International Journal of Pure and Applied Mathematics
Volume 108 No. 4 2016, 967-984
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v108i4.21
ijpam.eu

EXPLICIT MINIMUM POLYNOMIAL, EIGENVECTOR, AND INVERSE FORMULA FOR NONSYMMETRIC ARROWHEAD MATRIX

Wiwat Wanicharpichat
[1]Department of Mathematics
Faculty of Science
and
[2]Research Center for Academic Excellence in Mathematics
Naresuan University
Phitsanulok 65000, THAILAND

Abstract: In this paper, we treat the eigenvalue problem for a nonsymmetric arrowhead matrix which is the general form of a symmetric arrowhead matrix. The purpose of this paper is to present explicit formula of determinant, inverse, minimum polynomial, and eigenvector formula of some nonsymmetric arrowhead matrices are presented.

AMS Subject Classification: 15A09, 15A15, 15A18, 15A23, 65F15, 65F40
Key Words: arrowhead matrix, Krylov matrix, Schur complement, nonderogatory matrix

1. Introduction and Preliminaries

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For a positive integer n, let M_n be the set of all $n \times n$ matrices over $\mathbb{C}$. The set of all vectors, or $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$. A nonzero vector $\mathbf{v} \in \mathbb{C}^n$ is called an eigenvector of $A \in M_n$ corresponding to a

Received: June 6, 2016
Published: August 16, 2016

scalar $\lambda \in \mathbb{C}$ if $A\mathbf{v} = \lambda\mathbf{v}$, and the scalar λ is an eigenvalue of the matrix A. The set of eigenvalues of A is called as the spectrum of A and is denoted by $\sigma(A)$.

A matrix of the form

$$A = \begin{bmatrix} d & b_{n-1} & \cdots & b_2 & b_1 \\ b_{n-1} & s_{n-1} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ b_2 & \vdots & \ddots & s_2 & 0 \\ b_1 & 0 & \cdots & 0 & s_1 \end{bmatrix} \in M_n(\mathbb{R}) \tag{1}$$

is called real symmetric arrowhead matrices (also known as real symmetric bordered diagonal matrices) [16, 21]. This class of matrices appears in certain symmetric inverse eigenvalue and inverse Sturm-Liouville problems, which arise in many applications, including control theory and vibration analysis [2, 5, 17]. O'Leary and Stewart [15] have presented formulas and efficient algorithms for computing eigenvalues and eigenvectors of symmetric arrowhead matrices. The eigenvalue problem for a symmetric arrowhead matrix arises in the description of radiationless transitions in isolated molecules and of oscillators vibrationally coupled with a Fermi liquid [6]. The properties of eigenvectors of arrowhead matrices were studied in [15]. In physics, symmetric arrowhead matrices have been used to describe radiationless transitions in isolated molecules [1, 15].

Wilkinson [21, pp.95-97] has given some relations between the eigenvalues of the symmetric arrowhead matrix A and the diagonal entries s_i, $i = 1, 2, \ldots, n-1$. Stor et al. [20] has presented a new algorithm for solving an eigenvalue problem for a real symmetric arrowhead matrix, which algorithm computes all eigenvalues and all components of the corresponding eigenvectors with high relative accuracy in $O(n^2)$ operations. The algorithm is based on a shift-and-invert approach. Double precision is eventually needed to compute only one element of the inverse of the shifted matrix. Each eigenvalue and the corresponding eigenvector can be computed separately.

Montano et al. [13] have given the characteristic polynomial of the arrow matrix defined in (1) as

$$\Delta_B(t) = \det(tI_n - B) = (t-d)\prod_{i=1}^{n-1}(t-s_i) - \sum_{i=1}^{n-1} b_i^2 \prod_{\substack{k=1 \\ k\neq i}}^{n-1}(t-s_k). \tag{2}$$

A nonsymmetric eigensolver was discussed by Jessup in [10], about the relationship between a tridiagonal matrix and a nonsymmetric arrowhead matrix, and shown that a nonsymmetric arrowhead matrix is the sum of a diagonal matrix and a rank two nonsymmetric matrix. The characteristic polynomial and eigenvector formula of a special nonsymmetric arrowhead matrix with distinct diagonal entries was also presented.

Maybee [11, pp.536-537] has studied about the nonsymmetric arrowhead matrix of order $n+1$ real matrix which has the following format.

$$A = \begin{bmatrix} -a & b_1 & b_2 & \cdots & b_n \\ -b_1 & -c_1 & 0 & \cdots & 0 \\ -b_2 & 0 & -c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_n & 0 & 0 & \cdots & -c_n \end{bmatrix} \tag{3}$$

where $a \geq 0$, $c_j \geq 0$, $1 \leq j \leq n$, and b_j can be either positive or negative. This matrix is called the "bordered diagonal matrix of order $n+1$," and

$$\det(A) = (-1)^{n+1} a \prod_{j=1}^{n} c_j + (-1)^{n-1} \sum_{j=1}^{n} b_j^2 \prod_{k \neq j} c_k. \tag{4}$$

Thus $\det(A) = 0$ if 2 or more of the $c_j = 0$. Notice that if some members b_j in the matrix A is not 0, then A is a non-symmetric matrix.

In this paper, we emphasis on the eigenvalue problem for a nonsymmetric arrowhead matrix of order $n \times n$ which is zero, except for its main diagonal and the first row and the first column of the form

$$A = \begin{bmatrix} -d & a_{n-1} & \cdots & a_2 & a_1 \\ -b_{n-1} & -s_{n-1} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ -b_2 & \vdots & \ddots & -s_2 & 0 \\ -b_1 & 0 & \cdots & 0 & -s_1 \end{bmatrix}, \tag{5}$$

where $d, a_i, b_i, s_i \in \mathbb{C}^*$, $1 \leq i \leq n-1$. We will call A as a *nonsymmetric arrowhead matrix*.

For the sake of convenience, we write the nonsymmetric arrowhead matrix A in a partitioned form as the following:

$$A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\mathbf{q} & -\Lambda \end{bmatrix}_{(n,n)}, \tag{6}$$

where

$$\mathbf{p} = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_2 \\ a_1 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix} \in \mathbb{C}^{n-1},$$

where $d, a_i, b_i, s_i \in \mathbb{C}^*$, $1 \leq i \leq n-1$ and $\Lambda = \operatorname{diag}(s_{n-1}, s_{n-2}, \ldots, s_1)$ is a diagonal matrix of order $n-1$.

Definition 1. [9, Definition 1.3.1]. A matrix $B \in M_n$ is said to be *similar* to a matrix $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that $B = S^{-1}AS$.

Let

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(n,n)}$$

be a permutation matrix (the "backward identity matrix") of size $n \times n$ satisfying

$$J = J^{-1} = J^T.$$

We have

$$\begin{aligned} JAJ &= \begin{bmatrix} -s_1 & 0 & \cdots & 0 & -b_1 \\ 0 & -s_2 & \ddots & \vdots & -b_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & -s_{n-1} & -b_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & -d \end{bmatrix} =: \begin{bmatrix} -D & -\mathbf{b} \\ \mathbf{a}^T & -d \end{bmatrix}_{(n,n)} \\ &= \widetilde{A}, \end{aligned} \tag{7}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} \in \mathbb{C}^{n-1},$$

with $d, a_i, b_i, s_i \in \mathbb{C}^*$, $1 \leq i \leq n-1$ and $D = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$ is a diagonal matrix of order $n-1$. This show that the arrowhead symmetric matrix A

is similar to the matrix $\widetilde{A}$ via matrix J. The matrix $\widetilde{A}$ is another form of a nonsymmetric arrowhead matrix.

The purpose of this paper is to present the explicit formula of determinant, inverse, minimum polynomial and eigenvector formula of the nonsymmetric arrowhead matrix in (6).

We recall some well-known results that will be used sequel.

Solomon [19, Theorem 2] asserted that the companion matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{n-1} \end{bmatrix}$$

of $f(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$ over a ring with unity, the matrix C is similar to its transpose

$$C^T = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}$$

via an invertible symmetric matrix

$$P = \begin{bmatrix} c_1 & c_2 & \dots & c_{n-1} & 1 \\ c_2 & \dots & c_{n-1} & 1 & 0 \\ \vdots & \iddots & \iddots & 0 & 0 \\ c_{n-1} & 1 & \iddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{8}$$

Theorem 2 ([19, Theorem 2]). *Let R be a ring with 1, let $a_1 \dots, a_n$ be elements of R and let C be the companion matrix. There exists an invertible symmetric matrix P with coefficients in R such that*

$$PCP^{-1} = C^T. \tag{9}$$

For example, if $n = 3$ then

$$\begin{aligned} PCP^{-1} &= \begin{bmatrix} c_1 & c_2 & 1 \\ c_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -c_2 \\ 1 & -c_2 & c_2^2 - c_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{bmatrix} = C^T. \end{aligned}$$

Theorem 3. *[9, Theorem 1.4.8]. Let $A, B \in M_n$, if $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector corresponding to $\lambda \in \sigma(B)$ and if B is similar to A via S, then $S\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ.*

Theorem 4. *[9, Theorem 3.3.15]. A matrix $A \in M_n$ is similar to the companion matrix of its characteristic polynomial if and only if the minimal and characteristic polynomial of A are identical.*

Definition 5. [12, p. 644] A matrix $A \in M_n$ for which the characteristic polynomial $\Delta_A(t)$ equal to the minimum polynomial $m_A(t)$ are said to be a *nonderogatory* matrix.

2. The Determinant of Nonsymmetric Arrowhead Matrix

Let $A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\mathbf{q} & -\Lambda \end{bmatrix}_{(n,n)}$ be the arrowhead matrix as defined in (6). The matrix A is similar to $\widetilde{A}$ via J, as defined in (7), where

$$\widetilde{A} = \begin{bmatrix} -s_1 & 0 & \cdots & 0 & -b_1 \\ 0 & -s_2 & \ddots & \vdots & -b_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & -s_{n-1} & -b_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & -d \end{bmatrix} =: \begin{bmatrix} D & -\mathbf{b} \\ \mathbf{a}^T & -d \end{bmatrix}_{(n,n)}.$$

Two similar matrices have the same determinant, we have

$$\det(A) = \det(\widetilde{A}). \tag{10}$$

We use LU-decomposition by Hyman's Method in [8, p.280] to find the determinant of $\widetilde{A}$, that is also the determinant of A, as follows:

$$\begin{aligned}
\widetilde{A} &= \begin{bmatrix} -D & -\mathbf{b} \\ \mathbf{a}^T & -d \end{bmatrix}_{(n,n)} \\
&= \begin{bmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{a}^T(-D)^{-1} & 1 \end{bmatrix}_{(n,n)} \begin{bmatrix} -D & -\mathbf{b} \\ \mathbf{0}^T & -d - \mathbf{a}^T(-D)^{-1}(-\mathbf{b}) \end{bmatrix}_{(n,n)} \\
&=: LU.
\end{aligned}$$

We obtain that

$$\begin{aligned}
\det(\widetilde{A}) &= \det(-D) \times (-d - \mathbf{a}^T(-D)^{-1}(-\mathbf{b})) \\
&= \det(-D)(-d - \mathbf{a}^T(D)^{-1}(\mathbf{b})) \\
&= (-1)^{n-1} \prod_{i=1}^{n-1} s_i \big(-d - \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} \frac{1}{s_1} & 0 & \dots & 0 \\ 0 & \frac{1}{s_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{s_{n-1}} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}) \\
&= (-1)^{n-1} \prod_{i=1}^{n-1} s_i \left(-d - (\frac{a_1 b_1}{s_1} + \frac{a_2 b_2}{s_2} + \dots + \frac{a_{n-1} b_{n-1}}{s_{n-1}})\right) \\
&= (-1)^n \left(d \prod_{i=1}^{n-1} s_i + \sum_{i=1}^{n-1} a_i b_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} s_j \right).
\end{aligned}$$

Also, from (10)

$$\det(A) = \det(\tilde{A}) = (-1)^n \left(d \prod_{j=1}^{n-1} s_j + \sum_{i=1}^{n-1} a_i b_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} s_j \right).$$

We immediately obtain following theorem.

Theorem 6. *Let* $A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\mathbf{q} & -\Lambda \end{bmatrix}_{(n,n)}$ *be a nonsymmetric arrowhead matrix as defined in* (6)*. Then*

$$\det(A) = (-1)^n \left(d \prod_{j=1}^{n-1} s_j + \sum_{i=1}^{n-1} a_i b_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} s_j \right). \tag{11}$$

Considering as $\det(A) = 0$ if 2 or more of the $s_j = 0$. In particular if A is an order of $n+1$ and $\mathbf{p} = \mathbf{q}$ then the explicit determinant formula of A is the same as the illustration of Maybee [11] defined in (4).

For the arrowhead matrix A of order 4, if

$$A = \begin{bmatrix} -d & a_3 & a_2 & a_1 \\ -b_3 & -s_3 & 0 & 0 \\ -b_2 & 0 & -s_2 & 0 \\ -b_1 & 0 & 0 & -s_1 \end{bmatrix}, \tag{12}$$

then

$$\det A = a_1 b_1 s_2 s_3 + a_2 b_2 s_1 s_3 + a_3 b_3 s_1 s_2 + d s_1 s_2 s_3.$$

3. Inverse Formula of Nonsymmetric Arrowhead Matrix

In this section, we find the inverse of nonsymmetric arrowhead matrix.

Let us consider the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{13}$$

where the submatrix A is assumed to be square and nonsingular. The Schur complement of A in M, denoted by (M/A), is the matrix

$$(M/A) = D - CA^{-1}B.$$

When M is square and the submatrix is nonsingular,

$$\det M = \det A \cdot \det(M/A),$$

an identity first proved by Schur [18].

Brezinski [4, p.232] similarly define the following Schur complements, where the matrix which is inverted is assumed to be square and nonsingular

$$\begin{aligned} (M/B) &= C - DB^{-1}A \\ (M/C) &= B - AC^{-1}D \\ (M/D) &= A - BD^{-1}C. \end{aligned}$$

The following useful formula, due to Zhang [22].

Theorem 7. [22, p.20] *Let M be partitioned as in* (13) *and suppose both M and D are nonsingular. Then (M/D) is nonsingular and*

$$M^{-1} = \begin{bmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{bmatrix}. \tag{14}$$

By considering the nonsingular nonsymmetric arrowhead matrix as partitioned in form of $A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\mathbf{q} & -\Lambda \end{bmatrix}_{(n,n)}$ as defined in (6), where the Schur complement

$$(A/(-\Lambda)) = (-d) - \mathbf{p}^T(-\Lambda)^{-1}(-\mathbf{q}). \tag{15}$$

We immediately obtain following corollary.

Corollary 8. *Let $A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\mathbf{q} & -\Lambda \end{bmatrix}_{(n,n)}$ be a nonsymmetric arrowhead matrix as defined in* (6)*, and suppose A is nonsingular. Then $(A/(-\Lambda))$ is nonsingular and*

$$A^{-1} = \begin{bmatrix} (-d)^{-1} + (-d)^{-1}\mathbf{p}^T(A/(-\Lambda))(-\boldsymbol{q})(-d)^{-1} & -(-d)^{-1}\mathbf{p}^T(A/(-\Lambda))^{-1} \\ -(A/(-\Lambda))^{-1}(-\boldsymbol{q})(-d)^{-1} & (A/(-\Lambda))^{-1} \end{bmatrix},$$

where $(A/(-\Lambda)) = (-d) - \mathbf{p}^T(-\Lambda)^{-1}(-\boldsymbol{q})$, as defined in (15)*, and $\Lambda^{-1} = \operatorname{diag}(-\frac{1}{s_{n-1}}, -\frac{1}{s_{n-2}}, \dots, -\frac{1}{s_1})$.*

A straightforward computation, we have an explicit form of the inverse of the nonsymmetric arrowhead matrix $A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\mathbf{q} & -\Lambda \end{bmatrix}_{(n,n)}$ as defined in (6) as the following.

$$A^{-1} = \frac{(-1)^{n-1}}{\det A} \begin{bmatrix} \prod_{k=1}^{n-1} s_k & \mathbf{h}^T \\ -\mathbf{v} & -G \end{bmatrix}_{(n,n)},$$

where

$$\mathbf{v} = \begin{bmatrix} b_{n-1} \prod\limits_{\substack{k=1 \\ k \neq n-1}}^{n-1} s_k \\ \vdots \\ b_2 \prod\limits_{\substack{k=1 \\ k \neq 2}}^{n-1} s_k \\ b_1 \prod\limits_{\substack{k=1 \\ k \neq 1}}^{n-1} s_k \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} a_{n-1} \prod\limits_{\substack{k=1 \\ k \neq n-1}}^{n-1} s_k \\ \vdots \\ a_2 \prod\limits_{\substack{k=1 \\ k \neq 2}}^{n-1} s_k \\ a_1 \prod\limits_{\substack{k=1 \\ k \neq 1}}^{n-1} s_k \end{bmatrix} \in \mathbb{C}^{n-1},$$

and

$$G = [g_{ij}]_{(n-1,n-1)} \in M_{n-1},$$

where

$$g_{ij} = \begin{cases} d \prod\limits_{\substack{k=1 \\ k\neq n-i}}^{n-1} s_k + \sum\limits_{\substack{k=1 \\ k\neq n-i}}^{n-1} a_k b_k \prod\limits_{\substack{k=1 \\ k\neq i,n-i}}^{n-1} s_k, & \text{if } i = j, \\ a_{n-i} b_{n-j} \prod\limits_{\substack{k=1 \\ k\neq n-i,n-j}}^{n-1} s_k, & \text{if } i \neq j. \end{cases}$$

For the arrowhead matrix A of order 4 in (12), where $\det(A) \neq 0$, an inverse of A is in the following form:

$$A^{-1} = \frac{(-1)^{4-1}}{\det A} \times \begin{bmatrix} s_1 s_2 s_3 & a_3 s_1 s_2 & a_2 s_1 s_3 & a_1 s_2 s_3 \\ -b_3 s_1 s_2 & d s_1 s_2 + a_1 b_1 s_2 + a_2 b_2 s_1 & -a_2 b_3 s_1 & -a_1 b_3 s_2 \\ -b_2 s_1 s_3 & -a_3 b_2 s_1 & d s_1 s_3 + a_1 b_1 s_3 + a_3 b_3 s_1 & -a_1 b_2 s_3 \\ -b_1 s_2 s_3 & -a_3 b_1 s_2 & -a_2 b_1 s_3 & d s_2 s_3 + a_2 b_2 s_3 + a_3 b_3 s_2 \end{bmatrix}.$$

4. The Minimum Polynomial of Nonsymmetric Arrowhead Matrix

Let $S = \{s_1, s_2, \ldots, s_{n-1}\}$ be a set of the diagonal elements of the submatrix Λ of A as defined in (6). We define

$$\sum \prod_{\substack{\binom{n-1}{k} \\ 1\leq j\leq n-1}} s_j$$

as the sum of all terms of the products of all elements in S, which each forms of term n choose k different elements of the set S, where the symbol $\binom{n-1}{k}$ is " $n-1$ choose k."

For example: In case $n = 6$, $S = \{s_1, s_2, s_3, s_4, s_5\}$,

$$\sum \prod_{\substack{\binom{5}{4} \\ 1\leq j\leq 5}} s_j = s_1 s_2 s_3 s_4 + s_1 s_2 s_3 s_5 + s_1 s_2 s_4 s_5 + s_1 s_3 s_4 s_5 + s_2 s_3 s_4 s_5.$$

If we remove the ith entry from the set S, we write

$$\sum \prod_{\substack{j\neq i \\ \binom{n-2}{k} \\ 1\leq j\leq n-1}} s_j.$$

For instance, if $n = 6$ then

$$\sum_{\substack{j\neq 3 \\ \binom{4}{2} \\ 1\leq j\leq 5}} \prod s_j = s_1s_2 + s_1s_4 + s_1s_5 + s_2s_4 + s_2s_5 + s_4s_5.$$

Theorem 9. *The arrowhead matrix* $A = \begin{bmatrix} -d & \mathbf{p}^T \\ -\boldsymbol{q} & -\Lambda \end{bmatrix}_{(n,n)}$ *as defined in* (6) *is a nonderogatory matrix.*

Proof. Let

$$A = \begin{bmatrix} -d & a_{n-1} & \cdots & a_2 & a_1 \\ -b_{n-1} & -s_{n-1} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ -b_2 & \vdots & \ddots & -s_2 & 0 \\ -b_1 & 0 & \cdots & 0 & -s_1 \end{bmatrix}.$$

We will prove that the arrowhead matrix A is similar to a companion matrix. We shall prove by explicit constructing the existence of a nonsingular matrix K such that KAK^{-1} is also a companion matrix. Now, choose the nth Krylov matrix K associated with A,

$$K = \begin{bmatrix} \mathbf{e}_1 & A\mathbf{e}_1 & A^2\mathbf{e}_1 & \ldots & A^{n-1}\mathbf{e}_1 \end{bmatrix}, \tag{16}$$

where $\mathbf{e}_1 = [1, 0, \ldots, 0]^T \in \mathbb{C}^n$. By straightforward computing, we have

$K^{-1}AK$

$$
\begin{aligned}
&= \begin{bmatrix}
0 & 0 & \dots & 0 & 0 & 0 & -d\prod\limits_{j=1}^{n-1} s_j - \sum\limits_{i=1}^{n-1} a_i b_i \prod\limits_{\substack{j\neq i\\ \binom{n-2}{n-2}\\ 1\leq j\leq n-1}} s_j \\
1 & 0 & \dots & 0 & 0 & 0 & -d\sum \prod\limits_{\substack{\binom{n-1}{n-2}\\ 1\leq j\leq n-1}} s_j - \sum\limits_{i=1}^{n-1} a_i b_i \sum \prod\limits_{\substack{j\neq i\\ \binom{n-2}{n-3}\\ 1\leq j\leq n-1}} s_j - \sum \prod\limits_{j=1}^{n-1} s_j \\
0 & 1 & \dots & 0 & 0 & 0 & -d\sum \prod\limits_{\substack{\binom{n-1}{n-3}\\ 1\leq j\leq n-1}} s_j - \sum\limits_{i=1}^{n-1} a_i b_i \sum \prod\limits_{\substack{j\neq i\\ \binom{n-2}{n-4}\\ 1\leq j\leq n-1}} s_j - \sum \prod\limits_{\substack{\binom{n-1}{n-2}\\ 1\leq j\leq n-1}} s_j \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & -d\sum \prod\limits_{\substack{\binom{n-1}{2}\\ 1\leq j\leq n-1}} s_j - \sum\limits_{i=1}^{n-1} a_i b_i \sum \prod\limits_{\substack{j\neq i\\ \binom{n-2}{1}\\ 1\leq j\leq n-1}} s_j - \sum \prod\limits_{\substack{\binom{n-1}{3}\\ 1\leq j\leq n-1}} s_j \\
0 & 0 & \dots & 0 & 1 & 0 & -d\sum \prod\limits_{\substack{\binom{n-1}{1}\\ 1\leq j\leq n-1}} s_j - \sum\limits_{i=1}^{n-1} a_i b_i - \sum \prod\limits_{\substack{\binom{n-1}{2}\\ 1\leq j\leq n-1}} s_j \\
0 & 0 & \dots & 0 & 0 & 1 & -d - \sum \prod\limits_{\substack{\binom{n-1}{1}\\ 1\leq j\leq n-1}} s_j
\end{bmatrix} \\
&=: C^T. \qquad (17)
\end{aligned}
$$

We denote the matrix $C^T = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}$ is the desired companion matrix. We also denote the characteristic polynomial of C^T by $\Delta_{C^T}(t)$, and minimal polynomial which denoted by $m_{C^T}(t)$. Since A is similar to a companion matrix C^T, then A is a nonderogatory matrix, by Theorem 4, and

$$\Delta_A(t) = m_A(t) = t^n + c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \dots + c_1 t + c_0, \qquad (18)$$

where

$$c_k = d\sum \prod_{\substack{\binom{n-1}{n-1-k}\\ 1\leq j\leq n-1}} s_j + \sum_{i=1}^{n-1} a_i b_i \sum \prod_{\substack{j\neq i\\ \binom{n-2}{n-2-k}\\ 1\leq j\leq n-1}} s_j + \sum \prod_{\substack{\binom{n-1}{n-k}\\ 1\leq j\leq n-1}} s_j, \qquad (19)$$

$k = 0, 1, 2, \dots n-1$, we define

$$\sum \prod_{\substack{\binom{n-1}{n-1}\\ 1\leq j\leq n-1}} s_j = \prod_{j=1}^{n-1} s_j,$$

and

$$\sum \prod_{\substack{\binom{r}{s} \\ 1 \le j \le n-1}} s_j = \begin{cases} 1, & \text{if} \quad s = 0, \\ 0, & \text{if} \quad r < s, \end{cases}$$

which proves assertion. □

For example, let A be a nonsymmetric arrowhead matrix defined in (12), if $d, a_i, b_i, s_i \in \mathbb{C}^*$, $1 \le i \le 3$, then it is similar to the following companion matrix C via $K = [\mathbf{e}_1, A\mathbf{e}_1, A^2\mathbf{e}_1, A^3\mathbf{e}_1]$ as follows:

$$C^T = \begin{bmatrix} 0 & 0 & 0 & -a_1b_1s_2s_3 - a_2b_2s_1s_3 - a_3b_3s_1s_2 - ds_1s_2s_3 \\ 1 & 0 & 0 & -ds_1s_2 - ds_1s_3 - ds_2s_3 - a_1b_1s_2 - a_1b_1s_3 - a_2b_2s_1 - a_2b_2s_3 - a_3b_3s_1 - a_3b_3s_2 - s_1s_2s_3 \\ 0 & 1 & 0 & -ds_1 - ds_2 - ds_3 - a_1b_1 - a_2b_2 - a_3b_3 - s_1s_2 - s_1s_3 - s_2s_3 \\ 0 & 0 & 1 & -d - s_1 - s_2 - s_3 \end{bmatrix}.$$

5. Explicit Eigenvector of Nonsymmetric Arrowhead Matrix

A square matrix over complex numbers always has at least one nonzero eigenvector. O'Leary and Stewart in [15] have presented the formulas and efficient algorithm for computing eigenvector of symmetric arrowhead matrices. The procedure for computing the remaining eigenvalues and eigenvectors of symmetric arrowhead matrix follows basic steps which similar to those for the eigendecomposition of a diagonal plus symmetric rank one matrix developed in [7].

Jessup [10] first considered arrowhead matrix as in equation (7) in case where D has distinct diagonal elements $s_1 \neq s_2, \ldots, s_{n-1}$, but because $s_i \in \mathbb{C}^*$ not all distinct, the details are vary in several important ways. The purpose of this section is to present an explicit formula of some nonsymmetric arrowhead eigenvectors.

Now analogous as eigenvector of a companion matrix in [3, pp.630-631] and in [14, p.6], we obtain

Theorem 10. *Let λ be an eigenvalue of a nonsymmetric arrowhead matrix A as defined in* (5)*, if $d, a_i, b_i, s_i \in \mathbb{C}^*$, $1 \le i \le n-1$, and $-s_i \neq \lambda$, for all*

$1 \leq i \leq n-1$, *then*

$$\mathbf{v} = \begin{bmatrix} \prod_{i=1}^{n-1} (\lambda + s_i) \\ -b_{n-1} \prod_{\substack{i \neq n-1 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ -b_{n-2} \prod_{\substack{i \neq n-2 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ \vdots \\ -b_3 \prod_{\substack{i \neq 3 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ -b_2 \prod_{\substack{i \neq 2 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ -b_1 \prod_{\substack{i \neq 1 \\ 1 \leq j < n-1}} (\lambda + s_i) \end{bmatrix}$$

is an eigenvector of A corresponding to the eigenvalue λ.

Proof. From equation (17), the nonsymmetric arrowhead matrix A is similar to the companion matrix C^T. Then they have the same eigenvalues in common. Let λ be an eigenvalue of A, then λ is also an eigenvalue of C^T. Since λ is a root of the characteristic polynomial $\Delta_A(t)$ in (18), we have

$$\Delta_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 = 0. \tag{20}$$

Therefore

$$\lambda^n = -\left(c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0\right).$$

From Theorem 2, the companion matrix C is similar to its transpose matrix namely C^T via the matrix P as defined in (8). Put a vector $\mathbf{u} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$.

We will prove that this vector $\mathbf{u}$ is an eigenvector of C corresponding to the eigenvalue λ. Let us consider

$$\begin{aligned} C\mathbf{u} &= \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -c_0 & -c_1 & \dots & -c_{n-2} & -c_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ -c_0 - c_1\lambda - \dots - c_{n-2}\lambda^{n-2} - c_{n-1}\lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} \\ &= \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \lambda\mathbf{u}, \end{aligned}$$

it is easy to see that the first component in the vector $\mathbf{u}$ cannot be zero, the vector $\mathbf{u}$ is not a zero-vector, it is an eigenvector of C corresponding to λ.

Since $KC^TK^{-1} = A$, and $PCP^{-1} = C^T$ and from (9), we have

$$C = P^{-1}C^TP = P^{-1}(K^{-1}AK)P = (KP)^{-1}A(KP).$$

Therefore $C = (KP)^{-1}A(KP)$. Theorem 3 asserts that $(KP)\mathbf{u}$ is an eigenvector of A corresponding to the eigenvalue λ, where K is the nth Krylov matrix associated with A and $\mathbf{e}_1$ as defined in (16), and P as defined in (8). By straightforward computing, we have

$$\begin{aligned} &\mathbf{v} = (KP)\mathbf{u} \\ &= \begin{bmatrix} (\lambda+s_1)(\lambda+s_2)(\lambda+s_3)\dots(\lambda+s_{n-3})(\lambda+s_{n-2})(\lambda+s_{n-1}) \\ -b_{n-1}(\lambda+s_1)(\lambda+s_2)(\lambda+s_3)\dots(\lambda+s_{n-4})(\lambda+s_{n-3})(\lambda+s_{n-2}) \\ -b_{n-2}(\lambda+s_1)(\lambda+s_2)(\lambda+s_3)\dots(\lambda+s_{n-4})(\lambda+s_{n-3})(\lambda+s_{n-1}) \\ \vdots \\ -b_3(\lambda+s_1)(\lambda+s_2)(\lambda+s_4)\dots(\lambda+s_{n-4})(\lambda+s_{n-2})(\lambda+s_{n-1}) \\ -b_2(\lambda+s_1)(\lambda+s_3)(\lambda+s_4)\dots(\lambda+s_{n-4})(\lambda+s_{n-2})(\lambda+s_{n-1}) \\ -b_1(\lambda+s_2)(\lambda+s_3)(\lambda+s_4)\dots(\lambda+s_{n-4})(\lambda+s_{n-2})(\lambda+s_{n-1}) \end{bmatrix} \end{aligned}$$

$$\mathbf{v} = \begin{bmatrix} \prod_{i=1}^{n-1} (\lambda + s_i) \\ -b_{n-1} \prod_{\substack{i \neq n-1 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ -b_{n-2} \prod_{\substack{i \neq n-2 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ \vdots \\ -b_3 \prod_{\substack{i \neq 3 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ -b_2 \prod_{\substack{i \neq 2 \\ 1 \leq j < n-1}} (\lambda + s_i) \\ -b_1 \prod_{\substack{i \neq 1 \\ 1 \leq j < n-1}} (\lambda + s_i) \end{bmatrix}.$$

Since $-s_i \neq \lambda$, for all $1 \leq i \leq n-1$, we obtain that the first component of this vector is not zero, so that the vector $\mathbf{v}$ is not a zero vector. Hence the proof is complete. □

For example: If A is a nonsymmetric arrowhead matrix as defined in (12), and $\lambda \neq -s_i$, $1 \leq i \leq 3$ is an eigenvalue of the matrix A, then we have

$$\mathbf{v} = (KP)\,\mathbf{u},$$

where

$$K = \begin{bmatrix} 1 & -d & d^2 - a_1b_1 - a_2b_2 - a_3b_3 & 2da_1b_1 - d^3 + 2da_2b_2 + 2da_3b_3 + a_1b_1s_1 + a_2b_2s_2 + a_3b_3s_3 \\ 0 & -b_3 & db_3 + b_3s_3 & a_3b_3^2 - b_3s_3^2 - d^2b_3 - db_3s_3 + a_1b_1b_3 + a_2b_2b_3 \\ 0 & -b_2 & db_2 + b_2s_2 & a_2b_2^2 - b_2s_2^2 - d^2b_2 - db_2s_2 + a_1b_1b_2 + a_3b_2b_3 \\ 0 & -b_1 & db_1 + b_1s_1 & a_1b_1^2 - b_1s_1^2 - d^2b_1 - db_1s_1 + a_2b_1b_2 + a_3b_1b_3 \end{bmatrix},$$

is the 4th Krylov matrix associated with A and $\mathbf{e}_1 \in \mathbb{C}^4$;

$$P = \begin{bmatrix} c_1 & c_2 & c_3 & 1 \\ c_2 & c_3 & 1 & 0 \\ c_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

where c_i, $i = 1, 2, 3$ are the entries of the last column of the companion matrix C^T which defined in (4), and

$$\mathbf{u} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}.$$

We have

$$\mathbf{v} = (KP)\,\mathbf{u} = \begin{bmatrix} (\lambda + s_1)(\lambda + s_2)(\lambda + s_3) \\ -b_3(\lambda + s_1)(\lambda + s_2) \\ -b_2(\lambda + s_1)(\lambda + s_3) \\ -b_1(\lambda + s_2)(\lambda + s_3) \end{bmatrix}.$$

6. Conclusion

In this paper, we mainly study about the explicit formula of determinant, inverse, minimum polynomial and eigenvector of a nonsymmetric arrowhead matrix.

Acknowledgments

The author is very grateful to the anonymous referees for their comments and suggestions, which inspired the improvement of the manuscript. This work was supported by Naresuan University.

References

[1] M. Bixon, J. Jortner, Intramolecular radiationless transitions, *J. Chem. Physics*, **48** (2) (1968), 715-726.

[2] D. Boley, G. Golub, A survey of matrix inverse eigenvalue problem, *Inverse Problem*, **3** (1987), 595-622.

[3] L. Brand, The companion matrix, its properties, *The American Mathematical Monthly*, **71** (6) (1964), 629-634.

[4] C. Brezinski, Other Manifestations of the Schur Complement, *Linear Algebra Appl.*, **111** (1988), 231-247.

[5] M.T. Chu, G.H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms and Applications*, Oxford University Press, New York (2005).

[6] J.W. Gadzuk, Localized vibratinal modes in Ferrni liquids, General theory, *Phys. Rev. B*, **24** (4) (1981), 1651-1663.

[7] G.H. Golub, Some modified matrix eigenvalue problems. *SIAM Review*, **15** (1973), 318-334.

[8] N.J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2nd ed, Society for Industrial and Applied Mathematics, Philadelphia (2002).

[9] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, (1996).

[10] E.R. Jessup, A case against a divide and conquer approach to the nonsymmetric eigenvalue problem, *Appl. Numer. Math.*, **12** (1993), 403-420.

[11] J.S. Maybee, Combinatorially Symmetric Matrices, *Linear Algebra Appl.*, **8** (1974), 529-537.

[12] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, Philadelphia (2000).

[13] E. Montaño, M. Salas, R.L. Soto, Positive matrices with prescribed singular values, *Proyecciones*, **27** (3) (2008), 289-305.

[14] S. Moritsugu, K. Kuriyama, A linear algebra method for solving systems of algebraic equations, *J. Jap. Soc. Symb. Alg. Comp. (J. JSSAC)*, **7** (4) (2000), 2-22.

[15] D.P. O'Leary, G.W. Stewart, Computing the eigenvalues and eigenvectors of symmetric arrowhead matrices, *J. Comput. Phys.*, **90** (2) (1990), 497-505.

[16] J. Peng, X.Y. Hu, L. Zhang, Two inverse eigenvalue problems for a special kind of matrices, *Linear Algebra Appl.*, **416** (2006), 336-347.

[17] H. Pickmann, J. Egaña, R. L. Soto, Extremal inverse eigenvalue problem for bordered diagonal matrices, *Linear Algebra Appl.*, **427** (2007), 256-271.

[18] I. Schur, Über potenzreihen, die im Innern des Einheitskreises beschränkt sind, J. Reine Angew. Math., **147** (1917) 205-232.

[19] L. Solomon, Similarity of the companion matrix and its transpose, *Linear Algebra Appl.*, **302-303** (1999), 555-561.

[20] N.J. Stor, I. Slapniăr, J. Barlow, Accurate eigenvalue decomposition of arrowhead matrices and applications, *Linear Algebra Appl*, **464** (2015), 62-89.

[21] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford (1965).

[22] F. Zhang, *The Schur Complement and Its Applications*, Springer, Inc., New York (2005).