SOME COMPARISON RESULTS FOR
MOVING LEAST-SQUARE APPROXIMATIONS

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Abstract: Some properties of moving least-square approximations for two concrete weight functions are investigated.

The used technique is based on some properties of differential equations and applications of the theory of Lyapunov functions.

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1. Statement

Let us us remind the definition of moving least-squares approximation and some basic results.

Let:

1. \( \{x_1, \ldots, x_m\} \) be a set of points in bounded domain \( \mathcal{D} \subset \mathbb{R}^d \); and let \( x_i \neq x_j \), if \( i \neq j \).

2. \( f : \mathcal{D} \to \mathbb{R} \) be a continuous map.

3. \( \{p_1(x), \ldots, p_l(x)\} \) be a set of fundamental functions in \( \mathcal{D} \) (i.e. continuous and linearly independent) and let \( \mathcal{P}_l \) be their linear span.
Following [6], we will use the following definition. The moving least-squares approximation of order \( l \) at a point \( \mathbf{x} \) is the value of \( p^*(\mathbf{x}) \), where \( p^* \in \mathcal{P}_l \) is minimizing the least-squares error

\[
\sum_{i=1}^{m} W(\mathbf{x}, \mathbf{x}_i) (p(\mathbf{x}) - f(\mathbf{x}_i))^2
\]

among all \( p \in \mathcal{P}_l \).

The equivalent statement is the following constrained problem:

Find the minimum of

\[
Q = \sum_{i=1}^{m} w(\mathbf{x}, \mathbf{x}_i)a_i^2,
\]

subject to

\[
\sum_{i=1}^{m} a_ip_j(\mathbf{x}_i) = p_j(\mathbf{x}), \ j = 1, \ldots l.
\]

Here we assumed:

H1.1. \( W(\mathbf{x}_i, \mathbf{x}) > 0 \) if \( \mathbf{x}_i \neq \mathbf{x} \); \( w(\mathbf{x}_i, \mathbf{x}) = W^{-1}(\mathbf{x}_i, \mathbf{x}), i = 1, \ldots, m \).

H1.2. \( \text{rank}(E^t) = l \).

H1.3. \( 1 \leq l < m \).

We introduce the notations:

\[
E = \begin{pmatrix}
p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_l(\mathbf{x}_1) \\
p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_l(\mathbf{x}_2) \\
\vdots & \vdots & \ddots & \vdots \\
p_1(\mathbf{x}_m) & p_2(\mathbf{x}_m) & \cdots & p_l(\mathbf{x}_m)
\end{pmatrix},
\]

\[
a = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{pmatrix},
\]

\[
D = 2 \begin{pmatrix}
w(\mathbf{x}_1, \mathbf{x}) & 0 & \cdots & 0 \\
0 & w(\mathbf{x}_2, \mathbf{x}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & w(\mathbf{x}_m, \mathbf{x})
\end{pmatrix},
\]

\[
c = \begin{pmatrix}
p_1(\mathbf{x}) \\
p_2(\mathbf{x}) \\
\vdots \\
p_l(\mathbf{x})
\end{pmatrix}.
\]

**Theorem 1.1** (see [6]). Let the conditions (H1) hold true. Then:

1. The matrix

\[
A = \begin{pmatrix}
D & E^t \\
E^t & 0
\end{pmatrix}
\]

is non-singular.
2. The approximation defined by the moving least-squares method is

\[ \hat{L}(f) = \sum_{i=1}^{m} a_i f(x_i), \]  

(4)

where

\[ a = A_0 c \quad \text{and} \quad A_0 = D^{-1} E (E^T D^{-1} E)^{-1}. \]  

(5)

3. If \( w(x_i, x_i) = 0 \) for all \( i = 1, \ldots, m \) then the approximation is interpolatory.

For the approximation order of moving least-squares approximation (see [6] and [2]) it is not difficult to receive (for convenience we suppose \( P \) is the span of standard monomial basis, see [2]):

\[ \left| f(x) - \hat{L}(f)(x) \right| \leq \| f(x) - p^*(x) \|_{\infty} \left[ 1 + \sum_{i=1}^{m} |a_i| \right], \]  

(6)

and \((C_1=\text{const.})\)

\[ \| f(x) - p^*(x) \|_{\infty} \leq C_1 h^{l+1} \max \left\{ \left| f^{(l+1)}(x) \right| : x \in D \right\}. \]  

(7)

Of course, if \( D \) is a bounded domain in \( \mathbb{R}^d \) and the function \( f \) is \((l + 1)\)-continuously differentiable in \( D \), then there exists a constant \( C_2 \) such that

\[ \max \left\{ \left| f^{(l+1)}(x) \right| : x \in \overline{D} \right\} \leq C_2. \]  

Therefore, (6) and (7) yield

\[ \left| f(x) - \hat{L}(f)(x) \right| \leq C_1 C_2 h^{l+1} \left[ 1 + \sum_{i=1}^{m} |a_i| \right] \]  

\[ \leq C_1 C_2 h^{l+1} \left[ 1 + \| a_i \|_1 \right] \]  

\[ \leq \sqrt{m} C_1 C_2 h^{l+1} \left[ 1 + \| a_i \|_2 \right]. \]  

(8)

It follows from (8) that the error of moving least-squares approximation is upper-bounded of the 2-norm of coefficients of approximation \( a(x) \).

In the article, we will consider two families of weight-functions \((\alpha, \beta \geq 0)\):

\[ w_1(\alpha, x, y) = \exp (\alpha \| x - y \|^2) \]

and

\[ w_2(\alpha, \beta, x, y) = \exp (\alpha \| x - y \|^2) - \beta. \]
Usually the moving least-squares approximation generated by weight-function \( w_1 \) is called exp-moving least-squares approximation.

Our goal in this short note is to compare the upper bounds generated by the use of \( w_i, i = 1, 2 \).

Let us note the following facts:

1. If \( \alpha = 0 \) in \( w_1 \), then we receive classical least-squares approximation.

2. \( w_1(\alpha, x, y) = w_2(\alpha, 0, x, y) \).

3. The moving least-squares approximation generated by weight function \( w_2(\alpha, 1, x, y) \) is studied in Levin’s works, and we will call it Levin approach, see for example [6]. In this case the approximation in interpolatory.

For some application of moving least-squares approximation to predict chemical properties of oils see [15], [16], [17], and [18].

2. The Weight Family \( w_1 \) Generates “Decreasing Bounds” with Respect to \( \alpha \)

Through this section, we will suppose that conditions (H1) hold true and \( w(x, y) = w_1(\alpha, x, y) \).

Obviously \( A_0 = A_0(\alpha, x) \) and moreover

\[
a(\alpha, x) = D^{-1} E (E^t D^{-1} E)^{-1} c(x).
\] (9)

Here, in the right-hand side, only the matrix \( D \) depends on \( \alpha \) and \( x \).

Let us set

\[
H = 2 \begin{pmatrix} \|x - x_1\|^2 & 0 & \cdots & 0 \\ 0 & \|x - x_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|x - x_m\|^2 \end{pmatrix}.
\]

Then

\[
\frac{dD}{d\alpha} = 2 \begin{pmatrix} \frac{dw_1(\alpha, x, x_1)}{d\alpha} & 0 & \cdots & 0 \\ 0 & \frac{dw_1(\alpha, x, x_2)}{d\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{dw_1(\alpha, x, x_m)}{d\alpha} \end{pmatrix}.
\]
Theorem 2.1. Let the conditions (H1) hold true. Then for any fixed point \(x \in D \setminus \{x_1, \ldots, x_m\}\) there exists a constant \(\mu > 0\) such that for any two non-negative numbers \(\alpha_1, \alpha_2\) (\(\alpha_1 \leq \alpha_2\)), we have

\[
\|a(\alpha_2, x)\| \leq \mu \|a(\alpha_1, x)\|.
\]

Proof. Let \(x \in D \setminus \{x_1, \ldots, x_m\}\) be a fixed point. Let

\[
A_1(\alpha, x) = A_0 E^t = D^{-1} E \left( E^t D^{-1} E \right)^{-1} E^t, \quad A_2(\alpha, x) = A_1(\alpha, x) - I,
\]

where \(I\) is the identity \((m \times m)\)-matrix.

To simplify notations, we will write \(A_1 = A_1(\alpha, x), A_2 = A_2(\alpha, x), \) etc.

From equality

\[
a(\alpha, x) = A_0 c = D^{-1} E \left( E^t D^{-1} E \right)^{-1} c
\]

we obtain (differentiation with respect to \(\alpha\); only the matrix \(D\) depends from \(\alpha\)):

\[
\frac{da(\alpha, x)}{d\alpha} = \left( \frac{d}{d\alpha} D^{-1} E \left( E^t D^{-1} E \right)^{-1} \right) c
\]

\[
= \left( \frac{d}{d\alpha} D^{-1} \right) E \left( E^t D^{-1} E \right)^{-1} c + D^{-1} E \left( \frac{d}{d\alpha} \left( E^t D^{-1} E \right)^{-1} \right) c
\]

\[
= - HD^{-1} E \left( E^t D^{-1} E \right)^{-1} c
\]

\[
+ D^{-1} E \left( - \left( E^t D^{-1} E \right)^{-1} \left( \frac{d}{d\alpha} E^t D^{-1} E \right) \left( E^t D^{-1} E \right)^{-1} \right) c
\]
\[ \begin{align*}
&= -Ha \\
&+ D^{-1}E (E^t D^{-1}E)^{-1} (E^t HD^{-1}E) (E^t D^{-1}E)^{-1} c \\
&= -Ha \\
&+ D^{-1}E (E^t D^{-1}E)^{-1} (E^t H) \left( (E^t D^{-1}E)^{-1} \right) c \\
&= -Ha \\
&+ D^{-1}E (E^t D^{-1}E)^{-1} (E^t H) a \\
&= \left( (D^{-1}E (E^t D^{-1}E)^{-1} E^t - I) \right) Ha \\
&= A_2 Ha.
\end{align*} \]

Therefore \( a(\alpha) \) is a solution of the equation

\[ \frac{da(\alpha)}{d\alpha} = A_2(\alpha)Ha(\alpha). \quad (10) \]

Let us set:

\[ L(a) = \langle a, Ha \rangle, \quad a \in \mathbb{R}^m. \]

Our goal is to prove that \( L \) is a Lyapunov function for (10).

Indeed:

1. \( L(0) = 0 \).

2. Let \( \mu_* \) (resp. \( \mu^* \)) be the smallest (resp. largest) eigenvalue of \( H \), or equivalently smallest (resp. largest) entry of \( H \), because \( H \) is a diagonal matrix. Then

\[ \mu_* \|a\|^2 \leq L(a) = \langle a, Ha \rangle \leq \mu^* \|a\|^2, \quad (11) \]

for any \( a \in \mathbb{R}^m \).

3. For any \( a \in \mathbb{R}^m \), we have \( L(a) = \langle a, Ha \rangle \geq 0 \), because the matrix \( H \) is positive definite.

4. The derivatives:

\[ \frac{\partial L(a)}{\partial a} = 2Ha \quad \text{(because \( H \) is symmetric)}, \]

\[ \dot{L}(a) = \frac{dL(a(\alpha))}{d\alpha} = \left\langle \frac{\partial L(a)}{\partial a}, \dot{a}(\alpha) \right\rangle \]
\[ = 2 \langle Ha, A_2(\alpha)Ha \rangle \]
\[=2 \langle a_1, A_2(\alpha) a_1 \rangle \quad \text{(here } a_1 = H a)\]
\[=2 \langle a_1, (A_2(\alpha) D^{-1}) D^{1/2} D^{1/2} a_1 \rangle\]
\[=2 \langle D^{-1/2} a_2, (A_2(\alpha) D^{-1}) D^{1/2} a_2 \rangle \quad \text{(here } a_2 = D^{1/2} a_1)\]
\[=2 \langle a_2, D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2} a_2 \rangle.\]

The matrix \(A_2(\alpha) D^{-1}\) is symmetric with eigenvalues \(-1\) and \(0\), see [11].

The matrix \(D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2}\) is symmetric too:
\[
\left( D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2} \right)^t = D^{1/2} (A_2(\alpha) D^{-1})^t D^{-1/2}
\]
\[= D^{1/2} (A_2(\alpha) D^{-1})^t D^{-1/2}\]
\[= D^{-1/2} D (A_2(\alpha) D^{-1})^t D^{-1/2}\]
\[= D^{-1/2} (A_2(\alpha) D^{-1}) D D^{-1/2}\]
\[= D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2}.\]

Here, we used
\[D (A_2(\alpha) D^{-1})^t = (A_2(\alpha) D^{-1} D)^t = A_2(\alpha) = (A_2(\alpha) D^{-1}) D.\]

Moreover the matrices \(A_2(\alpha) D^{-1}\) and \(D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2}\) share one and the same characteristic polynomial \(\det(A_2(\alpha) D^{-1} - \lambda I) = 0\). Therefore the eigenvalues of \(D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2}\) are \(-1\) and \(0\).

Using Rayleigh-Ritz theorem, we obtain
\[
\hat{L}(a) = 2 \left\langle a_2, D^{-1/2} (A_2(\alpha) D^{-1}) D^{1/2} a_2 \right\rangle
\]
\[\leq 2 \max\{-1, 0\} \|a_2\|^2 \quad \text{(12)}
\]
\[\leq 0.\]

Therefore \(L\) is positive definite decrescent (and of course radially unbounded) Lyapunov function for (10).

Let \(\alpha_1 > 0\) and \(\alpha_2 > \alpha_1\). It follows from inequalities (12) that
\[L(a(\alpha_1)) \geq L(a(\alpha_2)). \quad \text{(13)}\]

Now, using (11), we obtain
\[
\mu_* \|a(\alpha_2)\|^2 \leq L(a(\alpha_2)) \leq L(a(\alpha_1)) \leq \mu^* \|a(\alpha_1)\|^2.
\]
or, if we set \( \mu = \sqrt{\frac{\mu^*}{\mu^*}} \), then
\[
\|a(\alpha_2)\| \leq \mu \|a(\alpha_1)\|.
\]

**Corollary 2.1.** Let the conditions (H1) hold true. Let \( x \) be a fixed point in \( D \).

Let \( \hat{L}_i(f) \), \( i = 1, 2 \) be two moving least-squares approximation of order \( l \) at a point \( x \), generated by the weight functions \( w(\alpha_i, x, y) \), respectively.

Then if \( \alpha_1 \leq \alpha_2 \) and
\[
\left| f(x) - \hat{L}_1(f)(x) \right| \leq C, \quad C = \text{const.}
\]
then
\[
\left| f(x) - \hat{L}_2(f)(x) \right| \leq \mu C,
\]
where the constant \( \mu \) is defined in the proof of Theorem 2.1.

The proof of Corollary 2.1 follows from (8) and Theorem 2.1.

### 3. The Weight Family \( w_2 \) Generates “Increasing Bounds” with Respect to \( \beta \in [0, 1] \)

Through this section, we will suppose that conditions (H1) hold true, \( w(x, y) = w_2(\alpha, \beta, x, y) \), and \( \alpha \) is a fixed non-negative number.

Obviously \( A_0 = A_0(\beta, x) \) and moreover
\[
a(\beta, x) = D^{-1} E \left( E^t D^{-1} E \right)^{-1} c(x).
\]
(14)

Here, in the right-hand side of the equality, only the matrix \( D \) depends on \( \beta \) and \( x \).

Obviously
\[
\frac{dD}{d\beta} = 2 \begin{pmatrix}
\frac{dw_2(\alpha, \beta, x, 1, x_1, x_2)}{d\beta} & 0 & \ldots & 0 \\
0 & \frac{dw_2(\alpha, \beta, x_2, x_2)}{d\beta} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{dw_2(\alpha, \beta, x_m, x_m)}{d\beta}
\end{pmatrix}
\]
\[
= -2I,
\]
\[
\frac{dD^{-1}}{d\beta} = -D^{-1} \frac{dD}{d\beta} D^{-1}
\]
\[
= 2D^{-1} D^{-1} = 2D^{-2}.
\]
Theorem 3.1. Let the conditions (H1) hold true. Then for any two numbers $\beta_1, \beta_2$, we have

$$\|a(\beta_1, x)\| \geq \|a(\beta_2, x)\|, \text{ if } 0 \leq \beta_1 \leq \beta_2 \leq 1.$$  

Proof. Let

$$A_1 = A_0 E^t = D^{-1} E (E^t D E^{-1})^{-1} E^t, \quad A_2 = A_1 - I.$$  

A differentiation of (14) with respect to $\beta$ yields:

$$\frac{da(\beta, x)}{d\beta} = \left( \frac{d}{d\beta} D^{-1} E (E^t D E^{-1})^{-1} \right) c$$

$$= \left( \frac{d}{d\beta} D^{-1} \right) E (E^t D E^{-1})^{-1} c
+ D^{-1} E \left( - (E^t D E^{-1})^{-1} \left( \frac{d}{d\beta} E^t D E^{-1} \right) (E^t D E^{-1})^{-1} \right) c$$

$$= 2 D^{-1} a
- D^{-1} E (E^t D E^{-1})^{-1} (E^t D E^{-1})^{-1} c
= 2 D^{-1} a$$

$$- 2 D^{-1} E (E^t D E^{-1})^{-1} (E^t D E^{-1}) \left( D^{-1} E (E^t D E^{-1})^{-1} \right) c$$

$$= 2 D^{-1} a$$

$$- 2 D^{-1} E (E^t D E^{-1})^{-1} (E^t D^{-1}) a$$

$$= 2 \left( I - D^{-1} E (E^t D E^{-1})^{-1} E^t \right) D^{-1} a$$

$$= - 2 A_2 D^{-1} a.$$  

Therefore $a(\beta)$ is a solution of

$$\frac{da(\beta)}{d\beta} = - 2 A_2 D^{-1} a(\beta). \quad (15)$$  

The matrix $-A_2 D^{-1}$ is symmetric and positive semi-definite (see [11]). Therefore,

$$L(a) = \langle a, a \rangle, \quad a \in \mathbb{R}^m$$

is a Lyapunov function for (15). Indeed

$$L(a) = \|a\|^2 \geq 0, \quad a \in \mathbb{R}^m, \quad (16)$$
\[
\frac{\partial L(a)}{\partial a} = 2a, \quad (17)
\]

\[
\dot{L}(a) = 2 \langle a, (-A_2 D^{-1}) a \rangle \geq 0 \quad a \in \mathbb{R}^m. \quad (18)
\]

Let \( x \) be a fixed point in \( D \). Let \( \beta_1, \beta_2 \in [0, 1] \) and \( \beta_1 < \beta_2 \). Then it follows from (18) that

\[
L(a(\beta_1, x)) \leq L(a(\beta_2, x)),
\]

and from (16), we receive

\[
\|a(\beta_1, x)\| \leq \|a(\beta_2, x)\|.
\]

Therefore the function \( \|a(\beta, x)\| \) is not decreasing with respect to \( \beta \in [0, 1] \).

**Example 3.1.** It is not difficult to see that the errors are increasing function of \( \beta \) — a little bit “strange fact”, because \( \beta = 1 \) is interpolatory approximation.

Let \( m = 4, l = 1 \), the given data

\[
\{(i, 2i) : i = 1, 3, 5, 7\}, \quad f(x) = 2x.
\]

Let \( \hat{L}_\beta(f) \) be the moving least-squares approximation of order \( l = 1 \) at a fixed point \( x \in [0, 7] \) with weight function \( w_2(1, \beta, x, y) \).

Then

\[
E = \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}, \quad a = \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}, \quad c = (1),
\]

\[
D_\beta(x) = 2 \begin{pmatrix}
w_2(1, \beta, x_1, x) & 0 & 0 & 0 \\
0 & w(1, \beta, x_2, x) & 0 & 0 \\
0 & 0 & w(1, \beta, x_3, x) & 0 \\
0 & 0 & 0 & w(1, \beta, x_4, x)
\end{pmatrix}.
\]

Then

\[
A_0 = D_\beta^{-1}(x) E \left( E^t D_\beta^{-1}(x) E \right)^{-1}
\]

and

\[
\hat{L}_\beta(f) = 2 \sum_{i=1}^{m} a_i(x) x_i.
\]

Using Maple 18, it is not hard to display the plots of \( \hat{L}_\beta(f), \beta = 0, \frac{1}{2}, 1 \), see Figure 1.
Figure 1: Plots of $\hat{L}_\beta(f)$, $x \in [0, 7]$.

References


