COMMON COUPLED FIXED POINTS FOR FOUR MAPS USING $\alpha$-ADMISSIBLE FUNCTIONS IN COMPLEX VALUED $b$-METRIC SPACES

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Abstract: In this paper, we obtain a unique common coupled fixed point theorem for four self maps using $\alpha$-admissible function in complex valued $b$-metric spaces. Also we give an example to illustrate our main theorem.

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1. Introduction and Preliminaries

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and
uniqeness of solutions to various mathematical models.

Azam et al. [1] introduced the notion of complex valued metric space which is a generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive condition. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer [3, 4, 6, 8, 11, 12, 14, 16, 20, 21, 28]. Recently some authors, for example [7, 13, 17, 22, 24, 25, 26], obtained coupled and common coupled fixed point theorems for a pair of mappings in complex valued metric spaces.

In this paper, we prove a unique common fixed point theorem for two pairs of mappings satisfying a contractive condition of rational type in the framework of complex valued $b$-metric spaces using $\alpha$-admissible function. The proved result generalizes and extends some of the results of [13, 22, 24, 25].

To begin with, we recall some basic definitions, notations and results.

Throughout this paper $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{C}$ denote the set of all real numbers, non-negative real numbers, positive integers and complex numbers respectively. First we refer the following preliminaries.

Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ follows:

$z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2), \text{Im}(z_1) \leq \text{Im}(z_2)$.

Thus $z_1 \preceq z_2$ if one of the following holds:

1. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
2. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
3. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,
4. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

Clearly $z_1 \preceq z_2 \Rightarrow |z_1| \leq |z_2|.$

**Definition 1.** ([1]) Let $X$ be a non empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on $X$ if for all $x, y, z \in X$ the following conditions are satisfied:

1. $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair $(X, d)$ is called a complex valued metric space.

Now, we briefly review the notation about complex valued $b$-metric spaces.
Definition 2. ([9]) Let \( X \) be a non empty set and \( s \geq 1 \). A function \( d : X \times X \to \mathbb{C} \) is called a complex valued \( b \)-metric on \( X \) if for all \( x, y, z \in X \) the following conditions are satisfied:

(i) \( 0 \preceq d(x, y) \) and \( d(x, y) = 0 \) if and only if \( x = y \);
(ii) \( d(x, y) = d(y, x) \);
(iii) \( d(x, y) \preceq s [d(x, z) + d(z, y)] \).

The pair \( (X, d) \) is called a complex valued \( b \)-metric space.

Remark 1. Let \( (X, d) \) be a complex valued \( b \)-metric space.

(i) \(
|d(x, y)| \text{ or } |d(u, v)| < |1 + d(x, y) + d(u, v)|, \forall x, y, u, v \in X.
\)
(ii) If \( x \neq y \) then \( |d(x, y)| > 0 \).
(iii) For \( 0 \leq k < 1 \) and \( z, w \in \mathbb{C} \), if \( |z| \leq k |w| \) and \( |w| \leq k |z| \) then \( z = w = 0 \).

Definition 3. ([9]) Let \( (X, d) \) be a complex valued \( b \)-metric space.

(i) A point \( x \in X \) is called interior point of a set \( A \subseteq X \) whenever there exists \( 0 \prec r \in \mathbb{C} \) such that \( B(x, r) = \{ y \in X : d(x, y) \prec r \} \subseteq A \).
(ii) A point \( x \in X \) is called a limit point of a set \( A \subseteq X \) whenever there exists \( 0 \prec r \in \mathbb{C} \) such that \( B(x, r) \cap (X - A) \neq \phi \).
(iii) A subset \( B \subseteq X \) is called open whenever each point of \( B \) is an interior point of \( B \).
(iv) A subset \( B \subseteq X \) is called closed whenever each limit point of \( B \) is in \( B \).
(v) The family \( F = \{ B(x, r) : x \in X \text{ and } 0 \prec r \} \) is a sub basis for a topology on \( X \). We denote this complex topology by \( \tau_c \). Indeed, the topology \( \tau_c \) is Hausdorff.

Let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \). If for every \( c \in \mathbb{C} \) with \( 0 \preceq c \) there is \( n_0 \in \mathcal{N} \) such that for all \( n > n_0, d(x_n, x) \prec c \), then \( \{x_n\} \) is said to be convergent to \( x \) and \( x \) is the limit point of \( \{x_n\} \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \). If for every \( c \in \mathbb{C} \) with \( 0 \prec c \) there is \( n_0 \in \mathcal{N} \) such that for all \( n > n_0, d(x_n, x_{n+m}) \prec c \), where \( m \in \mathcal{N} \), then \( \{x_n\} \) is called a Cauchy sequence in \( (X, d) \). If every Cauchy sequence is convergent in \( (X, d) \) then \( (X, d) \) is called a complete complex valued \( b \)-metric space. We require the following lemmas.
Lemma 1. ([9]) Let \((X, d)\) be a complex valued \(b\)-metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(|d(x_n, x)| \to 0\) as \(n \to \infty\).

Lemma 2. ([9]) Let \((X, d)\) be a complex valued \(b\)-metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if

\[|d(x_n, x_{n+m})| \to 0\] as \(n, m \to \infty\).

One can easily prove the following lemma

Lemma 3. Let \((X, d)\) be a complex valued \(b\)-metric space and let \(\{x_n\}\) and \(\{y_n\}\) be sequences in \(X\) converging to \(x\) and \(y\) respectively. Then

(i) \(\frac{1}{s} |d(x, z)| \leq \lim_{n \to \infty} |d(x_n, z)| \leq s \ |d(x, z)|\) for all \(z \in X\),

(ii) \(\frac{1}{s^2} |d(x, y)| \leq \lim_{n \to \infty} |d(x_n, y_n)| \leq s^2 \ |d(x, y)|\).

Recently Bhaskar and Lakshmikantham [23] introduced the concept of coupled fixed point and discussed some problems of the uniqueness of a coupled fixed point and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems. Later Lakshmikantham and Ciric [27] proved some coupled coincidence and coupled common fixed point results in partially ordered metric spaces.

Definition 4. Let \(X\) be a non empty set and \(F : X \times X \to X\) and \(S : X \to X\).

(i) ([27]) An element \((x, y) \in X \times X\) is called a common coupled fixed point of mappings \(F\) and \(S\) if \(x = Sx = F(x, y)\) and \(y = Sy = F(y, x)\).

(ii) ([10]) The pair \((F, S)\) is called \(w\)-compatible if \(S(F(x, y)) = F(Sx, Sy)\) and \(S(F(y, x)) = F(Sy, Sx)\), whenever \(Sx = F(x, y)\) and \(Sy = F(y, x)\).

Now we extend the definition of compatible maps introduced by Jungck [5] in metric spaces to complex valued \(b\)-metric spaces to the maps \(F : X \times X \to X\) and \(S : X \to X\) as follows.

Definition 5. Let \((X, d)\) be a complex valued \(b\)-metric space and \(F : X \times X \to X\) and \(S : X \to X\). Then the pair \((F, f)\) is said to be compatible if

(i) \(\lim_{n \to \infty} |d(S(F(x_n, y_n)), F(Sx_n, Sy_n))| = 0\)

(ii) \(\lim_{n \to \infty} |d(S(F(y_n, x_n)), F(Sy_n, Sx_n))| = 0\)

whenever there exist sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(F(x_n, y_n) \to t, Sx_n \to t\) and \(F(y_n, x_n) \to t', Sy_n \to t'\) for some \(t, t' \in X\).

Samet et al. [2] introduced the notion of \(\alpha\)-admissible mappings as follows
Definition 6. ([2]) Let $X$ be a non empty set, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be mappings. Then $T$ is called $\alpha$-admissible if for all $x, y \in X$, we have $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

In the sequel, Kaushik et al. [18] introduced the following $\alpha$-admissible mapping concept which is a generalization of the concept introduced by Mursaleen et al. [15].

Definition 7. ([18]) Let $X$ be a non-empty set and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be a function. Let $F : X \times X \to X$ and $S : X \to X$ be mappings. Then $F$ and $S$ are said to be $\alpha$-admissible if

$$\alpha((Sx, Sy), (Su, Sv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (u, v), F(v, u))) \geq 1,$$

for all $x, y, u, v \in X$.

If $S = I$(Identity map), then the above definition is the concept of Mursaleen et al. [15].

In this paper, we give an extension of the above definition for four maps.

Definition 8. Let $X$ be a non-empty set and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$. Let $F, G : X \times X \to X$ and $S, T : X \to X$ be mappings. Then we say that the pair $(S, T)$ is $\alpha$-admissible with respect to the pair $(F, G)$ if

(i) $\alpha((Sx, Sy), (Tu, Tv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (u, v), G(v, u))) \geq 1,

(ii) $\alpha((Tx, Ty), (Su, Sv)) \geq 1 \Rightarrow \alpha((G(x, y), G(y, x)), (u, v), F(v, u))) \geq 1$, for all $x, y, u, v \in X$.

Combining all the above notions, now we prove our main result.

2. Main Result

Theorem 9. Let $(X, d)$ be a complete complex valued b-metric space with $s \geq 1$. Let $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be a function and $F, G : X \times X \to X$ and $S, T : X \to X$ be mappings satisfying the following:

(9.1) $F(X \times X) \subseteq T(X)$, $G(X \times X) \subseteq S(X)$,

(9.2)

$$\alpha((Sx, Sy), (Tu, Tv)) s^4 d(F(x, y), G(u, v)) \leq a_1 d(Sx, Tu) + a_2 d(Sy, Tv) + a_3 d(Sx, F(x, y))$$
\begin{align*}
&+ a_4 \, d(Sy, F(y, x)) + a_5 \, d(Tu, G(u, v)) + a_6 \, d(Tv, G(v, u)) \\
&+ a_7 \, \frac{d(Sx, F(x, y)) \, d(Tu, G(u, v))}{1 + s^4 d(Sx, Tu) + s^4 d(Sy, Tv)} \\
&+ a_8 \, \frac{d(Tu, F(x, y)) \, d(Sx, G(u, v))}{1 + s^4 d(Sx, Tu) + s^4 d(Sy, Tv)},
\end{align*}

for all \( x, y, u, v \in X \), where \( a_i, i = 1, 2, \ldots, 8 \) are non-negative real numbers such that \( \sum_{i=1}^{8} a_i < \frac{1}{s} \).

(9.3) (a) \( \alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1, \)

(b) \( \alpha((Sy_1, Sx_1), (F(y_1, x_1), F(x_1, y_1))) \geq 1, \)

(c) \( \alpha((F(x_1, y_1), F(y_1, x_1)), (Sx_1, Sy_1)) \geq 1 \) and

(d) \( \alpha((F(y_1, x_1), F(x_1, y_1)), (Sy_1, Sx_1)) \geq 1 \) for some \( x_1, y_1 \in X \),

(9.4) the pair \( (S, T) \) is \( \alpha \)-admissible with respect to the pair \( (F, G) \),

(9.5)(a) the pair \( (F, S) \) is compatible, \( S \) is continuous and the pair \( (G, T) \) is \( w \)-compatible and if there exist sequences \( \{z_n\}, \{w_n\} \) in \( X \) such that

\begin{align*}
\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) &\geq 1, \\
\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) &\geq 1, \\
\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) &\geq 1, \\
\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) &\geq 1,
\end{align*}

with \( z_n \to z \) and \( w_n \to w \) imply

(i) \( \alpha((Sz_{2n}, Sw_{2n}), (z_{2n+1}, w_{2n+1})) \geq 1, \alpha((Sw_{2n}, Sz_{2n}), (w_{2n+1}, z_{2n+1})) \geq 1, \)

(ii) \( \alpha((z, w), (z_{2n+1}, w_{2n+1})) \geq 1, \alpha((w, z), (w_{2n+1}, z_{2n+1})) \geq 1, \)

(iii) \( \alpha(z, w), (z, w)) \geq 1, \alpha(w, z), (w, z)) \geq 1, \)

(iv) \( \alpha((z, w), (Tz, Tw)) \geq 1, \alpha((w, z), (Tw, Tz)) \geq 1 \) for all \( n \in \mathcal{N}. \)

(or)

(9.5)(b) the pair \( (G, T) \) is compatible, \( T \) is continuous and the pair \( (F, S) \) is \( w \)-compatible and if there exist sequences \( \{z_n\}, \{w_n\} \) in \( X \) such that

\begin{align*}
\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) &\geq 1, \\
\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) &\geq 1, \\
\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) &\geq 1,
\end{align*}
\[ \alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1, \]

with \( z_n \to z \) and \( w_n \to w \) imply

(i) \( \alpha((z_{2n}, w_{2n}), (Tz_{2n+1}, Tw_{2n+1})) \geq 1, \alpha((w_{2n}, z_{2n}), (Tw_{2n+1}, Tz_{2n+1})) \geq 1, \)

(ii) \( \alpha(z_{2n}, w_{2n}), (z, w)) \geq 1, \alpha(w_{2n}, z_{2n}), (w, z)) \geq 1, \)

(iii) \( \alpha((z, w), (z, w)) \geq 1, \alpha((w, z), (w, z)) \geq 1, \)

(iv) \( \alpha((Sz, Sw), (z, w)) \geq 1, \alpha((Sw, Sz), (w, z)) \geq 1 \) for all \( n \in \mathbb{N} \).

Then \( F, G, S \) and \( T \) have a common coupled fixed point.

(9.6) Further if we assume that \( \alpha((z, w), (z', w')) \geq 1 \) and
\( \alpha((w, z), (w', z')) \geq 1 \) whenever \( (z, w) \) and \( (z', w') \) are common coupled fixed points of \( F, G, S \) and \( T \) then \( F, G, S \) and \( T \) have a unique common coupled fixed point in \( X \times X \).

**Proof.** Let \( x_1 \) and \( y_1 \) be in \( X \) satisfying (9.3). Now define the sequences \( \{z_n\} \) and \( \{w_n\} \) from (9.1) as follows:

\[
\begin{align*}
    z_{2n+1} &= F(x_{2n+1}, y_{2n+1}) = Tx_{2n+2}, \\
    z_{2n+2} &= G(x_{2n+2}, y_{2n+2}) = Sx_{2n+3}, \\
    w_{2n+1} &= F(y_{2n+1}, x_{2n+1}) = Ty_{2n+2}, \\
    w_{2n+2} &= G(y_{2n+2}, x_{2n+2}) = Sy_{2n+3}, \quad \text{for } n = 0, 1, 2, 3, \ldots
\end{align*}
\]

From (9.3)(a), we have

\[
\begin{align*}
    \alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) &\geq 1 \\
    \Rightarrow \alpha((Sx_1, Sy_1), (Tx_2, Ty_2)) &\geq 1, \text{ from definition of } \{z_n\} \text{ and } \{w_n\} \\
    \Rightarrow \alpha((F(x_1, y_1), F(y_1, x_1)), (G(x_2, y_2), G(y_2, x_2))) &\geq 1, \text{ from (9.4)} \\
    \Rightarrow \alpha((z_1, w_1), (z_2, w_2)) &\geq 1 \text{ from definition of } \{z_n\} \text{ and } \{w_n\} \\
    \Rightarrow \alpha((Tx_2, Ty_2), (Sx_3, Sy_3)) &\geq 1, \text{ from definition of } \{z_n\} \text{ and } \{w_n\} \\
    \Rightarrow \alpha((G(x_2, y_2), G(y_2, x_2)), (F(x_3, y_3), F(y_3, x_3))) &\geq 1, \text{ from (9.4)} \\
    \Rightarrow \alpha((z_2, w_2), (z_3, w_3)) &\geq 1.
\end{align*}
\]

Continuing in this way, we have

\[
\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1, \quad \forall \ n.
\]

(1)

Similarly from (9.3)(c), (9.3)(b) and (9.3)(d) we can obtain

\[
\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1.
\]

(2)
Let $R_n = \max \{d(z_n, z_{n+1}), d(w_n, w_{n+1})\}$.

**Case (i)** Suppose $R_{2m} = 0$ for some $m$. Then $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$. Now

$$\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1. \quad (3)$$

$$\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1. \quad (4)$$

from (1),

$$d(z_{2m+1}, z_{2m+2}) = d(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2}))$$

$$\geq \alpha((Sx_{2m+1}, Sy_{2m+1}), (Tx_{2m+2}, Ty_{2m+2}))$$

$$\geq a_1 d(z_{2m}, z_{2m+1}) + a_2 d(w_{2m}, w_{2m+1}) + a_3 d(z_{2m}, z_{2m+1}) + a_4 d(w_{2m}, w_{2m+1}) + a_5 d(z_{2m+1}, z_{2m+2}) + a_6 d(w_{2m+1}, w_{2m+2})$$

$$+ a_7 \frac{d(z_{2m}, z_{2m+1})d(z_{2m+1}, z_{2m+2})}{1 + s^4d(z_{2m}, z_{2m+1}) + s^4d(w_{2m}, w_{2m+1})}$$

$$+ a_8 \frac{d(z_{2m+1}, z_{2m+1})d(z_{2m}, z_{2m+2})}{1 + s^4d(z_{2m}, z_{2m+1}) + s^4d(w_{2m}, w_{2m+1})}$$

$$= a_5 d(z_{2m+1}, z_{2m+2}) + a_6 d(w_{2m+1}, w_{2m+2}).$$

Thus $|d(z_{2m+1}, z_{2m+2})| \leq (a_5 + a_6)|R_{2m+1}|$.

Similarly using (3), we have $|d(w_{2m+1}, w_{2m+2})| \leq (a_5 + a_6)|R_{2m+1}|$.

Thus $|R_{2m+1}| \leq (a_5 + a_6)|R_{2m+1}|$ which in turn yields that $z_{2m+1} = z_{2m+2}$ and $w_{2m+1} = w_{2m+2}$.

Continuing in this way, we get $z_{2m} = z_{2m+1} = z_{2m+2} = \cdots$ and $w_{2m} = w_{2m+1} = w_{2m+2} = \cdots$.

Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in $(X, d)$.

**Case (ii):** Assume that $R_n \neq 0$ for all $n$.

As in Case (i), using Remark 1(i), we have

$$|d(z_{2n+1}, z_{2n+2})| \leq (a_1 + a_2 + a_3 + a_4)|R_{2n}| + (a_5 + a_6 + a_7)|R_{2n+1}|$$

and

$$|d(w_{2n+1}, w_{2n+2})| \leq (a_1 + a_2 + a_3 + a_4)|R_{2n}| + (a_5 + a_6 + a_7)|R_{2n+1}|.$$
Thus $|R_{2n+1}| \leq k |R_{2n}|$, where $k_1 = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_5 - a_6 - a_7}$.

Similarly using (2) and (4), we have $|R_{2n+2}| \leq k_2 |R_{2n+1}|$, where $k_2 = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_3 - a_4 - a_7}$. Let $k = \max \{k_1, k_2\}$. Thus

$$|R_n| \leq k |R_{n-1}|, \quad n = 2, 3, ....$$

(5)

For $m > n$, using (5) we have

$$|d(z_n, z_m)| \leq s |d(z_n, z_{n+1})| + s^2 |d(z_{n+1}, z_{n+2})| + .... + s^{m-n-1} |d(z_{m-1}, z_m)|$$

$$\leq [sk^{n-1} + s^2 k^n + .... + s^{m-n-1} k^{m-2}] |R_1|$$

$$\leq [(sk)^{n-1} + (sk)^n + .... + (sk)^{m-2}] |R_1|$$

$$\leq \frac{(sk)^{n-1}}{1 - sk} |R_1| \to 0 \text{ as } n, m \to \infty, \text{ since } sk < 1.$$  

Thus $\{z_n\}$ is a Cauchy sequence in $(X, d)$. Similarly we can show that $\{w_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is complete, there exist $z, w \in X$ such that $\lim_{n \to \infty} |d(z_n, z)| = 0$ and $\lim_{n \to \infty} |d(w_n, w)| = 0$ from Lemma 1. Thus

$$z = \lim_{n \to \infty} F(x_{2n+1}, y_{2n+1}) = \lim_{n \to \infty} G(x_{2n+2}, y_{2n+2})$$

$$= \lim_{n \to \infty} T x_{2n} = \lim_{n \to \infty} S x_{2n+1}$$

(6)

and

$$w = \lim_{n \to \infty} F(y_{2n+1}, x_{2n+1}) = \lim_{n \to \infty} G(y_{2n+2}, x_{2n+2})$$

$$= \lim_{n \to \infty} T y_{2n} = \lim_{n \to \infty} S y_{2n+1}$$

(7)

Suppose (9.5)(a) holds.

Since $S$ is continuous at $z$ and $w$ we have

$$SSx_{2n+1} \to Sz, SSy_{2n+1} \to Sw, S(F(x_{2n+1}, y_{2n+1})) \to Sz$$

and

$$S(F(y_{2n+1}, x_{2n+1})) \to Sw.$$  

Since $(F, S)$ is compatible and by Lemma 1, we have

$$|d(F(Sx_{2n+1}, Sy_{2n+1}), Sz)| \leq s |d(S(F(x_{2n+1}, y_{2n+1})), F(Sx_{2n+1}, Sy_{2n+1}))|$$

$$+ s |d(S(F(x_{2n+1}, y_{2n+1})), Sz)| \to 0.$$
Hence \( F(Sx_{2n+1}, Sy_{2n+1}) \to Sz \).

Similarly, we get \( F(Sy_{2n+1}, Sx_{2n+1}) \to Sw \). Now from (9.5)(a)(i) we have

\[
\alpha((SSx_{2n+1}, SSy_{2n+1}), (Tx_{2n+2}, Ty_{2n+2})) = \alpha((Sz_{2n}, Sw_{2n}), (z_{2n+1}, w_{2n+1})) \geq 1. \tag{8}
\]

Consider

\[
\begin{align*}
1 + s^2d(Sz, z) + s^2d(Sw, w) & \leq 1 + s^2[sd(Sz, SSx_{2n+1}) + s^2d(SSx_{2n+1}, Tx_{2n+2}) + s^2d(Tx_{2n+2}, z)] \\
& + s^2[sd(Sw, SSy_{2n+1}) + s^2d(SSy_{2n+1}, Ty_{2n+2}) + s^2d(Ty_{2n+2}, z)] \\
& = [1 + s^4d(SSx_{2n+1}, Tx_{2n+2}) + s^4d(SSy_{2n+1}, Ty_{2n+2})] \\
& + s^3d(Sz, SSx_{2n+1}) + s^4d(Tx_{2n+2}, z) + s^3d(Sw, SSy_{2n+1}) + s^4d(Ty_{2n+2}, w).
\end{align*}
\]

Letting \( n \to \infty \) and using Lemma 1, we get

\[
\begin{align*}
1 + s^2d(Sz, z) + s^2d(Sw, w) & \leq \lim_{n \to \infty} [1 + s^4d(SSx_{2n+1}, Tx_{2n+2}) + s^4d(SSy_{2n+1}, Ty_{2n+2})]. \tag{9}
\end{align*}
\]

Now we consider

\[
\begin{align*}
s^2 |d(Sz, z)| & = s^4 \frac{1}{s^2} |d(Sz, z)| \\
& \leq s^4 \lim_{n \to \infty} |d(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2}))| \\
& \leq \lim_{n \to \infty} \alpha((SSx_{2n+1}, SSy_{2n+1}), (Tx_{2n+2}, Ty_{2n+2}))s^4 \\
& |d(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2}))| \\
& \leq \lim_{n \to \infty} |a_1 |d(SSx_{2n+1}, Tx_{2n+2})| + a_2 |d(SSy_{2n+1}, Ty_{2n+2})| \\
& + a_3 |d(SSx_{2n+1}, F(Sx_{2n+1}, Sy_{2n+1}))| \\
& + a_4 |d(SSy_{2n+1}, F(Sy_{2n+1}, Sx_{2n+1}))| \\
& + a_5 |d(Tx_{2n+2}, G(x_{2n+2}, y_{2n+2}))| + a_6 |d(Ty_{2n+2}, G(y_{2n+2}, x_{2n+2}))| \\
& + a_7 \frac{|d(SSx_{2n+1}, F(Sx_{2n+1}, Sy_{2n+1}))| |d(Tx_{2n+2}, G(x_{2n+2}, y_{2n+2}))|}{|1 + s^4d(SSx_{2n+1}, Tx_{2n+2}) + s^4d(SSy_{2n+1}, Ty_{2n+2})|} \\
& + a_8 \frac{|d(Tx_{2n+2}, F(Sx_{2n+1}, Sy_{2n+1}))| |d(SSx_{2n+1}, G(x_{2n+2}, y_{2n+2}))|}{|1 + s^4d(SSx_{2n+1}, Tx_{2n+2}) + s^4d(SSy_{2n+1}, Ty_{2n+2})|} \\
& \leq a_1 s^2 |d(Sz, z)| + a_2 s^2 |d(Sw, w)| + a_8 \frac{s^2 |d(Sz, z)| s^2 |d(Sz, z)|}{|1 + s^2d(Sz, z) + s^2d(Sw, w)|},
\end{align*}
\]
from Lemma 3 and (9)
\[ \langle (a_1 + a_8) s^2 |d(Sz, z)| + a_2 s^2 |d(Sw, w)|, \]
from Remark 1(i).

Thus \( |d(z, Sz)| < \frac{a_2}{1 - a_1 - a_8} |d(w, Sw)|. \)

Similarly using (9.5)(a)(i), we have
\[ |d(w, Sw)| < \frac{a_2}{1 - a_1 - a_8} |d(z, Sz)|. \]

From Remark 1(iii), we have \( Sz = z \) and \( Sw = w. \)

From (9.5)(a)(ii), we have
\[ \alpha((S_z, Sw), (T x_{2n+2}, T y_{2n+2})) = \alpha((z, w), (z_{2n+1}, w_{2n+1})) \geq 1. \]

From Lemma 3:
\[
\frac{1}{s^4} |d(F(z, w), z)|
\leq \lim_{n \to \infty} |d(F(z, w), G(x_{2n+2}, y_{2n+2}))|
\leq \lim_{n \to \infty} \alpha((S_z, Sw), (T x_{2n+2}, T y_{2n+2})) s^4 |d(F(z, w), G(x_{2n+2}, y_{2n+2}))|
\leq \lim_{n \to \infty} \left[ \frac{a_1 |d(z, z_{2n+1})| + a_2 |d(w, w_{2n+1})| + a_3 |d(z, F(z, w))|}{a_4 |d(w, F(w, z))| + a_5 |d(z_{2n+1}, z_{2n+2})| + a_6 |d(w_{2n+1}, w_{2n+2})|}
+ a_7 \frac{|d(z, F(z, w))| |d(z_{2n+1}, z_{2n+2})| |d(z, F(z, w))|}{|d(z_{2n+1}, z_{2n+2})| + |d(w_{2n+1}, w_{2n+2})|}
+ a_8 \frac{a_1 |d(z_{2n+1}, F(z, w))| |d(z_{2n+1}, F(z, w))|}{|d(z_{2n+1}, F(z, w))| + |d(w_{2n+1}, w_{2n+2})|}ight]
\leq a_3 |d(z, F(z, w))| + a_4 |d(w, F(w, z))|, \text{ by Lemmas 1 and 3.}
\]

Thus \( |d(z, F(z, w))| \leq \frac{a_4 s^4}{a_3} |d(w, F(w, z))|. \)

Similarly using (9.5)(a)(ii), we have
\[ |d(w, F(w, z))| \leq \frac{a_4 s^4}{a_3} |d(z, F(z, w))|. \]

From Remark 1(iii), we have \( z = F(z, w) \) and \( F(w, z) = w. \) Thus \( Sz = z = F(z, w), Sw = w = F(w, z). \) Since \( z = F(z, w) \in F(X \times X) \subseteq T(X) \) and \( w = F(w, z) \in F(X \times X) \subseteq T(X), \) there exist \( u \) and \( v \) in \( X \) such that \( z = Tu \) and \( w = Tv. \) From (9.5)(a)(iii), we have
\[ \alpha((S_z, Sw), (Tu, Tv)) = \alpha((z, w), (z, w)) \geq 1. \]

Now from (9.2), we have
\[
d(z, G(u, v)) = d(F(z, w), G(u, v))
\leq \alpha((S_z, Sw), (Tu, Tv)) s^4 d(F(z, w), G(u, v))
\leq a_1 d(z, z) + a_2 d(w, w) + a_3 d(z, z) + a_4 d(w, w) + a_5 d(z, G(u, v))
+ a_6 d(w, G(v, u)) + a_7 \frac{d(z, z) d(z, G(u, v))}{1 + s^4 d(z, z) + s^4 d(w, w)}
+ a_8 \frac{d(z, z) d(z, G(u, v))}{1 + s^4 d(z, z) + s^4 d(w, w)}
= a_5 d(z, G(u, v)) + a_6 d(w, G(v, u)).
\]
Thus \(|d(z, G(u, v))| \leq \frac{a_6}{1 - a_5} |d(w, G(v, u))|\).

Similarly using (9.5)(a)(iii), we have

\(|d(w, G(v, u))| \leq \frac{a_6}{1 - a_5} |d(z, G(u, v))|\).

From Remark 1(iii), we have \(z = G(u, v)\) and \(w = G(v, u)\).

Since the pair \((G, T)\) is \(w\)-compatible, we have \(Tz = G(z, w)\) and \(w = G(w, z)\). From (9.5)(a)(iv), we have \(\alpha((Sz, Sw), (Tz, Tw)) = \alpha((z, w), (Tz, Tw)) \geq 1\). From (9.2), we have

\(|d(z, Tz)| = d(F(z, w), G(z, w))
\leq \alpha((Sx, Sy), (Tw, Tw)) s^4 |d(F(z, w), G(z, w))|
\leq a_1 |d(z, Tz)| + a_2 |d(w, Tw)| + a_3 |d(z, z)| + a_4 |d(w, w)|
+ a_5 |d(Tz, Tz)| + a_6 |d(Tw, Tw)| + a_7 \frac{|d(z, z)| |d(Tz, G(z, w))|}{1 + s^4 |d(Tz, G(z, w))| + s^4 |d(Tz, Tw)|}
+ a_8 \frac{|d(z, Tz)| |d(z, Tz)|}{1 + s^4 |d(Tz, Tz)| + s^4 |d(w, Tw)|}
\leq a_1 |d(z, Tz)| + a_2 |d(w, Tw)| + a_3 s^4 \frac{|d(z, Tz)| |d(z, Tz)|}{1 + s^4 |d(Tz, Tz)| + s^4 |d(w, Tw)|}
\leq a_1 |d(z, Tz)| + a_2 |d(w, Tw)| + a_8 |d(z, Tz)|, from Remark 1(i).

Thus \(|d(z, Tz)| < \frac{a_2}{1 - a_1 - a_8} |d(w, Tw)|\).

Similarly using (9.5)(a)(iv), we have

\(|d(w, Tw)| < \frac{a_2}{1 - a_1 - a_8} |d(z, Tz)|\).

Thus from Remark 1(iii), we have \(Tz = z\) and \(Tw = w\). Thus \((z, w)\) is a common coupled fixed point of \(F, G, S\) and \(T\).

Uniqueness of common coupled fixed point of \(F, G, S\) and \(T\) follows easily from (9.6) and (9.2).

Now we give an example to illustrate Theorem 9.

**Example 1.** Let \(X = [0, 2]\) and \(d(x, y) = i|x - y|^2\), \(\forall x, y \in X\).

Define \(F, G : X \times X \to X\) by \(F(x, y) = \frac{x^2 + y^2}{24}\) and \(G(x, y) = \frac{x^2 + y^2}{36}\) and \(S, T : X \to X\) by \(Sx = \frac{x^2}{2}\) and \(Tx = \frac{x^2}{3}\).

Define \(\alpha : X^2 \times X^2 \to \mathbb{R}^+\) by

\[
\alpha((x, y), (u, v)) = \begin{cases} 
1, & \text{if } x, y, u, v \in [0, \sqrt{3}], \\
0, & \text{otherwise}
\end{cases}
\]
Clearly \( F(X \times X) \subseteq T(X) \) and \( G(X \times X) \subseteq S(X) \). To verify (9.2) we consider the following two cases.

**Case (a)** Suppose \( x, y, u, v \in [0, \sqrt{3}] \).
Then \( \alpha((Sx, Sy), (Tu, Tv)) = \alpha((x^2/2, y^2/2), (u^2/3, v^2/3)) = 1 \),

\[
\alpha((Sx, Sy), (Tu, Tv)) \cdot \delta^4 d(F(x, y), G(u, v)) \leq 16i \left| \frac{x^2 + y^2}{24} - \frac{u^2 + v^2}{36} \right|
\]

\[
= \frac{16}{144} \left| \frac{3x^2 - 2u^2 + 3y^2 - 2v^2}{6} \right| \leq \frac{2}{9} \left[ \left| \frac{3x^2 - 2u^2}{6} \right|^2 + \left| \frac{3y^2 - 2v^2}{6} \right|^2 \right]
\]

\[
= \frac{2}{9} \left[ d(Sx, Tu) + d(Sy, Tv) \right].
\]

Here \( a_1 = a_2 = \frac{2}{9}, a_3 = a_4 = 0 \). Clearly \( \sum_{i=1}^{4} a_i < \frac{1}{s} \).

**Case (b):** Atleast one of \( x, y, u, v \not\in [0, \sqrt{3}] \).
Then \( \alpha((Sx, Sy), (Tu, Tv)) = \alpha((x^2/2, y^2/2), (u^2/3, v^2/3)) \).

**Sub case:** If \( \frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3} \in (1, \sqrt{3}] \) then

\[
\alpha((Sx, Sy), (Tu, Tv)) = 1.
\]

The inequality (9.2) holds as in case (a).

**Sub case:** If atleast one of \( \frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3} \in (\sqrt{3}, 2] \) then

\[
\alpha((Sx, Sy), (Tu, Tv)) = 0.
\]

Hence (9.2) holds.

By definition of \( \alpha \), the condition (9.3) with \( x_1 = 0 = y_1 \) and (9.4) are satisfied clearly.

To verify the compatibility of the pair \( (F, S) \), let us consider the sequences 
\( \{x_n\}, \{y_n\} \) in \( X \) such that \( F(x_n, y_n) \to t, Sx_n \to t, F(y_n, x_n) \to t' \) and \( Sx_n \to t' \) for some \( t, t' \in X \), \( F(x_n, y_n) \to t \Rightarrow \frac{x_n^2 + y_n^2}{24} \to t \) and \( F(y_n, x_n) \to t' \Rightarrow \frac{y_n^2 + x_n^2}{24} \to t' \). Hence \( t = t' \), \( Sx_n \to t \Rightarrow \frac{x_n^2}{2} \to t \) and \( Sy_n \to t' \Rightarrow \frac{y_n^2}{2} \to t' = t \).
Now \( \frac{x_n^2 + y_n^2}{24} \to \frac{4t^2 + 4t^2}{24} = \frac{t^2}{3} \). Hence \( \frac{t^2}{3} = t \Rightarrow t = 0 \). Thus \( x_n^2, y_n^2 \to 0 \).

\[
|d(S(F(x_n, y_n)), F(Sx_n, Sy_n))| = \left| \frac{1}{2} \left( \frac{x_n^2 + y_n^2}{24} \right)^2 - \frac{1}{24} \left( \frac{x_n^4}{4} + \frac{y_n^4}{4} \right)^2 \right| \to 0
\]
as \( n \to \infty \).

Similarly \( |d(S(F(y_n, x_n)), F(Sy_n, Sx_n))| \to 0 \). Thus the pair \( (F, S) \) is compatible. Also it is clear that the pair \( (G, T) \) is \( w \)-compatible.

By definition of \( \alpha \), one can easily verify the conditions (9.5) and (9.6).

Thus all the conditions of Theorem 9 are satisfied and \((0, 0)\) is the unique common fixed point of \( F, G, S \) and \( T \).

If \( \alpha((x, y), (u, v)) = 1 \) for all \( x, y, u, v \in X \) and \( s = 1 \) in Theorem 9, we have the following corollary.

**Corollary 10.** Let \( (X, d) \) be a complete complex valued metric space. Let \( F, G : X \times X \to X \) and \( S, T : X \to X \) be mappings satisfying the following:

(10.1) \( F(X \times X) \subseteq T(X) \), \( G(X \times X) \subseteq S(X) \),

(10.2)
\[
d(F(x, y), G(u, v)) \\
\leq a_1 d(Sx, Tu) + a_2 d(Sy, Tv) + a_3 d(Sx, F(x, y)) \\
+ a_4 d(Sy, F(y, x)) + a_5 d(Tu, G(u, v)) \\
+ a_6 d(Tv, G(v, u)) + a_7 \frac{d(Sx, F(x, y)) d(Tu, G(u, v))}{1 + d(Sx, Tu) + d(Sy, Tv)} \\
+ a_8 \frac{d(Tu, F(x, y)) d(Sx, G(u, v))}{1 + d(Sx, Tu) + d(Sy, Tv)}
\]
for all \( x, y, u, v \in X \), where \( a_i, i = 1, 2, \ldots, 8 \) are non-negative real numbers such that \( \sum_{i=1}^{8} a_i < 1 \).

(10.3)(a) the pair \( (F, S) \) is compatible, \( S \) is continuous and the pair \( (G, T) \) is \( w \)-compatible

(or)

(b) the pair \( (G, T) \) is compatible, \( T \) is continuous and the pair \( (F, S) \) is \( w \)-compatible

Then \( F, G, S \) and \( T \) have a unique common coupled fixed point.
References


