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ON $g\alpha r$ CLOSED SET IN TOPOLOGICAL SPACES

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Abstract: In this paper, we introduce a new class of sets called generalized α regular-closed sets in topological spaces (briefly $g\alpha r$-closed set). Also we discuss some of their properties and investigate the relations between the associated topology.

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1. Introduction

In 1970, Levine introduced the concept of generalized closed set and discussed the properties of sets, closed and open maps, compactness, normal and separation axioms. Later in 1996, the investigation on generalization of closed set has lead to significant contribution to the theory of separation axiom, generalization of continuity and covering properties. A.A.Omari and M.S.M. Noorani made an analytical study and gave the concepts of generalized b closed sets in topological spaces

In this paper, a new class of closed set called generalized α regular - closed set is introduced to prove that the class forms a topology. The notion of generalized α regular - closed set and its different characterizations are given in

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this paper. Throughout this paper (X, τ) and (Y, σ) represent the non - empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively.

2. Preliminaries

Definition 2.1. Let A subset of A of a topological space (X, τ) is called

1. α - open set [9] if $A \subseteq int(cl(int(A)))$.
2. generalized closed set (briefly g - closed) [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
3. weakly closed set(briefly w - closed) [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open.
4. generalized $*$ closed set(briefly $g^{\square}$ - closed) [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g - open.
5. generalized α - closed set (briefly $g\alpha$ - closed) [8] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α - open.
6. an α - generalized closed set (briefly αg - closed) [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
7. generalized b - closed set (briefly gb - closed) [1] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
8. semi generalized b - closed set (briefly sgb - closed) [4] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.
9. generalized αb closed set (briefly $g\alpha b$ - closed) [14] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is α open in X.
10. regular generalized b - closed set (briefly rgb - closed) [9] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.
11. generalized pre regular closed set (briefly gpr - closed) [3] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

3. Generalized α Regular-Closed Sets

In this section, we introduce generalized α regular - closed set and investigate some of its properties.

Definition 3.1. A subset A of a topological space (X, τ), is called generalized α regular - closed set (briefly $g\alpha r$ - closed set) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

Theorem 3.2. *Every closed set is $g\alpha r$ - closed set.*

Proof. Let A be any closed set in X such that $A \subseteq U$, where U is regular open. Since $\alpha cl(A) \subseteq cl(A) = A$. Therefore $\alpha cl(A) \subseteq U$. Hence A is $g\alpha r$ - closed set in X. □

The converse of above theorem need not be true as seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not closed set.

Theorem 3.4. *Every $g\alpha$ - closed set is $g\alpha r$ - closed set.*

Proof. Let A be any $g\alpha$ - closed set in X. Let $A \subseteq U$ and U is regular open set. Then U is $g\alpha$ open. Therefore $\alpha cl(A) \subseteq U$. Hence A is$g\alpha r$ - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.5. Let$X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not $g\alpha$ - closed set.

Theorem 3.6. *Every αg - closed set is $g\alpha r$ - closed set.*

Proof. Let A be any αg - closed set in X. Let $A \subseteq U$ and U is regular open set. Then U is open. Therefore $\alpha cl(A) \subseteq U$. Hence A is $g\alpha r$ - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.7. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not αg - closed set.

Theorem 3.8. *Every $g\alpha r$ - closed set is gpr - closed set.*

Proof. Let A be any $g\alpha r$ - closed set in X and U be any regular open set containing A. Then $pcl(A) \subseteq \alpha cl(A) \subseteq U$. Therefore $pcl(A) \subseteq U$. Hence A is gpr - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.9. Let $X = \{a, b, c, d, e\}$ with

$$\tau = \{X, \phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}.$$

The set $\{a\}$ is gpr - closed set but not $g\alpha r$ - closed set.

Theorem 3.10. *Every g - closed set is $g\alpha r$ - closed set.*

Proof. Let A be any g - closed set in X and U be any regular open set containing A. Since every regular open set is open, $\alpha cl(A) \subseteq cl(A) \subseteq U$. There f ore $\alpha cl(A) \subseteq U$. Hence A is $g\alpha r$ closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.11. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not g - closed set.

Theorem 3.12. *Every w - closed set is $g\alpha r$ - closed set.*

Proof. Let A be any w - closed set in X and U be any regular open set containing A. Since every regular open set is semi open, $\alpha cl(A) \subseteq cl(A) \subseteq U$. Therefore $\alpha cl(A) \subseteq U$. Hence A is $g\alpha r$ - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.13. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not w - closed set.

Theorem 3.14. *Every $g^{\square}$ - closed set is $g\alpha r$ - closed set.*

Proof. Let A be any $g^{\square}$ - closed set in X and U be any regular open set containing A. Since every regular open is g - open, $\alpha cl(A) \subseteq cl(A) \subseteq U$. Therefore $\alpha cl(A) \subseteq U$. Hence A is $g\alpha r$ - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.15. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. The set $\{a\}$ is $g\alpha r$ - closed set but not $g^{\square}$ - closed set.

Theorem 3.16. *Every $g\alpha^{\square}$ - closed set is $g\alpha r$ - closed set.*

Proof. Let A be $g\alpha^{\square}$ - closed set in X and U be any regular open set containing A. Since every regular open set is α open set, we have $\alpha cl(A) \subseteq int(U) \subseteq U$. Therefore $\alpha cl(A) \subseteq U$. Hence A is $g\alpha r$ - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.17. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a\}$ is $g\alpha r$ - closed set but not $g\alpha^{\square}$ - closed set.

4. Characteristics of $g\alpha r$-Closed Sets

Theorem 4.1. *If A and B are$g\alpha r$ - closed sets in X then $A \cup B$ is $g\alpha r$ - closed set in X.*

Proof. Let A and B are $g\alpha r$ - closed sets in X and U be any regular open set such that $A \cup B \subseteq U$. Therefore $\alpha cl(A) \subseteq U, \alpha cl(B) \subseteq U$. Hence $\alpha cl(A \cup B) = \alpha cl(A) \cup \alpha cl(B) \subseteq U$. Therefore $A \cup B$ is $g\alpha r$ - closed set in X. □

Theorem 4.2. *If a set A is $g\alpha r$ - closed set then $\alpha cl(A) - A$ contains no non empty regular closed set.*

Proof. Let F be a regular closed set in X such that $F \subseteq \alpha cl(A) - A$. Then $A \subseteq X - F$. Since A is $g\alpha r$ closed set and $X - F$ is regular open then $\alpha cl(A) \subseteq X - F$. (i.e.) $F \subseteq X - \alpha cl(A)$. So $F \subseteq (X - \alpha cl(A)) \cap (\alpha cl(A) - A)$. Therefore $F = \phi$ □

Theorem 4.3. *If $A \subseteq Y \subseteq X$ and suppose that A is $g\alpha r$ closed set in X then A is $g\alpha r$ - closed set relative to Y.*

Proof. Given that $A \subseteq Y \subseteq X$ and A is $g\alpha r$ - closed set in X. To prove that A is $g\alpha r$ - closed set relative to Y. Let us assume that $A \subseteq Y \cap U$, where U is regular - open in X. Since A is $g\alpha r$ - closed set, $A \subseteq U$ implies $\alpha cl(A) \subseteq U$. It follows that $Y \cap \alpha cl(A) \subseteq Y \cap U$. That is A is $g\alpha r$ - closed set relative to Y. □

Theorem 4.4. *For $x \in X$, then the set $X - \{x\}$ is a $g\alpha r$ - closed set or regular - open.*

Proof. Suppose that $X - \{x\}$ is not regular open, then X is the only regular open set containing $X - \{x\}$. (i.e.) $\alpha cl(X - \{x\}) \subseteq X$. Then $X - \{x\}$ is $g\alpha r$ - closed in X. □

Remark 4.5. The intersection of any two subsets of $g\alpha r$ - closed sets in X is not $g\alpha r$ - closed set in X.

Theorem 4.6. *If A is both regular open and $g\alpha r$ - closed set in X, then A is α closed set.*

Proof. Since A is regular open and $g\alpha r$ closed in X, $\alpha cl(A) \subseteq A$. But always $A \subseteq \alpha cl(A)$. Therefore $A = \alpha cl(A)$. Hence A is α closed set. □

Note 4.7. gs - closed set and $g\alpha r$ - closed set are independent to each other as seen from the following examples.

Example 4.8. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not gs - closed set.

Example 4.9. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. The set $\{c\}$ is gs - closed set but not $g\alpha r$ - closed set.

Note 4.10. gb closed set and $g\alpha r$ - closed set are independent to each other as seen from the following examples

Example 4.11. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a\}$ is gb closed set but not $g\alpha r$ - closed set.

Example 4.12. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is $g\alpha r$ - closed set but not gb - closed set.

Note 4.13. sgb closed set and $g\alpha r$ - closed set are independent to each other as seen from the following examples

Example 4.14. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{b\}$ is sgb closed set but not $g\alpha r$ - closed set.

Example 4.15. Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{b, c\}$ is $g\alpha r$ - closed set but not sgb - closed set.

5. Generalized α Regular-Open Sets and Generalized α Regular-Neighbourhoods

In this section, we introduce generalized α regular - open sets (briefly $g\alpha r$ - open) and generalized α regular - neighbourhoods (briefly $g\alpha r$ - neighbourhood) in topological spaces by using the notions of $g\alpha r$ - open sets and study some of their properties.

Definition 5.1. A subset A of a topological space (X, τ), is called generalized α regular - open set (briefly $g\alpha r$ - open set) if A^c is $g\alpha r$ - closed in X. We denote the family of all $g\alpha r$ - open sets in X by $g\alpha r - O(X)$.

Theorem 5.2. *If A and B are $g\alpha r$ - open sets in a space X. Then $A \cap B$ is also $g\alpha r$ - open set in X.*

Proof. If A and B are $g\alpha r$ - open sets in a space X. Then A^c and B^c are $g\alpha r$ - closed sets in a space X. $A^c \cup B^c$ is also $g\alpha r$ - closed set in X. (i.e.) $A^c \cup B^c = (A \cap B)^c$ is a $g\alpha r$ - closed set in X. Therefore $A \cap B$, $g\alpha r$ - open set in X. □

Theorem 5.3. *If $int(A) \subseteq B \subseteq A$ and if A is $g\alpha r$ - open in X, then B is $g\alpha r$ - open in X.*

Proof. Suppose that $int(A) \subseteq B \subseteq A$ and A is $g\alpha r$ - open in X then $A^c \subseteq B^c \subseteq cl(A^c)$. Since A^c is $g\alpha r$ - closed in X. Therefore B is $g\alpha r$ - open in X. □

Definition 5.4. Let x be a point in a topological space X and let $xinX$. A subset N of X is said to be a $g\alpha r$ - neighbourhood of x iff there exists a $g\alpha r$ - open set G such that x in $G \subset N$.

Definition 5.5. A subset N of space X is called a $g\alpha r$ - neighbourhood of $A \subset X$ iff there exists a $g\alpha r$ - open set G such that $A \subset G \subset N$.

Theorem 5.6. *Every neighbourhood N of x in X is a $g\alpha r$ - neighbourhood of x.*

Proof. Let N be a neighbourhood of point x in X. To prove that N is a $g\alpha r$ - neighbourhood of x. By Definition of neighbourhood, there exists an open set G such that x in $G \subset N$. Hence N is a $g\alpha r$ - neighbourhood of x. □

Remark 5.7. In general, a $g\alpha r$ - neighbourhood of x in X need not be a neighbourhood of x in X.

Remark 5.8. The $g\alpha r$ - neighbourhood N of x in X need not be a $g\alpha r$ - open in X.

Theorem 5.9. *If a subset N of a space X is $g\alpha r$ - open, then N is $g\alpha r$ - neighbourhood of each of its points.*

Proof. Suppose N is $g\alpha r$ - open. Let x in N. We claim that N is $g\alpha r$ - neighbourhood of x. For N is a $g\alpha r$ - open set such that x in $N \subset N$. Since x is an arbitrary point of N, it follows that N is a $g\alpha r$ - neighbourhood of each of its points. □

Theorem 5.10. *Let X be a topological space. If F is $g\alpha r$ - closed subset of X and x in F^c. Then there exists a $g\alpha r$ - neighbourhood N of x such that $N \cap F = \phi$.*

Proof. Let F be $g\alpha r$ - closed subset of X and x in F^c. Then F^c is $g\alpha r$ - open set of X. So by Theorem, F^c contains a $g\alpha r$ - neighbourhood of each of its points. Hence there exists a $g\alpha r$ - neighbourhood N of x such that $N \subset F^c$. (i.e.) $N \cap F = \phi$. □

Definition 5.11. Let x be a point in a topological space X. The set of all $g\alpha r$ - neighbourhood of x is called the $g\alpha r$ - neighbourhood system at x, and is denoted by $g\alpha r - N(x)$.

Theorem 5.12. *Let x be a point in a topological space and each $x \in X$, Let $g\alpha r - N(X, \tau)$ be the collection of all $g\alpha r$ - neighbourhood of x. Then we have the following results.*

(i) $\forall x \in X$, $g\alpha r - N(x) \neq \phi$.

(ii) $N \in g\alpha r - N(x) \Rightarrow x \in N$.

(iii) $N \in g\alpha r - N(x), M \supset N \Rightarrow M \in g\alpha r - N(x)$.

(iv) $N \in g\alpha r - N(x), M \in g\alpha r - N(x) \Rightarrow N \cap M \in g\alpha r - N(x)$, *if finite intersection of $g\alpha r$ open set is $g\alpha r$ open.*

(v) $N \in g\alpha r - N(x) \Rightarrow$ *there exists* $M \in g\alpha r - N(x)$ *such that* $M \subset N$ *and* $M \in g\alpha r - N(y)$ *for every* $y \in M$.

Proof. (i) Since X is $g\alpha r$ - open set, it is a $g\alpha r$ - neighbourhood of every $x \in X$. Hence there exists at least one $g\alpha r$ - neighbourhood (namely - X) for each $x \in X$. Therefore $g\alpha r - N(x) \neq \phi$ for every $x \in X$.

(ii) If $N \in g\alpha r - N(x)$, then N is $g\alpha r$ - neighbourhood of x. By Definition of $g\alpha r$ - neighbourhood, $x \in N$.

(iii) Let $N \in g\alpha r - N(x)$ and $M \supset N$. Then there is a $g\alpha r$ - open set G such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so M is $g\alpha r$ - neighbourhood of x. Hence $M \in g\alpha r - N(x)$.

(iv) Let $N \in g\alpha r - N(x)$, $M \in g\alpha r - N(x)$. Then by Definition of $g\alpha r$ - neighbourhood, there exists $g\alpha r$ - open sets G_1 and G_2 such that $x \in G_1 \subset N$ and $x \in G_2 \subset M$. Hence

$$x \in G_1 \cap G_2 \subset N \cap M \tag{1}$$

Since $G_1 \cap G_2$ is a $g\alpha r$ - open set, it follows from (1) that $N \cap M$ is a $g\alpha r$ - neighbourhood of x. Hence $N \cap M$ in $g\alpha r - N(x)$.

(v) Let N in $g\alpha r - N(x)$. Then there is a $g\alpha r$ - open set M such that $x \in M \subset N$. Since M is $g\alpha r$ - open set, it is $g\alpha r$ - neighbourhood of each of its points. Therefore $M \in g\alpha r - N(y)$ for every $y \in M$.

□

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