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SOME COMPARISON RESULTS FOR MOVING LEAST-SQUARE APPROXIMATIONS

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Abstract: Some properties of moving least-square approximations for two concrete weight functions are investigated.

The used thecnique is based on some properties of differential equations and applications of the theory of Lyapunov functions.

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Dedicated to the memory of our teacher and friend Prof. Drumi Bainov

1. Statement

Let us us remind the definition of moving least-squares approximation and some basic results.

Let:

1. $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}$ be a set of points in bounded domain $\mathcal{D} \subset \mathbb{R}^d$; and let $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, if $i \neq j$.
2. $f : \mathcal{D} \to \mathbb{R}$ be a continuous map.
3. $\{p_1(\boldsymbol{x}), \dots, p_l(\boldsymbol{x})\}$ be a set of fundamental functions in $\mathcal{D}$ (i.e. continuous and linearly independent) and let $\mathcal{P}_l$ be their linear span.

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4. $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smooth function.

Following [6], we will use the following definition. The *moving least-squares approximation* of order l at a point $\boldsymbol{x}$ is the value of $p^*(\boldsymbol{x})$, where $p^* \in \mathcal{P}_l$ is minimizing the least-squares error

$$\sum_{i=1}^{m} W(\boldsymbol{x}, \boldsymbol{x}_i)\left(p(\boldsymbol{x}) - f(\boldsymbol{x}_i)\right)^2$$

among all $p \in \mathcal{P}_l$.

The equivalent statement is the following constrained problem:

$$\text{Find the minimum of } Q = \sum_{i=1}^{m} w(\boldsymbol{x}, \boldsymbol{x}_i) a_i^2, \tag{1}$$

$$\text{subject to } \sum_{i=1}^{m} a_i p_j(\boldsymbol{x}_i) = p_j(\boldsymbol{x}),\ j = 1, \dots l. \tag{2}$$

Here we assumed:

H1.1. $W(\boldsymbol{x}_i, \boldsymbol{x}) > 0$ if $\boldsymbol{x}_i \neq \boldsymbol{x}$; $w(\boldsymbol{x}_i, \boldsymbol{x}) = W^{-1}(\boldsymbol{x}_i, \boldsymbol{x})$, $i = 1, \dots, m$.

H1.2. $\operatorname{rank}(E^t) = l$.

H1.3. $1 \leq l < m$.

We introduce the notations:

$$E = \begin{pmatrix} p_1(\boldsymbol{x}_1) & p_2(\boldsymbol{x}_1) & \cdots & p_l(\boldsymbol{x}_1) \\ p_1(\boldsymbol{x}_2) & p_2(\boldsymbol{x}_2) & \cdots & p_l(\boldsymbol{x}_2) \\ \vdots & \vdots & & \vdots \\ p_1(\boldsymbol{x}_m) & p_2(\boldsymbol{x}_m) & \cdots & p_l(\boldsymbol{x}_m) \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix},$$

$$D = 2\begin{pmatrix} w(\boldsymbol{x}_1, \boldsymbol{x}) & 0 & \cdots & 0 \\ 0 & w(\boldsymbol{x}_2, \boldsymbol{x}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & w(\boldsymbol{x}_m, \boldsymbol{x}) \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ \vdots \\ p_l(\boldsymbol{x}) \end{pmatrix}.$$

Theorem 1.1 (see [6])**.** Let the conditions (H1) hold true. Then:

1. The matrix

$$A = \begin{pmatrix} D & E \\ E^t & 0 \end{pmatrix} \tag{3}$$

is non-singular.

2. The approximation defined by the moving least-squares method is

$$\hat{L}(f) = \sum_{i=1}^{m} a_i f(\boldsymbol{x}_i), \tag{4}$$

where

$$\boldsymbol{a} = A_0 \boldsymbol{c} \quad \text{and} \quad A_0 = D^{-1}E\left(E^t D^{-1} E\right)^{-1}. \tag{5}$$

3. If $w(\boldsymbol{x}_i, \boldsymbol{x}_i) = 0$ for all $i = 1, \ldots, m$ then the approximation is interpolatory.

For the approximation order of moving least-squares approximation (see [6] and [2]) it is not difficult to receive (for convenience we suppose $\mathcal{P}$ is the span of standard monomial basis, see [2]):

$$\left| f(\boldsymbol{x}) - \hat{L}(f)(\boldsymbol{x}) \right| \leq \|f(\boldsymbol{x}) - p^*(\boldsymbol{x})\|_\infty \left[1 + \sum_{i=1}^{m} |a_i|\right], \tag{6}$$

and (C_1=const.)

$$\|f(\boldsymbol{x}) - p^*(\boldsymbol{x})\|_\infty \leq C_1 h^{l+1} \max\left\{\left|f^{(l+1)}(\boldsymbol{x})\right| : \boldsymbol{x} \in \mathcal{D}\right\}. \tag{7}$$

Of course, if $\mathcal{D}$ is a bounded domain in $\mathbb{R}^d$ and the function f is $(l+1)$-continuously differentiable in $\mathcal{D}$, then there exists a constant C_2 such that $\max\left\{\left|f^{(l+1)}(x)\right| : x \in \overline{\mathcal{D}}\right\} \leq C_2$. Therefore, (6) and (7) yield

$$\begin{aligned} \left| f(\boldsymbol{x}) - \hat{L}(f)(\boldsymbol{x}) \right| &\leq C_1 C_2 h^{l+1} \left[1 + \sum_{i=1}^{m} |a_i|\right] \\ &\leq C_1 C_2 h^{l+1} \left[1 + \|\boldsymbol{a}_i\|_1\right] \\ &\leq \sqrt{m} C_1 C_2 h^{l+1} \left[1 + \|\boldsymbol{a}_i\|_2\right]. \end{aligned} \tag{8}$$

It follows from (8) that the error of moving least-squares approximation is upper-bounded of the 2-norm of coefficients of approximation $\boldsymbol{a}(\boldsymbol{x})$.

In the article, we will consider two families of weight-functions ($\alpha, \beta \geq 0$):

$$w_1(\alpha, \boldsymbol{x}, \boldsymbol{y}) = \exp\left(\alpha \|\boldsymbol{x} - \boldsymbol{y}\|^2\right)$$

and

$$w_2(\alpha, \beta, \boldsymbol{x}, \boldsymbol{y}) = \exp\left(\alpha \|\boldsymbol{x} - \boldsymbol{y}\|^2\right) - \beta.$$

Usually the moving least-squares approximation generated by weight-function w_1 is called exp-*moving least-squares approximation*.

Our goal in this short note is to compare the upper bounds generated by the use of w_i, $i = 1, 2$.

Let us note the following facts:

1. If $\alpha = 0$ in w_1, then we receive classical least-squares approximation.
2. $w_1(\alpha, \boldsymbol{x}, \boldsymbol{y}) = w_2(\alpha, 0, \boldsymbol{x}, \boldsymbol{y})$.
3. The moving least-squares approximation generated by weight function $w_2(\alpha, 1, \boldsymbol{x}, \boldsymbol{y})$ is studied in Levin's works, and we will call it *Levin approach*, see for example [6]. In this case the approximation in interpolatory.

For some application of moving least-squares approximation to predict chemical properties of oils see [15], [16], [17], and [18].

2. The Weight Family w_1 Generates "Decreasing Bounds" with Respect to α

Through this section, we will suppose that conditions (H1) hold true and $w(\boldsymbol{x}, \boldsymbol{y}) = w_1(\alpha, \boldsymbol{x}, \boldsymbol{y})$.

Obviously $A_0 = A_0(\alpha, \boldsymbol{x})$ and moreover

$$\boldsymbol{a}(\alpha, \boldsymbol{x}) = D^{-1}E\left(E^t D^{-1} E\right)^{-1} \boldsymbol{c}(\boldsymbol{x}). \tag{9}$$

Here, in the right-hand side, only the matrix D depends on α and $\boldsymbol{x}$.

Let us set

$$H = 2\begin{pmatrix} \|\boldsymbol{x}-\boldsymbol{x}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\boldsymbol{x}-\boldsymbol{x}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \|\boldsymbol{x}-\boldsymbol{x}_m\|^2 \end{pmatrix}.$$

Then

$$\frac{dD}{d\alpha} = 2\begin{pmatrix} \frac{dw_1(\alpha,\boldsymbol{x},\boldsymbol{x}_1)}{d\alpha} & 0 & \cdots & 0 \\ 0 & \frac{dw_1(\alpha,\boldsymbol{x},\boldsymbol{x}_2)}{d\alpha} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{dw_1(\alpha,\boldsymbol{x},\boldsymbol{x}_m)}{d\alpha} \end{pmatrix}$$

$$
\begin{aligned}
&=2\begin{pmatrix} \|\boldsymbol{x}-\boldsymbol{x}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\boldsymbol{x}-\boldsymbol{x}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \|\boldsymbol{x}-\boldsymbol{x}_m\|^2 \end{pmatrix} \\
&\quad\times\begin{pmatrix} e^{\alpha\|\boldsymbol{x}-\boldsymbol{x}_i\|^2} & 0 & \cdots & 0 \\ 0 & e^{\alpha\|\boldsymbol{x}-\boldsymbol{x}_i\|^2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\alpha\|\boldsymbol{x}-\boldsymbol{x}_i\|^2} \end{pmatrix} \\
&=HD,
\end{aligned}
$$

$$
\begin{aligned}
\frac{dD^{-1}}{d\alpha} &= -D^{-1}\frac{dD}{d\alpha}D^{-1} \\
&= -D^{-1}(HD)D^{-1} = -HD^{-1}.
\end{aligned}
$$

Theorem 2.1. Let the conditions (H1) hold true.

Then for any fixed point $\boldsymbol{x} \in \mathcal{D}\setminus\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_m\}$ there exists a constant $\mu > 0$ such that for any two non-negative numbers α_1, α_2 ($\alpha_1 \le \alpha_2$), we have

$$\|\boldsymbol{a}(\alpha_2,\boldsymbol{x})\| \le \mu\|\boldsymbol{a}(\alpha_1,\boldsymbol{x})\|.$$

Proof. Let $\boldsymbol{x} \in \mathcal{D}\setminus\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_m\}$ be a fixed point. Let

$$A_1(\alpha,\boldsymbol{x}) = A_0E^t = D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t, \quad A_2(\alpha,\boldsymbol{x}) = A_1(\alpha,\boldsymbol{x}) - I,$$

where I is the identity $(m\times m)$-matrix.

To simplify notations, we will write $A_1 = A_1(\alpha,\boldsymbol{x})$, $A_2 = A_2(\alpha,\boldsymbol{x})$, etc.

From equality

$$\boldsymbol{a}(\alpha,\boldsymbol{x}) = A_0\boldsymbol{c} = D^{-1}E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c}$$

we obtain (differentiation with respect to α; only the matrix D depends from α):

$$
\begin{aligned}
\frac{d\boldsymbol{a}(\alpha,\boldsymbol{x})}{d\alpha} &= \left(\frac{d}{d\alpha}D^{-1}E\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= \left(\frac{d}{d\alpha}D^{-1}\right)E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} + D^{-1}E\left(\frac{d}{d\alpha}\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= -HD^{-1}E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} \\
&\quad + D^{-1}E\left(-\left(E^tD^{-1}E\right)^{-1}\left(\frac{d}{d\alpha}E^tD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c}
\end{aligned}
$$

$$
\begin{aligned}
&= - H\boldsymbol{a} \\
&\quad + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\left(E^tHD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} \\
&= - H\boldsymbol{a} \\
&\quad + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\left(E^tH\right)\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= - H\boldsymbol{a} \\
&\quad + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\left(E^tH\right)\boldsymbol{a} \\
&= \left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t - I\right)H\boldsymbol{a} \\
&= A_2H\boldsymbol{a}.
\end{aligned}
$$

Therefore $\boldsymbol{a}(\alpha)$ is a solution of the equation

$$
\frac{d\boldsymbol{a}(\alpha)}{d\alpha} = A_2(\alpha)H\boldsymbol{a}(\alpha). \tag{10}
$$

Let us set:

$$
L(\boldsymbol{a}) = \langle \boldsymbol{a}, H\boldsymbol{a}\rangle, \quad \boldsymbol{a} \in \mathbb{R}^m.
$$

Our goal is to prove that L is a Lyapunov function for (10).
Indeed:

1. $L(\mathbf{0}) = 0$.

2. Let μ_* (resp. μ^*) be the smallest (resp. largest) eigenvalue of H, or equivalently smallest (resp. largest) entry of H, because H is a diagonal matrix. Then

$$
\mu_*\|\boldsymbol{a}\|^2 \le L(\boldsymbol{a}) = \langle \boldsymbol{a}, H\boldsymbol{a}\rangle \le \mu^*\|\boldsymbol{a}\|^2, \tag{11}
$$

   for any $\boldsymbol{a} \in \mathbb{R}^m$.

3. For any $\boldsymbol{a} \in \mathbb{R}^m$, we have $L(\boldsymbol{a}) = \langle \boldsymbol{a}, H\boldsymbol{a}\rangle \ge 0$, because the matrix H is positive definite.

4. The derivatives:

$$
\begin{aligned}
\frac{\partial L(\boldsymbol{a})}{\partial \boldsymbol{a}} &= 2H\boldsymbol{a} \qquad (\text{because } H \text{ is symmetric}), \\
\dot{L}(\boldsymbol{a}) &= \frac{dL(\boldsymbol{a}(\alpha))}{d\alpha} = \left\langle \frac{\partial L(\boldsymbol{a})}{\partial \boldsymbol{a}}, \dot{\boldsymbol{a}}(\alpha)\right\rangle \\
&= 2\left\langle H\boldsymbol{a}, A_2(\alpha)H\boldsymbol{a}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&=2\langle \boldsymbol{a}_1, A_2(\alpha)\boldsymbol{a}_1\rangle \qquad (\text{here } \boldsymbol{a}_1 = H\boldsymbol{a})\\
&=2\left\langle \boldsymbol{a}_1, \left(A_2(\alpha)D^{-1}\right) D^{1/2}D^{1/2}\boldsymbol{a}_1\right\rangle\\
&=2\left\langle D^{-1/2}\boldsymbol{a}_2, \left(A_2(\alpha)D^{-1}\right) D^{1/2}\boldsymbol{a}_2\right\rangle \qquad (\text{here } \boldsymbol{a}_2 = D^{1/2}\boldsymbol{a}_1)\\
&=2\left\langle \boldsymbol{a}_2, D^{-1/2}\left(A_2(\alpha)D^{-1}\right) D^{1/2}\boldsymbol{a}_2\right\rangle.
\end{aligned}
$$

The matrix $A_2(\alpha)D^{-1}$ is symmetric with eigenvalues -1 and 0, see [11]. The matrix $D^{-1/2}\left(A_2(\alpha)D^{-1}\right)D^{1/2}$ is symmetric too:

$$
\begin{aligned}
\left(D^{-1/2}\left(A_2(\alpha)D^{-1}\right)D^{1/2}\right)^t &= D^{t/2}\left(A_2(\alpha)D^{-1}\right)^t D^{-t/2}\\
&= D^{1/2}\left(A_2(\alpha)D^{-1}\right)^t D^{-1/2}\\
&= D^{-1/2}D\left(A_2(\alpha)D^{-1}\right)^t D^{-1/2}\\
&= D^{-1/2}\left(A_2(\alpha)D^{-1}\right) D D^{-1/2}\\
&= D^{-1/2}\left(A_2(\alpha)D^{-1}\right) D^{1/2}.
\end{aligned}
$$

Here, we used

$$
D\left(A_2(\alpha)D^{-1}\right)^t = \left(A_2(\alpha)D^{-1}D\right)^t = A_2(\alpha) = \left(A_2(\alpha)D^{-1}\right)D.
$$

Moreover the matrices $A_2(\alpha)D^{-1}$ and $D^{-1/2}\left(A_2(\alpha)D^{-1}\right)D^{1/2}$ share one and the same characteristic polynomial $\det(A_2(\alpha)D^{-1} - \lambda I) = 0$. Therefore the eigenvalues of $D^{-1/2}\left(A_2(\alpha)D^{-1}\right)D^{1/2}$ are -1 and 0.

Using Rayleigh-Ritz theorem, we obtain

$$
\begin{aligned}
\dot{L}(\boldsymbol{a}) &= 2\left\langle \boldsymbol{a}_2, D^{-1/2}\left(A_2(\alpha)D^{-1}\right) D^{1/2}\boldsymbol{a}_2\right\rangle\\
&\le 2\max\{-1,0\}\|\boldsymbol{a}_2\|^2\\
&\le 0.
\end{aligned}
\tag{12}
$$

Therefore L is positive definite decrescent (and of course radially unbounded) Lyapunov function for (10).

Let $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$. It follows from inequalities (12) that

$$
L(\boldsymbol{a}(\alpha_1)) \ge L(\boldsymbol{a}(\alpha_2)). \tag{13}
$$

Now, using (11), we obtain

$$
\mu_*\|\boldsymbol{a}(\alpha_2)\|^2 \le L(\boldsymbol{a}(\alpha_2)) \le L(\boldsymbol{a}(\alpha_1)) \le \mu^*\|\boldsymbol{a}(\alpha_1)\|^2
$$

or, if we set $\mu = \sqrt{\frac{\mu^*}{\mu_*}}$, then

$$\|\boldsymbol{a}(\alpha_2)\| \leq \mu \|\boldsymbol{a}(\alpha_1)\|. \qquad \square$$

Corollary 2.1. Let the conditions (H1) hold true. Let $\boldsymbol{x}$ be a fixed point in $\mathcal{D}$.

Let $\hat{L}_i(f)$, $i = 1, 2$ be two moving least-squares approximation of order l at a point $\boldsymbol{x}$, generated by the weight functions $w(\alpha_i, \boldsymbol{x}, \boldsymbol{y})$, respectively.

Then if $\alpha_1 \leq \alpha_2$ and

$$\left| f(\boldsymbol{x}) - \hat{L}_1(f)(\boldsymbol{x}) \right| \leq C, \; C = \text{const}.$$

then

$$\left| f(\boldsymbol{x}) - \hat{L}_2(f)(\boldsymbol{x}) \right| \leq \mu C,$$

where the constant μ is defined in the proof of Theorem 2.1.

The proof of Corollary 2.1 follows from (8) and Theorem 2.1.

3. The Weight Family w_2 Generates "Increasing Bounds" with Respect to $\beta \in [0, 1]$

Through this section, we will suppose that conditions (H1) hold true, $w(\boldsymbol{x}, \boldsymbol{y}) = w_2(\alpha, \beta, \boldsymbol{x}, \boldsymbol{y})$, and α is a fixed non-negative number.

Obviously $A_0 = A_0(\beta, \boldsymbol{x})$ and moreover

$$\boldsymbol{a}(\beta, \boldsymbol{x}) = D^{-1} E \left(E^t D^{-1} E \right)^{-1} \boldsymbol{c}(\boldsymbol{x}). \tag{14}$$

Here, in the right-hand side of the equality, only the matrix D depends on β and $\boldsymbol{x}$.

Obviously

$$\begin{aligned}
\frac{dD}{d\beta} =& 2 \begin{pmatrix} \frac{dw_2(\alpha,\beta,x_1,x)}{d\beta} & 0 & \cdots & 0 \\ 0 & \frac{dw_2(\alpha,\beta,x_2,x)}{d\beta} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{dw_2(\alpha,\beta,x_m,x)}{d\beta} \end{pmatrix} \\
=& -2I, \\
\frac{dD^{-1}}{d\beta} =& -D^{-1} \frac{dD}{d\beta} D^{-1} \\
=& 2D^{-1}D^{-1} = 2D^{-2}.
\end{aligned}$$

Theorem 3.1. Let the conditions (H1) hold true.
Then for any two numbers β_1, β_2, we have

$$\|\boldsymbol{a}(\beta_1, \boldsymbol{x})\| \geq \|\boldsymbol{a}(\beta_2, \boldsymbol{x})\|, \text{ if } 0 \leq \beta_1 \leq \beta_2 \leq 1.$$

Proof. Let

$$A_1 = A_0 E^t = D^{-1}E\left(E^t D^{-1} E\right)^{-1} E^t, \quad A_2 = A_1 - I.$$

A differentiation of (14) with respect to β yields:

$$\begin{aligned}
\frac{d\boldsymbol{a}(\beta, \boldsymbol{x})}{d\beta} &= \left(\frac{d}{d\beta} D^{-1}E\left(E^t D^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= \left(\frac{d}{d\beta}D^{-1}\right) E\left(E^t D^{-1}E\right)^{-1}\boldsymbol{c} + D^{-1}E\left(\frac{d}{d\beta}\left(E^t D^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= 2D^{-2}E\left(E^t D^{-1}E\right)^{-1}\boldsymbol{c} \\
&\quad + D^{-1}E\left(-\left(E^t D^{-1}E\right)^{-1}\left(\frac{d}{d\beta}E^t D^{-1}E\right)\left(E^t D^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= 2D^{-1}\boldsymbol{a} \\
&\quad - D^{-1}E\left(E^t D^{-1}E\right)^{-1}\left(E^t 2D^{-2}E\right)\left(E^t D^{-1}E\right)^{-1}\boldsymbol{c} \\
&= 2D^{-1}\boldsymbol{a} \\
&\quad - 2D^{-1}E\left(E^t D^{-1}E\right)^{-1}\left(E^t D^{-1}\right)\left(D^{-1}E\left(E^t D^{-1}E\right)^{-1}\right)\boldsymbol{c} \\
&= 2D^{-1}\boldsymbol{a} \\
&\quad - 2D^{-1}E\left(E^t D^{-1}E\right)^{-1}\left(E^t D^{-1}\right)\boldsymbol{a} \\
&= 2\left(I - D^{-1}E\left(E^t D^{-1}E\right)^{-1}E^t\right)D^{-1}\boldsymbol{a} \\
&= -2A_2 D^{-1}\boldsymbol{a}.
\end{aligned}$$

Therefore $\boldsymbol{a}(\beta)$ is a solution of

$$\frac{d\boldsymbol{a}(\beta)}{d\beta} = -2A_2 D^{-1}\boldsymbol{a}(\beta). \tag{15}$$

The matrix $-A_2 D^{-1}$ is symmetric and positive semi-definite (see [11]). Therefore,

$$L(\boldsymbol{a}) = \langle \boldsymbol{a}, \boldsymbol{a} \rangle, \quad \boldsymbol{a} \in \mathbb{R}^m$$

is a Lyapunov function for (15). Indeed

$$L(\boldsymbol{a}) = \|\boldsymbol{a}\|^2 \geq 0, \quad \boldsymbol{a} \in \mathbb{R}^m, \tag{16}$$

$$\frac{\partial L(\boldsymbol{a})}{\partial \boldsymbol{a}} = 2\boldsymbol{a}, \tag{17}$$

$$\dot{L}(\boldsymbol{a}) = 2\left\langle \boldsymbol{a}, \left(-A_2 D^{-1}\right) \boldsymbol{a}\right\rangle \geq 0 \quad \boldsymbol{a} \in \mathbb{R}^m. \tag{18}$$

Let $\boldsymbol{x}$ be a fixed point in $\mathcal{D}$. Let $\beta_1, \beta_2 \in [0,1]$ and $\beta_1 < \beta_2$. Then it follows from (18) that

$$L(\boldsymbol{a}(\beta_1, \boldsymbol{x})) \leq L(\boldsymbol{a}(\beta_2, \boldsymbol{x})),$$

and from (16), we receive

$$\|\boldsymbol{a}(\beta_1, \boldsymbol{x})\| \leq \|\boldsymbol{a}(\beta_2, \boldsymbol{x})\|.$$

Therefore the function $\|\boldsymbol{a}(\beta, \boldsymbol{x})\|$ is not decreasing with respect to $\beta \in [0,1]$. □

Example 3.1. *It is not difficult to see that the errors are increasing function of β — a little bit "strange fact", because $\beta = 1$ is interpolatory approximation.*

Let $m = 4$, $l = 1$, the given data

$$\{(i, 2i) : i = 1, 3, 5, 7\}, \quad f(x) = 2x.$$

Let $\hat{L}_\beta(f)$ be the moving least-squares approximation of order $l = 1$ at a fixed point $x \in [0,7]$ with weight function $w_2(1, \beta, x, y)$.

Then

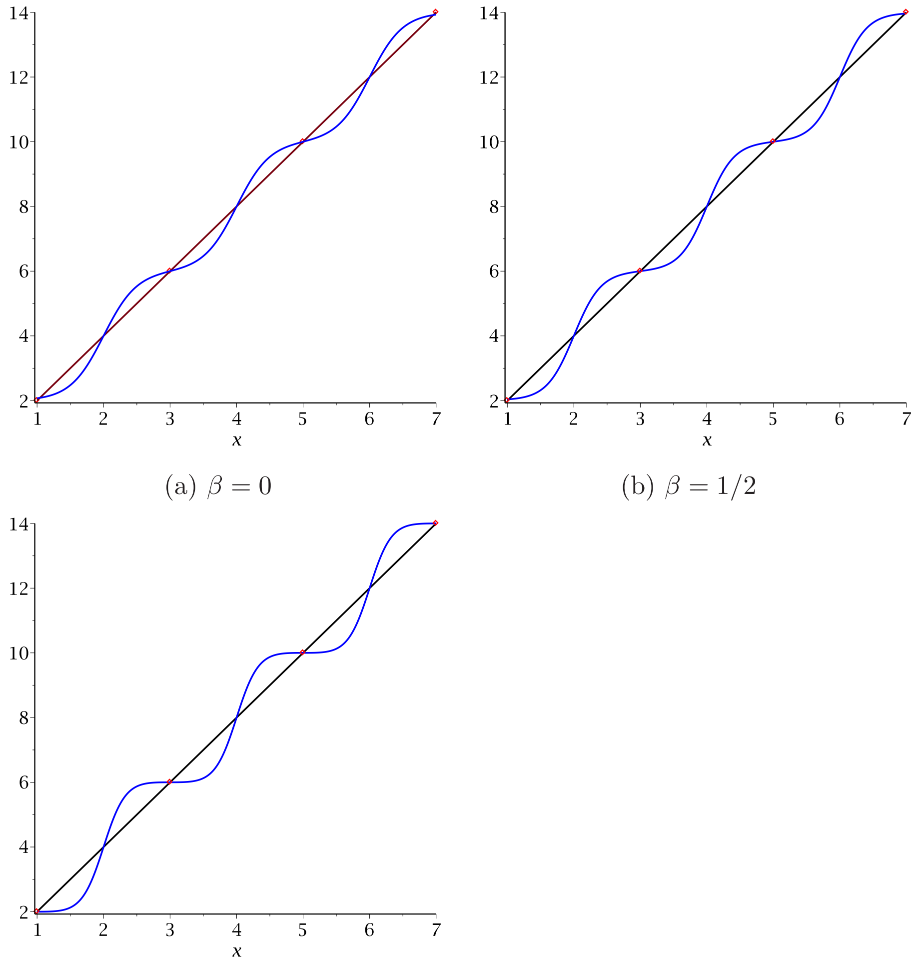
$$E = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \; c = \begin{pmatrix} 1 \end{pmatrix},$$

$$D_\beta(x) = 2\begin{pmatrix} w_2(1,\beta,x_1,x) & 0 & 0 & 0 \\ 0 & w(1,\beta,x_2,x) & 0 & 0 \\ 0 & 0 & w(1,\beta,x_3,x) & 0 \\ 0 & 0 & 0 & w(1,\beta,x_4,x) \end{pmatrix}.$$

Then $A_0 = D_\beta^{-1}(x) E \left(E^t D_\beta^{-1}(x) E\right)^{-1}$ and

$$\hat{L}_\beta(f) = 2\sum_{i=1}^{m} a_i(x) x_i.$$

Using Maple 18, it is not hard to display the plots of $\hat{L}_\beta(f)$, $\beta = 0, \frac{1}{2}, 1$, see Figure 1.



(a) $\beta = 0$

(b) $\beta = 1/2$

(c) $\beta = 1$

Figure 1: Plots of $\hat{L}_\beta(f)$, $x \in [0, 7]$.

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