$\alpha((x,y),(u,v)) = \begin{cases} 1, & \text{if } x,y,u,v \in [0,\sqrt{3}], \\ 0, & \text{otherwise} \end{cases}$$

Clearly $F(X \times X) \subseteq T(X)$ and $G(X \times X) \subseteq S(X)$. To verify (9.2) we consider the following two cases.

Case (a) Suppose $x, y, u, v \in [0, \sqrt{3}]$.
Then $\alpha((Sx, Sy), (Tu, Tv)) = \alpha((\frac{x^2}{2}, \frac{y^2}{2}), (\frac{u^2}{3}, \frac{v^2}{3})) = 1$,

$$\begin{aligned}\alpha((Sx, Sy), (Tu, Tv))s^4 d(F(x, y), G(u, v)))&\\ &= 16i\left|\frac{x^2+y^2}{24} - \frac{u^2+v^2}{36}\right|^2\\ &= \frac{16}{144}i\left|\frac{3x^2 - 2u^2 + 3y^2 - 2v^2}{6}\right|^2\\ &\leq \frac{2}{9}i[\left|\frac{3x^2 - 2u^2}{6}\right|^2 + \left|\frac{3y^2 - 2v^2}{6}\right|^2]\\ &= \frac{2}{9}[d(Sx, Tu) + d(Sy, Tv)].\end{aligned}$$

Here $a_1 = a_2 = \frac{2}{9}, a_3 = a_4 = 0$. Clearly $\sum_{i=1}^{4} a_i < \frac{1}{s}$.

Case (b): Atleast one of $x, y, u, v \notin [0, \sqrt{3}]$.
Then $\alpha((Sx, Sy), (Tu, Tv)) = \alpha((\frac{x^2}{2}, \frac{y^2}{2}), (\frac{u^2}{3}, \frac{v^2}{3}))$.

Sub case: If $\frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3} \in (1, \sqrt{3}]$ then

$$\alpha((Sx, Sy), (Tu, Tv)) = 1.$$

The inequality (9.2) holds as in case (a).

Sub case: If atleast one of If $\frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3} \in (\sqrt{3}, 2]$ then

$$\alpha((Sx, Sy), (Tu, Tv)) = 0.$$

Hence (9.2) holds.

By definition of α, the condition (9.3) with $x_1 = 0 = y_1$ and (9.4) are satisfied clearly.

To verify the compatibility of the pair(F, S), let us consider the sequences $\{x_n\}$, $\{y_n\}$ in X such that $F(x_n, y_n) \to t$, $Sx_n \to t$, $F(y_n, x_n) \to t'$ and $Sx_n \to t'$ for some $t, t' \in X$, $F(x_n, y_n) \to t \Rightarrow \frac{x_n^2 + y_n^2}{24} \to t$ and $F(y_n, x_n) \to t' \Rightarrow \frac{y_n^2 + x_n^2}{24} \to t'$. Hence $t = t'$, $Sx_n \to t \Rightarrow \frac{x_n^2}{2} \to t$ and $Sy_n \to t' \Rightarrow \frac{y_n^2}{2} \to t' = t$.

Now $\frac{x_n^2+y_n^2}{24} \to \frac{4t^2+4t^2}{24} = \frac{t^2}{3}$. Hence $\frac{t^2}{3} = t \Rightarrow t = 0$. Thus $x_n^2, y_n^2 \to 0$.

$$|d(S(F(x_n, y_n)), F(Sx_n, Sy_n))| = \left|\frac{1}{2}\left(\frac{x_n^2 + y_n^2}{24}\right)^2 - \frac{1}{24}\left(\frac{x_n^4}{4} + \frac{y_n^4}{4}\right)\right|^2 \to 0$$

as $n \to \infty$.

Similarly $|d(S(F(y_n, x_n)), F(Sy_n, Sx_n))| \to 0$. Thus the pair (F, S) is compatible. Also it is clear that the pair (G, T) is w-compatible.

By definition of α, one can easily verify the conditions (9.5) and (9.6).

Thus all the conditions of Theorem 9 are satisfied and $(0, 0)$ is the unique common fixed point of F, G, S and T.

If $\alpha((x, y), (u, v)) = 1$ for all $x, y, u, v \in X$ and $s = 1$ in Theorem 9, we have the following corollary.

Corollary 10. *Let (X, d) be a complete complex valued metric space. Let $F, G : X \times X \to X$ and $S, T : X \to X$ be mappings satisfying the following :*

(10.1) $F(X \times X) \subseteq T(X)$, $G(X \times X) \subseteq S(X)$,

(10.2)
$$\begin{aligned} &d(F(x, y), G(u, v)) \\ &\quad\precsim a_1\ d(Sx, Tu) + a_2\ d(Sy, Tv) + a_3\ d(Sx, F(x, y)) \\ &\qquad + a_4\ d(Sy, F(y, x)) + a_5\ d(Tu, G(u, v)) \\ &\quad + a_6\ d(Tv, G(v, u)) + a_7 \tfrac{d(Sx, F(x,y))\ d(Tu, G(u,v))}{1 + d(Sx, Tu) + d(Sy, Tv)} \\ &\qquad + a_8 \tfrac{d(Tu, F(x,y))\ d(Sx, G(u,v))}{1 + d(Sx, Tu) + d(Sy, Tv)} \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i, i = 1, 2, ..., 8$ are non-negative real numbers such that $\sum_{i=1}^{8} a_i < 1$,

(10.3)(a) *the pair (F, S) is compatible, S is continuous and the pair (G, T) is w-compatible*

(or)

(b) *the pair (G, T) is compatible, T is continuous and the pair (F, S) is w-compatible*

Then F, G, S and T have a unique common coupled fixed point.

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