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SOFT L-FUZZY QUASI-UNIFORMITIES AND SOFT L-FUZZY TOPOGENOUS ORDERS

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Abstract: In this paper, we introduce soft L-fuzzy topogenous orders and soft L-fuzzy quasi-uniformities in complete residuated lattice. We obtain two soft L-fuzzy bitopogenous structures induced by a soft L-fuzzy quasi-uniformity. We give their examples.

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1. Introduction

Molodtsov [9] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Pawlak's rough set [10,11] can be viewed as a special case of soft rough sets [3]. The topological structures of soft sets have been developed by many researchers [1,2,12-15].

On the other hand, Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [4,5].

In this paper, we introduce soft L-fuzzy topogenous orders and soft L-fuzzy quasi-uniformities in complete residuated lattice. From Theorem 14, we obtain two soft L-fuzzy bitopogenous structures induced by a soft L-fuzzy quasi-

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uniformity. We give their examples.

2. Preliminaries

Definition 1. [4,5] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \rightarrow, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution * which is defined by $x \oplus y = (x^* \odot y^*)^*$ unless otherwise specified and we denote $L_0 = L - \{0\}$.

Lemma 2. [4,5] *For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.*

(1) $1 \rightarrow x = x$, $0 \odot x = 0$, $x \rightarrow y = 1$ iff $x \leq y$.

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$.

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$.

(7) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$.

(8) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$.

(9) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$.

(10) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$.

(11) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(12) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

(13) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$.

(14) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)$.

(15) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

(16) $x \odot y \odot (z \odot w) \leq (x \odot z) \oplus (y \odot w)$.

Definition 3. [7] Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F, A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \rightarrow L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 4. [7] Let (F, A) and (G, A) be two fuzzy soft sets over a common universe X.

(1) (F, A) is a fuzzy soft subset of (G, A), denoted by $(F, A) \leq (G, A)$ if $F(\epsilon) \leq G(\epsilon)$, for each $\epsilon \in A$.

(2) $(F, A) \wedge (G, A) = (F \wedge G, A)$ if $(F \wedge G)(\epsilon) = F(\epsilon) \wedge G(\epsilon)$ for each $\epsilon \in A$.

(3) $(F, A) \vee (G, A) = (F \vee G, A)$ if $(F \vee G)(\epsilon) = F(\epsilon) \vee G(\epsilon)$ for each $\epsilon \in A$.

(4) $(F, A) \odot (G, A) = (F \odot G, A)$ if $(F \odot G)(\epsilon) = F(\epsilon) \odot G(\epsilon)$ for each $\epsilon \in A$.

(5) $(F, A)^* = (F^*, A)$ if $F^*(\epsilon) = (F(\epsilon))^*$ for each $\epsilon \in A$.

(6) $(F, A) \oplus (G, A) = (F \oplus G, A)$ if $(F \oplus G)(\epsilon) = (F^*(\epsilon) \odot G^*(\epsilon))^*$ for each $\epsilon \in A$.

3. Soft L-Fuzzy Quasi Uniformities and Soft L-Fuzzy Topogenous Orders

Definition 5. A mapping $\xi : S(X, A) \times S(X, A) \to L$ is called a soft L-fuzzy semi-topogenous order on X if it satisfies the following axioms.

(ST1) $\xi((1_X, A), (1_X, A)) = \xi((0_X, A), (0_X, A)) = 1$.

(ST2) If $\xi((F, A), (G, A)) \neq 0$, then $(F, A) \leq (G, A)$.

(ST3) If $(F_1, A) \leq (F, A)$, $(G, A) \leq (G_1, A)$, then $\xi((F, A), (G, A)) \leq \xi((F_1, A), (G_1, A))$.

A mapping ξ is called a soft strong L-fuzzy semi-topogenous order on X if it satisfies (ST1), (ST3) and the following axiom.

(S) $\xi((F, A), (G, A)) \leq S((F, A), (G, A))$ where

$$S((F, A), (G, A)) = \bigwedge_{a \in A} \bigwedge_{x \in X} (F(a)(x) \to G(a)(x)).$$

Remark 6. If ξ is a soft (resp. strong) L-fuzzy semi-topogenous order on X. Define a mapping $\xi^s : S(X, A) \times S(X, A) \to L$ as $\xi^s((F, A), (G, A)) = \xi((G, A)^*, (F, A)^*)$. Then ξ^s is a soft (resp. strong) L-fuzzy semi-topogenous order on X.

Definition 7. A soft (resp. strong) L-fuzzy semi-topogenous order ξ is called:

(1) soft (resp. strong) L-fuzzy topogenous if (T)

$$\begin{aligned} &\xi((F_1, A) \odot (F_2, A), (G_1, A) \odot (G_2, A)) \\ &\geq \xi((F_1, A), (G_1, A)) \odot \xi((F_2, A), (G_2, A)). \end{aligned}$$

(2) soft (resp. strong) L-fuzzy cotopogenous if (CT)

$$\begin{aligned}&\xi((F_1,A)\oplus(F_2,A),(G_1,A)\oplus(G_2,A))\\ &\geq \xi((F_1,A),(G_1,A))\odot\xi((F_2,A),(G_2,A)),\end{aligned}$$

(3) soft (resp. strong) L-fuzzy bitopogenous if ξ are soft (resp. strong) L-fuzzy topogenous and soft (resp. strong) L-fuzzy cotopogenous.

Definition 8. A soft (resp. strong) L-fuzzy topogenous (resp. cotopogenous) order ξ on X is said to be a soft (resp. strong) L-fuzzy topogenous (resp. cotopogenous) structure if $\xi\circ\xi\geq\xi$, where

$$\begin{aligned}&(\xi\circ\xi)((F,A),(H,A))\\ &=\bigvee_{(G,A)\in S(X,A)}\xi((F,A),(G,A))\odot\xi((G,A),(H,A)).\end{aligned}$$

Example 9. Let $H=\{h_i\mid i=\{1,...,6\}\}$ with h_i=house and $E=\{e,b,w,c,i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings.

Define a binary operation $\odot$ on $[0,1]$ by

$$x\odot y=\max\{0,x+y-1\},\ x\to y=\min\{1-x+y,1\}$$

$$x\oplus y=\min\{1,x+y\},\ x^*=1-x$$

Then $([0,1],\wedge,\to,0,1)$ is a complete residuated lattice (ref.[4,5]). Let $A=\{b,c,i\}\subset E$ and $X=\{h^1,h^4,h^5,h^6\}$. Put (H,A) be a fuzzy soft set as follow:

(H,A)	h^1	h^4	h^5	h^6
b	0.5	0.6	0.2	0.6
c	0.1	0.5	0.5	0.6
i	0.4	0.6	0.6	0.5

$(H,A)\odot(H,A)$	h^1	h^4	h^5	h^6
b	0.0	0.2	0.0	0.2
c	0.0	0.0	0.0	0.2
i	0.0	0.2	0.2	0.0

(H^*,A)	h^1	h^4	h^5	h^6
b	0.5	0.4	0.8	0.4
c	0.9	0.5	0.5	0.4
i	0.6	0.4	0.4	0.5

$(H^*, A) \oplus (H^*, A)$	h^1	h^4	h^5	h^6
b	1.0	0.8	1.0	0.8
c	1.0	1.0	1.0	0.8
i	1.0	0.8	0.8	1.0

(1) Define a soft L-fuzzy topogenous order $\xi : S(X, A) \times S(X, A) \to L$ as follows

$$\xi((F, A), (G, A)) = \begin{cases} 1, & \text{if } (F, A) = (0_X, A) \text{ or } (G, A) = (1_X, A) \\ 0.6, & \text{if } (F, A) \le (H, A) \le (G, A), \\ & (F, A) \not\le (H, A) \odot (H, A) \\ 0.3, & \text{if } (0_X, A) \ne (F, A) \le (H, A) \odot (H, A) \\ & \le (G, A), (H, A) \not\le (G, A), \\ 0, & \text{otherwise.} \end{cases}$$

But it is not a soft L-fuzzy topogenous structure because

$$\begin{aligned} &\bigvee_{(F,A) \in S(X,A)} (\xi((H, A) \odot (H, A), (F, A)) \\ &\odot \xi((F, A), (H, A) \odot (H, A))) = 0 \\ &\not\ge \xi((H, A) \odot (H, A), (H, A) \odot (H, A)) = 0.3. \end{aligned}$$

(2) By Remark 6, we obtain a soft L-fuzzy cotopogenous order $\xi^s : S(X, A) \times S(X, A) \to L$ as follows

$$\xi^s((F, A), (G, A)) = \begin{cases} 1, & \text{if } (F, A) = (0_X, A) \text{ or } (G, A) = (1_X, A) \\ 0.6, & \text{if } (F, A) \le (H, A)^* \le (G, A), \\ & (G, A) \not\ge (H, A)^* \oplus (H, A)^* \\ 0.3, & \text{if } (F, A) \le (H, A)^* \oplus (H, A)^* \\ & \le (G, A) \ne (1_X, A), (F, A) \not\le (H, A)^*, \\ 0, & \text{otherwise,} \end{cases}$$

But it is not a soft L-fuzzy cotopogenous structure because

$$\begin{aligned} &\bigvee_{(F,A) \in S(X,A)} (\xi^s((H, A)^* \oplus (H, A)^*, (F, A)) \\ &\odot \xi^s((F, A), (H, A)^* \oplus (H, A)^*)) = 0 \\ &\not\ge \xi^s((H, A)^* \oplus (H, A)^*, (H, A)^* \oplus (H, A)^*) = 0.3. \end{aligned}$$

Lemma 10. *Define a binary mapping $S : S(X, A) \times S(X, A) \to L$ by*

$$S((F, A), (G, A)) = \bigwedge_{a \in A} \bigwedge_{x \in X} (F(a)(x) \to G(a)(x)).$$

Then the following statements hold.

(1) $(F,A) \leq (G,A)$ *iff* $S((F,A),(G,A)) = 1$.

(2) *If* $(F,A) \leq (G,A)$, *then*

$$S((G,A),(H,A)) \leq S((F,A),(H,A)),$$
$$S((H,A),(F,A)) \leq S((H,A),(G,A)).$$

(3) $S((F,A),(G,A)) \odot S((G,A),(H,A))$
$\leq S((F,A),(H,A))$.

(4) $S((F,A),(G,A)) \odot S((H,A),(K,A))$
$\leq S((F,A) \odot (H,A),(G,A) \odot (K,A))$.

(5) $S((F,A),(G,A)) \odot S((H,A),(K,A))$
$\leq S((F,A) \oplus (H,A),(G,A) \oplus (K,A))$.

(6) S *is a soft strong* L*-fuzzy bitopogenous order.*

Proof. (1) By Lemma 2(1), we obtain

$$S((F,A),(G,A)) = \top$$
$$\text{iff } \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x) \to G(a)(x)) = \top$$
$$\text{iff } (F,A) \leq (G,A).$$

(2) It is easily proved from the definition of S.

(3) By Lemma 2(15), we have

$$(F(a)(x) \to G(a)(x)) \odot (G(a)(x) \to H(a)(x))$$
$$\leq (F(a)(x) \to H(a)(x)).$$

(4) By Lemma 2(13), we have

$$(F(a)(x) \to G(a)(x)) \odot (H(a)(x) \to K(a)(x))$$
$$\leq (F(a)(x) \odot H(a)(x) \to G(a)(x) \odot H(a)(x)).$$

(5) By Lemma 2(14), we have

$$(F(a)(x) \to G(a)(x)) \odot (H(a)(x) \to K(a)(x))$$
$$\leq (F(a)(x) \oplus H(a)(x) \to G(a)(x) \oplus H(a)(x)).$$

(6) We easily prove it from (1)-(5).

Definition 11. A mapping $\mathcal{U} : S(X \times X, A) \to L$ is called a soft L-fuzzy pre-uniformity on X iff it satisfies the properties.

(SU1) There exists $(U,A) \in S(X \times X, A)$ such that $\mathcal{U}((U,A)) = 1$,

(SU2) If $(V,A) \leq (U,A)$, then $\mathcal{U}((V,A)) \leq \mathcal{U}((U,A))$,
(SU3) For every $(U,A),(V,A) \in S(X \times X, A)$,

$$\mathcal{U}((U,A) \odot (V,A)) \geq \mathcal{U}((U,A)) \odot \mathcal{U}((V,A))$$

(SU4) If $\mathcal{U}((U,A)) \neq 0$, then $(1_{\triangle}, A) \leq (U,A)$, where

$$1_{\Delta}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

A soft L-fuzzy pre-uniformity $\mathcal{U}$ is called a soft L-fuzzy quasi-uniformity if

$$\text{(Q)}\ \ \mathcal{U}(U,A) \leq \bigvee \{\mathcal{U}((V,A)) \odot \mathcal{U}((W,A)) \mid (V,A) \circ (W,A) \leq (U,A)\},$$

where, for all $x, y \in X, a \in A$,

$$(V(a) \circ W(a))(x,y) = \bigvee_{z \in X} (V(a)(z,x) \odot W(a)(x,y)).$$

The triple $(X, A, \mathcal{U})$ is called a soft L-fuzzy pre-(resp. quasi-) uniform space.

Remark 12. Let $(X, A, \mathcal{U})$ be a soft L-fuzzy pre-uniform space, then by (SU1) and (SU2), we have $\mathcal{U}(1_{X \times X}) = 1$ because $(U,A) \leq (1_{X \times X}, A)$ for all $(U,A) \in S(X \times X, A)$.

Lemma 13. *Let $(X, A, \mathcal{U})$ be a soft L-fuzzy pre-uniform space. For each $(U,A) \in S(X \times X, A)$ and $(F,A) \in S(X,A)$, we define , for all $x \in X, a \in A$,*

$$(U,A)[(F,A)](a)(x) = \bigvee_{y \in X} (F(a)(y) \odot U(a)(y,x)),$$

$$(U,A)[[(F,A)]](a)(x) = \bigvee_{y \in X} (F(a)(y) \odot U(a)(x,y)).$$

For each $(U,A),(V,A),(U_1,A),(U_2,A) \in S(X \times X, A)$ and $(F,A),(G,A)$, $(F_1,A),(F_2,A),(F_i,A) \in S(X,A)$, the following properties hold.
(1) *For $\mathcal{U}((U,A)) > 0$, $(F,A) \leq (U,A)[(F,A)]$ and $(F,A) \leq (U,A)[[(F,A)]]$.*
(2) $(U,A) \leq (U,A) \circ (U,A)$, *for $\mathcal{U}((U,A)) > 0$.*
(3) $((V,A) \circ (U,A))[(F,A)] = (V,A)[(U,A)[(F,A)]]$
$((V,A) \circ (U,A))[[(F,A)]] = (V,A)[[(U,A)[[(F,A)]]]]$.
(4) $(U,A)[\bigvee_i (F_i,A)] = \bigvee_i (U,A)[(F_i,A)]$ *and*
$(U,A)[[\bigvee_i (F_i,A)]] = \bigvee_i (U,A)[[(F_i,A)]]$.
(5) $((U_1,A) \odot (U_2,A))[(F_1,A) \odot (F_2,A)] \leq (U_1,A)[(F_1,A)] \odot (U_2,A)[(F_2, A)]$.

(6) $((U_1,A)\odot(U_2,A))[[(F_1,A)\odot(F_2,A)]] \le (U_1,A)[[(F_1,A)]]\odot(U_2,A)[[(F_2, A)]]$.

(7) $((U_1,A)\odot(U_2,A))[(F_1,A)\oplus(F_2,A)] \le (U_1,A)[(F_1,A)]\oplus(U_2,A)[(F_2, A)]$.

(8) $((U_1,A)\odot(U_2,A))[[(F_1,A)\oplus(F_2,A)]] \le (U_1,A)[[(F_1,A)]]\oplus(U_2,A)[[(F_2, A)]]$.

Proof. (1) Since $\mathcal{U}((U,A)) > 0$, by (U4), $(1_\triangle, A) \le (U,A)$. It implies $(F,A) = (1_\triangle,A)[(F,A)] \le (U,A)[(F,A)]$.

(2) Since $(1_\triangle,A) \le (U,A)$ from (1), $(U,A) = (1_\triangle,A)\circ(U,A) \le (U,A)\circ(U,A)$.

(3) By Lemma 2 (5), we have

$$\begin{aligned}
&((V,A)\circ(U,A))[(F,A)](a)(x)\\
&= \bigvee_{y\in X}\{F(a)(y)\odot((V,A)\circ(U,A))(a)(y,x)\}\\
&= \bigvee_{y\in X}\{F(a)(y)\odot\bigvee_{z\in X}(U(a)(y,z)\odot V(a)(z,x))\}\\
&= \bigvee_{y\in X}\bigvee_{z\in X}\{F(a)(y)\odot U(a)(y,z)\odot V(a)(z,x)\}\\
&= \bigvee_{z\in X}\{\bigvee_{y\in X}(F(a)(y)\odot U(a)(y,z))\odot V(a)(z,x)\}\\
&= \bigvee_{z\in X}\{(U,A)[(F,A)](a)(z)\odot V(a)(z,x)\}\\
&= (V,A)[(U,A)[(F,A)]](a)(x).
\end{aligned}$$

(4) By Lemma 2 (5), we have

$$\begin{aligned}
&(U,A)[\bigvee_i(F_i,A)](a)(x) = \bigvee_{y\in X}\{(\bigvee_i F_i(a)(y)\odot U(a)(y,x)\}\\
&= \bigvee_{y\in X}\{\bigvee_i\{F_i(a)(y)\odot U(a)(y,x))\}\}\\
&= \bigvee_i\{\bigvee_{y\in X}\{F_i(a)(y)\odot U(a)(y,x)\}\} = \bigvee_i(U,A)[(F_i,A)](a)(x).
\end{aligned}$$

(5)

$$\begin{aligned}
&(U_1,A)[(F_1,A)](a)(x)\odot(U_2,A)[(F_2,A)](a)(x)\\
&= \Big(\bigvee_{y\in X}(F_1(a)(y)\odot U_1(a)(y,x))\Big)\\
&\odot\Big(\bigvee_{z\in X}(F_2(a)(z)\odot U_2(a)(z,x))\Big)\\
&\ge \bigvee_{y\in X}\Big((F_1(a)(y)\odot U_1(a)(y,x))\odot(F_2(a)(y)\odot U_2(a)(y,x))\Big)\\
&= \bigvee_{y\in X}\Big(((F_1,A)\odot(F_2,A))(a)(y)\odot((U_1,A)\odot(U_2,A))(a)(y,x)\Big)\\
&= ((U_1,A)\odot(U_2,A))[(F_1,A)\odot(F_2,A)](a)(x).
\end{aligned}$$

(7)

$$\begin{aligned}
&(U_1, A)[(F_1, A)](a)(x) \oplus (U_2, A)[(F_2, A)](a)(x) \\
&= \Big(\bigvee_{y\in X}((F_1, A)(a)(y) \odot (U_1, A)(a)(y, x))\Big) \\
&\oplus\Big(\bigvee_{z\in X}((F_2, A)(a)(z) \odot (U_2, A)(a)(z, x))\Big) \\
&\geq \bigvee_{y\in X} \Big\{((F_1, A)(a)(y) \odot (U_1, A)(a)(y, x)) \\
&\oplus((F_2, A)(a)(y) \odot (U_2, A)(a)(y, x))\Big\} \\
&\text{(by Lemma 2 (16))} \\
&\geq \bigvee_{y\in X} \Big(((F_1, A)(a) \oplus (F_2, A))(a)(y) \\
&\odot((U_1, A)(a) \odot (U_2, A))(a)(y, x)\Big) \\
&= ((U_1, A) \odot (U_2, A))[(F_1, A) \oplus (F_2, A)](a)(x).
\end{aligned}$$

(6) and (8) are similarly proved as (5) and (7), respectively.

Theorem 14. *Let $(X, A,\mathcal{U})$ be a soft L-fuzzy pre-uniform space. Define mappings $\xi^r_{\mathcal{U}}, \xi^l_{\mathcal{U}} : S(X, A) \times S(X, A) \to L$ by*

$$\begin{aligned}
\xi^r_{\mathcal{U}}((F, A), (G, A)) &= \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[(F, A)] \leq (G, A)\}, \\
\xi^l_{\mathcal{U}}((F, A), (G, A)) &= \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[[(F, A)]] \leq (G, A)\}.
\end{aligned}$$

Then $\xi^r_{\mathcal{U}}$ and $\xi^l_{\mathcal{U}}$ are soft L-fuzzy bitopogenous orders. If $(X, A,\mathcal{U})$ be a soft L-fuzzy quasi-uniform space, then $\xi^r_{\mathcal{U}}$ and $\xi^l_{\mathcal{U}}$ are soft L-fuzzy bitopogenous stuctures.

Proof. (ST1) Since $(U, A)[(0_X, A)] = (0_X, A)$ and $(U, A)[(1_X, A)] = (1_X, A)$, for $\mathcal{U}((U, A)) = 1$, we have

$$\xi^r_{\mathcal{U}}((0_X, A), (0_X, A)) = \xi^r_{\mathcal{U}}((1_X, A), (1_X, A)) = 1.$$

(ST2) If $\xi^r_{\mathcal{U}}((F, A), (G, A)) \neq 0$, there exists $(U, A) \in S(X \times X, A)$ with $\mathcal{U}((U, A)) > 0$. By Lemma 13(1), $(F, A) \leq (U, A)[(F, A)] \leq (G, A)$. Hence $(F, A) \leq (G, A)$.

(ST3) If $(F_1, A) \leq (F, A)$, $(G, A) \leq (G_1, A)$, then

$$\begin{aligned}
\xi^r_{\mathcal{U}}((F, A), (G, A)) &= \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[(F, A)] \leq (G, A)\} \\
&\leq \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[(F, A)] \leq (G_1, A)\} \\
&\leq \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[(F_1, A)] \leq (G_1, A)\} \\
&= \xi^r_{\mathcal{U}}((F_1, A), (G_1, A)).
\end{aligned}$$

(T) By Lemma 13(5),

$$\begin{aligned}&\xi^r_{\mathcal{U}}((F_1,A),(G_1,A))\odot\xi^r_{\mathcal{U}}((F_2,A),(G_2,A))\\&=\bigvee\{\mathcal{U}((U,A))\mid(U,A)[(F_1,A)]\leq(G_1,A)\}\\&\odot\bigvee\{\mathcal{U}((V,A))\mid(V,A)[(F_2,A)]\leq(G_2,A)\}\\&\leq\bigvee\{\mathcal{U}((U,A))\odot\mathcal{U}((V,A))\mid(U,A)[(F_1,A)]\\&\odot(V,A)[(F_2,A)]\leq(G_1,A)\odot(G_2,A)\}\\&\leq\bigvee\{\mathcal{U}((U,A)\odot(V,A))\mid((U,A)\odot(V,A))\\&[(F_1,A)\odot(F_2,A)]\leq(G_1,A)\odot(G_2,A)\}\\&\leq\bigvee\{\mathcal{U}((W,A))\mid(W,A)[(F_1,A)\odot(F_2,A)]\leq(G_1,A)\odot(G_2,A)\}\\&=\xi^r_{\mathcal{U}}((F_1,A)\odot(F_2,A),(G_1,A)\odot(G_2,A)).\end{aligned}$$

(CT) By Lemma 13(7),

$$\begin{aligned}&\xi^r_{\mathcal{U}}((F_1,A),(G_1,A))\odot\xi^r_{\mathcal{U}}((F_2,A),(G_2,A))\\&=\bigvee\{\mathcal{U}((U,A))\mid(U,A)[(F_1,A)]\leq(G_1,A)\}\\&\odot\bigvee\{\mathcal{U}((V,A))\mid(V,A)[(F_2,A)]\leq(G_2,A)\}\\&\leq\bigvee\{\mathcal{U}((U,A))\odot\mathcal{U}((V,A))\mid(U,A)[(F_1,A)]\\&\oplus(V,A)[(F_2,A)]\leq(G_1,A)\oplus(G_2,A)\}\\&\leq\bigvee\{\mathcal{U}((U,A)\odot(V,A))\mid((U,A)\odot(V,A))\\&[(F,A)\oplus(F,A)]\leq(G_1,A)\oplus(G_2,A)\}\\&\leq\bigvee\{\mathcal{U}((W,A))\mid(W,A)[(F,A)]\leq(G_1,A)\odot(G_2,A)\}\\&=\xi^r_{\mathcal{U}}((F_1,A)\odot(F_2,A),(G_1,A)\odot(G_2,A)).\end{aligned}$$

Hence $\xi^r_{\mathcal{U}}$ is a soft L-fuzzy bitopogenous order. Similarly, $\xi^r_{\mathcal{U}}$ is a soft L-fuzzy bitopogenous order.

Let $(X,A,\mathcal{U})$ be a soft L-fuzzy quasi-uniform space. Then (TS) For each $(U,A)\in S(X\times X,A)$ such that $(U,A)[(F,A)]\leq(G,A)$, by (Q), we have

$$\mathcal{U}((U,A))\leq\bigvee\{\mathcal{U}((V,A))\odot\mathcal{U}((W,A))\mid(V,A)\circ(W,A)\leq(U,A)\}.$$

Thus,

$$\begin{aligned}&\bigvee\{\mathcal{U}((U,A))\mid(U,A)[(F,A)]\leq(G,A)\}\\&\leq\bigvee\{\mathcal{U}((V,A))\odot\mathcal{U}((W,A))\mid(V,A)\circ(W,A)\\&[(F,A)]=(V,A)[(W,A)[(F,A)]]\leq(G,A)\}\\&\leq\bigvee_{(H,A)\in S(X,A)}\{\bigvee\{\mathcal{U}((V,A))\odot\mathcal{U}((W,A))\mid\\&(W,A)[(F,A)]\leq(H,A),\ (V,A)[(H,A)]\leq(G,A)\}\}\\&\leq\bigvee_{(H,A)\in S(X,A)}\{\bigvee\{\mathcal{U}((V,A))\mid(V,A)[(H,A)]\leq\\&(G,A)\}\odot\bigvee\{\mathcal{U}((W,A))\mid(W,A)[(F,A)]\leq(H,A)\}\}\\&=\bigvee_{(H,A)\in S(X,A)}\xi^r_{\mathcal{U}}((F,A),(H,A))\odot\xi^r_{\mathcal{U}}((H,A),(G,A)).\end{aligned}$$

Hence $\xi^r_{\mathcal{U}}$ is a soft L-fuzzy bitopogenous structure. Similarly, $\xi^l_{\mathcal{U}}$ is a soft L-fuzzy bitopogenous structure.

Example 15. Let H, E and $([0,1],\wedge,\rightarrow,0,1)$ as Example 9. Let $A=\{b,c\}\subset E$ and $X=\{h^1,h^4,h^5\}\subset H$. Put a fuzzy soft set (U,A) as follow:

$$(U,\{b\})=\begin{pmatrix} b & h^1 & h^4 & h^5\\ h^1 & 1 & 0.6 & 0.7\\ h^4 & 0.1 & 1 & 0.5\\ h^5 & 0.4 & 0.6 & 1\end{pmatrix}\qquad (U,\{c\})=\begin{pmatrix} c & h^1 & h^4 & h^5\\ h^1 & 1 & 0.3 & 0.6\\ h^4 & 0.1 & 1 & 0.6\\ h^5 & 0.7 & 0.5 & 1\end{pmatrix}$$

Then we obtain $(U\odot U,A)$ as

$$(U\odot U,\{b\})=\begin{pmatrix} b & h^1 & h^4 & h^5\\ h^1 & 1 & 0.2 & 0.4\\ h^4 & 0 & 1 & 0\\ h^5 & 0 & 0.2 & 1\end{pmatrix}\qquad (U\odot U,\{c\})=\begin{pmatrix} c & h^1 & h^4 & h^5\\ h^1 & 1 & 0 & 0.2\\ h^4 & 0 & 1 & 0.2\\ h^5 & 0.4 & 0 & 1\end{pmatrix}$$

Define $\mathcal{U}:S(X\times X,A)\rightarrow L$ as follows

$$\mathcal{U}((V,A))=\begin{cases} 1, & \text{if } (V,A)=(\top_{X\times X},A),\\ 0.6, & \text{if } (U,A)\le (V,A)\neq(\top_{X\times X},A),\\ 0.3, & \text{if } (U,A)\odot(U,A)\le (V,A)\not\ge (U,A),\\ 0, & \text{otherwise.}\end{cases}$$

Since $0.3=\mathcal{U}(((W\odot W),A)\ge \mathcal{U}((W,A))\odot\mathcal{U}((W,A))=0.2$, $\mathcal{U}$ is a soft L-fuzzy pre-uniformity on X. But $\mathcal{U}$ is not a soft L-fuzzy quasi-uniformity on X because

$$\begin{aligned}&0.3=\mathcal{U}((U,A)\odot(U,A))\\ &\not\le \bigvee\{\mathcal{U}((V,A))\odot\mathcal{U}((W,A))\mid (V,A)\circ(W,A)\le (U,A)\odot(U,A)\}\\ &=0.\end{aligned}$$

By Theorem 14, we obtain soft L-fuzzy bitopogenous orders $\xi^r_{\mathcal{U}},\xi^l_{\mathcal{U}}:S(X,A)\times S(X,A)\rightarrow L$ as follows

$$\xi^r_{\mathcal{U}}((F,A),(H,A))=\begin{cases} 1, & \text{if } (\top_{X\times X},A)[F,A]\le (H,A)\\ 0.6, & \text{if } (W,A)[(F,A)]\le (H,A),\\ & (\top_{X\times X},A)[F,A]\not\le (H,A)\\ 0.3, & \text{if } (W\odot W,A)[(F,A)]\le (H,A),\\ & (W,A)[(F,A)]\not\le (H,A),\\ 0, & \text{otherwise,}\end{cases}$$

$$\xi^l_{\mathcal{U}}((F,A),(H,A)) = \begin{cases} 1, & \text{if } (\top_{X\times X},A)[[F,A]] \leq (H,A) \\ 0.6, & \text{if } (W,A)[[(F,A)]] \leq (H,A), \\ & (\top_{X\times X},A)[[F,A]] \not\leq (H,A) \\ 0.3, & \text{if } (W\odot W,A)[[(F,A)]] \leq (H,A), \\ & (W,A)[[(F,A)]] \not\leq (H,A), \\ 0, & \text{otherwise.} \end{cases}$$

But $\xi^r_{\mathcal{U}}$ is not a soft L-fuzzy bitopogenous structure because

$$\begin{aligned} 0 &= (\xi^r_{\mathcal{U}} \circ \xi^r_{\mathcal{U}})((W\odot W,A)[(F,A)],(W\odot W,A)[(F,A)]) \\ &\not\geq \xi^r_{\mathcal{U}}((W\odot W,A)[(F,A)],(W\odot W,A)[(F,A)]) = 0.3. \end{aligned}$$

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