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ON PERTURBED FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract: In this paper, we present the sufficient criteria for the existence of solutions for perturbed fractional differential equations and inclusions with generalized Riemann-Liouville fractional integral boundary conditions. We make use of a nonlinear alternative, which deals with the sum of completely continuous and contractive single-valued or multi-valued operators, to obtain the desired results.

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In Memory of Drumi Bainov
July 2, 1933 – July 1, 2011.

1. Introduction

The subject of fractional differential equations has emerged as an interesting and popular field of research in view of its extensive applications in applied and technical sciences. One can easily observe the role and importance of fractional calculus in several diverse disciplines such as physics, chemical processes, population dynamics, biotechnology, economics, etc. For examples and recent development on the topic, see [1]-[19] and the references cited therein. The significance of fractional derivatives owes to the fact that they serve as an excellent tool for the description of memory and hereditary properties of various materials and processes. One can notice that fractional derivatives are defined via fractional integrals. Among several types of fractional integrals found in the literature, Riemann-Liouville and Hadamard fractional integrals are the extensively studied ones. A new fractional integral, called *generalized Riemann-Liouville fractional integral*, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, was introduced in [20] (see Definition 2.6). For more details of this integral and similar ones, we refer the reader to the text [21] and the papers [22, 23].

In this paper, we study perturbed boundary value problems of fractional differential equations and inclusions supplemented with generalized Riemann-Liouville fractional integral conditions. In precise terms, we consider the single-valued and multi-valued nonlocal problems respectively given by

$$\begin{cases} D^{\alpha}x(t) = f(t,x(t)) + g(t,x(t)), \quad t \in [0,T], \\ x(0) = 0, \\ x(T) = \beta \dfrac{\rho^{1-q}}{\Gamma(q)} \displaystyle\int_0^{\xi} \frac{s^{\rho-1}}{(\xi^{\rho}-s^{\rho})^{1-q}} x(s)ds := \beta \, {}^{\rho}I^{q}x(\xi), \quad 0<\xi<T, \end{cases} \tag{1}$$

and

$$\begin{cases} D^{\alpha}x(t) \in F(t,x(t)) + G(t,x(t)), \quad t \in [0,T], \\ x(0) = 0, \quad x(T) = \beta \, {}^{\rho}I^{q}x(\xi), \quad 0<\xi<T, \end{cases} \tag{2}$$

where D^{α} is the Caputo fractional derivative of order $1 < \alpha \le 2$, $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, ${}^{\rho}I^{q}$ is the generalized Riemann-Liouville fractional integral of orders $q > 0$, $\rho > 0$, and $F, G : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are multi-valued functions, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

The paper is organized as follows: In Section 2 we present some preliminary concepts of fractional calculus and lemmas. Section 3 contains the existence result for problem (1) which is established via a fixed point theorem due to Burton and Kirk [24]. In Section 4, we discuss the existence of solutions for problem (2) by means of a nonlinear alternative for contractive maps [25].

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present some results needed in the sequel.

Definition 2.1. The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$ and Γ is the gamma function given by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)ds, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of a real number q, provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3. The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^cD^q f(t) = D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < q < n.$$

Remark 2.4. If $f(t) \in C^n[0, \infty)$, then

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \ t > 0, \ n-1 < q < n.$$

Lemma 2.5. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

In view of Lemma 2.5, it follows that

$$I^q \, {}^cD^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{3}$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Definition 2.6. [20] The generalized Riemann-Liouville fractional integral of order $q > 0$ and $\rho > 0$, of a function $f(t)$, for all $0 < t < \infty$, is defined as

$$^{\rho}I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Remark 2.7. For $\rho = 1$ in the above definition, we arrive at the standard Riemann-Liouville fractional integral, which is used to define both the Riemann-Liouville and Caputo fractional derivatives, while, in the limit $\rho \to 0$, we have

$$\lim_{\rho \to 0} {}^{\rho}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds,$$

which is the famous Hadamard fractional integral. See [20].

Lemma 2.8. *Let q and p be the positive constants. Then*

$$^{\rho}I^q t^p = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)} \frac{t^{p+\rho q}}{\rho^q}. \tag{4}$$

Proof. From Definition 2.6, we have

$$\begin{aligned} ^{\rho}I^q t^p &= \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} s^p}{(t^\rho - s^\rho)^{1-q}} \, ds = \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} \int_0^1 \frac{u^{\frac{p}{\rho}}}{(1-u)^{1-q}} \, du \\ &= \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} B\left(\frac{p+\rho}{\rho}, q \right) = \frac{t^{p+\rho q}}{\rho^q} \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}. \end{aligned}$$

This completes the proof. □

Lemma 2.9. *For any $y \in C([0,T], \mathbb{R})$, a function $x \in C^2([0,T], \mathbb{R})$ is a solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), \quad 1 < \alpha \leq 2, \\ x(0) = 0, \quad x(T) = \beta \, {}^{\rho}I^q x(\xi), \quad 0 < \xi < T, \end{cases} \tag{5}$$

if and only if

$$x(t) = J^{\alpha}y(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}y(\xi) - J^{\alpha}y(T)\Big\}, \tag{6}$$

where

$$\Lambda = T - \beta\frac{\xi^{\rho q+1}}{\rho^{q}}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})} \neq 0. \tag{7}$$

Proof. It is well known that the general solution of the fractional differential equation in (5) can be written as

$$x(t) = c_0 + c_1 t + J^{\alpha}y(t), \tag{8}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Using the first condition ($x(0) = 0$) given by (5) in (8), we get $c_0 = 0$. Applying the generalized fractional integral operator on (8) and using Lemma 2.8, we obtain

$${}^{\rho}I^{q}x(t) = {}^{\rho}I^{q}J^{\alpha}y(t) + c_1\frac{t^{\rho q+1}}{\rho^{q}}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}, \tag{9}$$

which, together with the second condition of (5), yields

$$J^{\alpha}y(T) + c_1 T = \beta\ {}^{\rho}I^{q}J^{\alpha}y(\xi) + \beta c_1\frac{\xi^{\rho q+1}}{\rho^{q}}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}. \tag{10}$$

Using the notation (7) in (10), we find that

$$c_1 = \frac{1}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}y(\xi) - J^{\alpha}y(T)\Big\}.$$

Substituting the values of c_0 and c_1 in (8), we obtain (6). Conversely, it can easily be shown by direct computation that x given by the integral equation (6) satisfies the problem (5). This completes the proof. □

Throughout this paper, for convenience of proving, we let the notations

$$J^{z}f(s,x(s))(y) = \frac{1}{\Gamma(z)}\int_0^y (y-s)^{z-1}f(s,x(s))ds,$$

$${}^{\rho}I^{z}f(s,x(s))(y) = \frac{\rho^{1-z}}{\Gamma(z)}\int_0^y \frac{s^{\rho-1}f(s,x(s))}{(y^{\rho}-s^{\rho})^{1-z}}ds,$$

where $z > 0$ and $y \in [0,T]$.

3. Existence result for problem (1)

We denote by $\mathcal{C} = C([0,T], \mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. Also by $L^1([0,T], \mathbb{R})$ we denote the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

Our main result in this section is based upon the following fixed point theorem due to Burton and Kirk [24].

Theorem 3.1. *Let X be a Banach space, and $A, B : X \to X$ two operators such that:*

(i) A is a contraction, and

(ii) B is completely continuous.

Then either

(a) the operator equation $y = A(y) + B(y)$ has a solution, or

(b) the set $\mathcal{E} = \left\{u \in X : \lambda A\left(\frac{u}{\lambda}\right) + \lambda B(u) = u\right\}$ is unbounded for $\lambda \in (0,1)$.

Theorem 3.2. *Assume that $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In addition we suppose that:*

(H_1) there exist a function $k(t) \in L^1(J, \mathbb{R}_+)$ such that

$$|g(t,u) - g(t,\overline{u})| \leq k(t)|u - \overline{u}|, \quad \text{for a.e. } t \in J, \ u, \ \overline{u} \in \mathbb{R},$$

with

$$\gamma := J^\alpha k(s)(T) + \frac{T}{|\Lambda|} J^\alpha k(s)(T) + \frac{|\beta| T}{|\Lambda|} \, {}^\rho I^q J^\alpha k(s)(\xi) < \frac{1}{2}; \tag{11}$$

(H_2) there exists a function $p \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t), \quad \text{for a.e. } t \in J, \text{ and each } u \in \mathbb{R}.$$

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. Let us transform the problem (1) into a fixed point problem. For that, we introduce an operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ associated with the problem (1) as

$$\begin{aligned}(\mathcal{P}x)(t) \;&= J^{\alpha}[f(s,x(s)) + g(s,x(s))](t) \\ &+\frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}[f(s,x(s)) + g(s,x(s))](\xi) \\ &-J^{\alpha}[f(s,x(s)) + g(s,x(s))](T)\Big\}, \quad t \in [0,T].\end{aligned} \tag{12}$$

Then we split the operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ defined by (12) as

$$(\mathcal{P}x)(t) = (\mathcal{P}_1 x)(t) + (\mathcal{P}_2 x)(t), \quad t \in [0,T], \tag{13}$$

where $\mathcal{P}_{1,2} : \mathcal{C} \to \mathcal{C}$ are given by

$$(\mathcal{P}_1 x)(t) = J^{\alpha} f(s,x(s))(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha} f(s,x(s))(\xi) - J^{\alpha} f(s,x(s))(T)\Big\}, \tag{14}$$

$$(\mathcal{P}_2 x)(t) = J^{\alpha} g(s,x(s))(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha} g(s,x(s))(\xi) - J^{\alpha} g(s,x(s))(T)\Big\}. \tag{15}$$

Obviously the existence of a fixed point for the operator $\mathcal{P}$ is equivalent to the existence of one for $\mathcal{P}_1 + \mathcal{P}_2$. So we shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all conditions of Theorem 3.1. The proof will be given in several steps.

Step 1. The operator $\mathcal{P}_1$ defined by (14) is continuous.

Let $\{x_n\} \subset B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,T] \times [-r,r]$, it follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ is uniformly valid on $[0,T]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$, which proves the continuity of $\mathcal{P}_1$.

Step 2. The operator $\mathcal{P}_1$ maps bounded sets into bounded sets in $\mathcal{C}$.

It is enough to show that for any $r > 0$, there exists a positive constant L such that for each $x \in B_r$, we have $\|\mathcal{P}_1 x\| \le L$. Let $x \in B_r$. Then

$$\begin{aligned}\|\mathcal{P}_1 x\| \;&\le\; J^{\alpha}|f(s,x(s))|(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^{\rho}I^{q}J^{\alpha}|f(s,x(s))|(\xi) + J^{\alpha}|f(s,x(s))|(T)\Big\} \\ &\le\; J^{\alpha}p(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}p(s)(T) := L.\end{aligned}$$

Step 3. The operator $\mathcal{P}_1$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.

Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then, for each $x \in B_r$, we obtain

$$\begin{aligned}
& |(\mathcal{P}_1 x)(\tau_2) - (\mathcal{P}_1 x)(\tau_1)| \\
\leq\ & |J^\alpha f(s,x(s))(\tau_2) - J^\alpha f(s,x(s))(\tau_1)| + \frac{|\tau_2-\tau_1|}{|\Lambda|} J^\alpha |f(s,x(s))|(T) \\
& + \frac{|\beta||\tau_2-\tau_1|}{|\Lambda|} |{}^\rho I^q J^\alpha |f(s,x(s))|(\xi) \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}] p(s) ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} p(s) ds \right| \\
& + \frac{|\tau_2-\tau_1|}{|\Lambda|} \Big(J^\alpha p(s)(T) + |\beta|\ {}^\rho I^q J^\alpha p(s)(\xi) \Big),
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{P}_1$ is equicontinuous. Thus, by the above three steps, the operator $\mathcal{P}_1$ is completely continuous.

Step 4. The operator $\mathcal{P}_2$ defined by (15) is a contraction.

For $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, from the definition of $\mathcal{P}_2$ and assumption (H_1), we obtain

$$\begin{aligned}
& |(\mathcal{P}_2 x)(t) - (\mathcal{P}_2 y)(t)| \\
\leq\ & J^\alpha |g(s,x(s)) - g(s,y(s))|(t) + \frac{T}{|\Lambda|} J^\alpha |g(s,x(s)) - g(s,y(s))|(T) \\
& + \frac{|\beta| T}{|\Lambda|}\ {}^\rho I^q J^\alpha |g(s,x(s)) - g(s,y(s))|(\xi) \\
\leq\ & \Big\{ J^\alpha k(s)(T) + \frac{T}{|\Lambda|} J^\alpha k(s)(T) + \frac{|\beta| T}{|\Lambda|}\ {}^\rho I^q J^\alpha k(s)(\xi) \Big\} \|x-y\|.
\end{aligned}$$

Hence

$$\|(\mathcal{P}_2 x) - (\mathcal{P}_2 y)\| \leq \gamma \|x-y\|.$$

As $\gamma < 1$ by (11), the operator $\mathcal{P}_2$ is a contraction map from the Banach space $\mathcal{C}$ into itself.

Step 5. A priori bounds on solutions.

Now it remains to show that the set $\mathcal{E} = \Big\{ u \in \mathcal{C} : \lambda \mathcal{P}_1\left(\frac{u}{\lambda}\right) + \lambda \mathcal{P}_2(u) = u \Big\}$ is unbounded for some $\lambda \in (0,1)$.

Let $\lambda \in (0,1)$ and $x \in \mathcal{E}$ be a solution of the integral equation

$$\begin{aligned} & x(t) \\ = \; & \lambda J^\alpha f(s,x(s))(t) + \lambda \frac{t}{\Lambda}\Big\{\beta \; {}^\rho I^q J^\alpha f(s,x(s))(\xi) - J^\alpha f(s,x(s))(T)\Big\} \\ & + \lambda J^\alpha g\Big(s, \frac{x(s)}{\lambda}\Big)(t) + \lambda \frac{t}{\Lambda}\Big\{\beta \; {}^\rho I^q J^\alpha g\Big(s, \frac{x(s)}{\lambda}\Big)(\xi) - J^\alpha g\Big(s, \frac{x(s)}{\lambda}\Big)(T)\Big\}, \\ & t \in [0,T]. \end{aligned}$$

Then, for each $t \in [0,T]$, we have

$$\begin{aligned} |x(t)| \;\leq\; & J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \; {}^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T) \\ & + \lambda J^\alpha \left[\left| g\Big(s, \frac{x(s)}{\lambda}\Big) - g(s,0)\right| + |g(s,0)|\right](T) \\ & + \frac{|\beta| T}{|\Lambda|} \; {}^\rho I^q J^\alpha \left[\left| g\Big(s, \frac{x(s)}{\lambda}\Big) - g(s,0)\right| + |g(s,0)|\right](\xi) \\ & + \frac{T}{|\Lambda|} J^\alpha \left[\left| g\Big(s, \frac{x(s)}{\lambda}\Big) - g(s,0)\right| + |g(s,0)|\right](T) \\ \leq\; & J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \; {}^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T) \\ & + \|x\| \left\{ J^\alpha k(s)(T) + \frac{|\beta| T}{|\Lambda|} \; {}^\rho I^q J^\alpha k(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha k(s)(T)\right\} \\ & + g^* \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\}, \end{aligned}$$

where $g^* = \sup\{|g(t,0)|, t \in [0,T]\}$, which yields

$$\begin{aligned} (1-\gamma)\|x\| \leq\; & J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \; {}^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T) \\ & + g^* \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\}. \end{aligned}$$

Consequently we have

$$\|x\| \;\leq\; M := \frac{1}{(1-\gamma)}\left[J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \; {}^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T) \right.$$

$$+g^*\left\{\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\}\Bigg],$$

which means that the set $\mathcal{E}$ is bounded, since $\gamma < 1/2$. Hence, part (b) of Theorem 3.1 does not hold. Thus part (a) holds, that is, $\mathcal{P}$ has a fixed point in $[0,T]$, and consequently the problem (1) has a solution. This completes the proof. □

4. Existence result for problem (2)

First of all, we recall some basic concepts for multi-valued maps [26, 27, 28].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;

(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E;

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;

(vi) is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable;

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multi-valued operator G will be denoted by $FixG$.

Definition 4.1. A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T], \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a. e. $t \in [0,T]$.

For each $x \in \mathcal{C}$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1([0,T], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall two useful results regarding closed graphs and upper-semicontinuity.

Lemma 4.2. *([26, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.3. *([29]) Let X be a Banach space. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T], X)$ to $C([0,T], X)$. Then the operator*

$$\Theta \circ S_{F,x} : C([0,T], X) \to \mathcal{P}_{cp,c}(C([0,T], X)), \quad x \mapsto (\Theta \circ S_{F,x})(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,T], X) \times C([0,T], X)$.

To prove our main result in this section, we use the following form of the nonlinear alternative for contractive maps [25, Corollary 3.8].

Theorem 4.4. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators such that*

(a) Z_1 is contraction, and

(b) Z_2 is upper semi-continuous and compact.

Then, if $Q = Z_1 + Z_2$, *either*

(i) Q *has a fixed point in* $\bar{D}$ *or*

(ii) *there is a point* $u \in \partial D$ *and* $\lambda \in (0,1)$ *with* $u \in \lambda Q(u)$.

Definition 4.5. A function $x \in C([0,T],\mathbb{R})$ possessing a fractional derivative of order $\alpha \in (1,2]$ is a solution of the problem (2) if $x(0) = 0$, $x(T) = \beta\ {}^{\rho}I^{q}x(\xi)$, and there exist functions $f, g \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ and $g(t) \in G(t,x(t))$ a.e. on $[0,T]$ and

$$x(t) = J^{\alpha}[f(s)+g(s)](t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}[f(s)+g(s)](\xi) - J^{\alpha}[f(s)+g(s)](T)\Big\}. \quad (16)$$

Theorem 4.6. *Assume that:*

(A_1) $F : [0,T]\times\mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory;*

(A_2) *there exists a continuous nondecreasing function* $\Phi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\Phi(\|x\|) \quad \text{for each } (t,x) \in [0,T]\times\mathbb{R};$$

(A_3) *the multi-valued map* $t \to G(t,x)$ *is measurable for each* $x \in \mathbb{R}$ *and integrably bounded, i.e., there exists a function* $M \in L^1([0,T],\mathbb{R})$ *such that*

$$|G(t,x)| := \sup\{|g| : g(t) \in G(t,x)\} \le M(t), \quad \text{for a.e. } t \in [0,T] \text{ and } x \in \mathbb{R};$$

(A_4) *for* $G : [0,T]\times\mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$, *there exists a function* $\ell \in L^1([0,T],\mathbb{R})$ *such that*

$$H_d(G(t,x) - G(t,y)) \le \ell(t)|x-y|, \ t \in [0,T]$$

for all $x, y \in \mathbb{R}$ *with*

$$J^{q}\ell(s)(T) + \frac{T}{|\Lambda|}\left(|\beta|\ {}^{\rho}I^{q}J^{q}\ell(s)(\xi_i) + J^{q}\ell(s)(T)(T)\right) < 1; \quad (17)$$

(A_5) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\Phi(M)\Psi_1 + \Psi_2} > 1, \tag{18}$$

where

$$\Psi_1 = J^{\alpha}p(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}p(s)(T)$$

and

$$\Psi_2 = J^{\alpha}M(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}M(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}M(s)(T).$$

Then the boundary value problem (2) has at least one solution on $[0, T]$.

Proof. To transform the problem (2) into a fixed point problem, we define an operator $\mathcal{N} : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C})$ by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in \mathcal{C} : \\ h(t) = \left\{ \begin{array}{l} J^{\alpha}[f(s) + g(s)](t) + \dfrac{t}{\Lambda}\Big\{\alpha^{\rho}I^{q}J^{\alpha}[f(s) + g(s)](\xi) \\ -J^{\alpha}[f(s) + g(s)](T)\Big\}, \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$ and $g \in S_{G,x}$.

Next we introduce the multi-valued operators

$$\mathcal{A}, \mathcal{B} : C([0, T], \mathbb{R}) \longrightarrow \mathcal{P}(C([0, T], \mathbb{R}))$$

by

$$\mathcal{A}(x) = \left\{h \in \mathcal{C} : h(t) = J^{\alpha}f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}f(s)(\xi) - J^{\alpha}f(s)(T)\Big\}\right\} \tag{19}$$

for $f \in S_{F,x}$ and

$$\mathcal{B}(x) = \left\{z \in \mathcal{C} : z(t) = J^{\alpha}g(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}g(s)(\xi) - J^{\alpha}g(s)(T)\Big\}\right\} \tag{20}$$

for $g \in S_{G,x}$. Observe that $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.4 on $[0, T]$. First, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ define the multivalued operators, that is, $\mathcal{A}, \mathcal{B} : B_r \to \mathcal{P}_{cp,c}(\mathcal{C})$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ is a bounded set in $\mathcal{C}$. First we prove that $\mathcal{A}$ is compact-valued on B_r. Note that the operator $\mathcal{A}$ is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1([0,T],\mathbb{R})$ into $\mathcal{C}$, defined by

$$\mathcal{L}(v)(t) = J^\alpha v(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha v(s)(\xi) - J^\alpha v(s)(T)\Big\}.$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t))$ for almost all $t \in [0,T]$. Since $F(t,x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ pointwise on $[0,T]$.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
& |\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)| \\
\le\ & |J^\alpha v_n(s)(t_2) - J^\alpha v_n(s)(t_1)| + \frac{|t_2 - t_1|}{|\Lambda|} J^q |v_n(s)|(T) \\
& + \frac{|\beta||t_2 - t_1|}{|\Lambda|} |{}^\rho I^q J^q |v_n(s)|(\xi) \\
\le\ & \frac{\psi(r)}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] p(s) ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} p(s) ds \right| \\
& + \psi(r) \frac{|t_2 - t_1|}{|\Lambda|} \Bigg(J^q p(s)(T) + |\beta| |{}^\rho I^q J^q p(s)(\xi) \Bigg).
\end{aligned}$$

We see that the right hand of the above inequality tends to zero, independently of x, as $t_2 \to t_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equi-continuous and by using the Arzelá-Ascoli theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that, $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{A}(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{A}(x)$ is compact.

Now, we show that $\mathcal{A}(x)$ is convex for all $x \in \mathcal{C}$. Let $h_1, h_2 \in \mathcal{A}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$h_i(t) = J^\alpha f_i(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f_i(s)(\xi) - J^\alpha f_i(s)(T)\Big\}, \quad i = 1, 2,$$

for almost all $t \in [0,T]$. Let $0 \le \lambda \le 1$. Then, we have

$$[\lambda h_1 + (1-\lambda)h_2](t) = J^\alpha[\lambda f_1(s) + (1-\lambda) f_2(s)](t)$$

$$+\frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}[\lambda f_1(s)+(1-\lambda)f_2(s)](\xi)-J^{\alpha}[\lambda f_1(s)+(1-\lambda)f_2(s)](s)(T)\Big\}.$$

Since F has convex values, so $S_{F,x}$ is convex and $\lambda f_1(s)+(1-\lambda)f_2(s)\in S_{F,x}$. Thus

$$\lambda h_1+(1-\lambda)h_2\in\mathcal{A}(x).$$

Consequently, $\mathcal{A}$ is convex-valued. For $\mathcal{B}$ we work in a similar way.

The rest of the proof consists of several steps and claims.

Step 1: We show that $\mathcal{B}$ *is a contraction on* $\mathcal{C}$. Let $x,\bar{x}\in C([0,T],\mathbb{R})$ and $z_1\in\mathcal{G}(x)$. Then there exists $g_1(t)\in G(t,x(t))$ such that, for each $t\in[0,T]$,

$$z_1(t)=J^{q}g_1(s)(t)+\frac{t^{q-1}}{\Lambda}\Big(\beta\ {}^{\rho}I^{q}J^{\alpha}g_1(s)(\xi_i)-J^{q}g_1(s)(T)\Big).$$

By (A_4), we have

$$H_d(G(t,x),G(t,\bar{x}))\le\ell(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w\in G(t,\bar{x}(t))$ such that

$$|v_1(t)-w|\le\ell(t)|x(t)-\bar{x}(t)|,\quad t\in[0,T].$$

Define $U:[0,T]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le\ell(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t)\cap G(t,\bar{x}(t))$ is measurable (Proposition III.4 [30]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in G(t,\bar{x}(t))$ and for each $t\in[0,T]$, we have $|v_1(t)-v_2(t)|\le\ell(t)|x(t)-\bar{x}(t)|$.

For each $t\in[0,T]$, let us define

$$z_2(t)=J^{q}g_2(s)(t)+\frac{t}{\Lambda}\Big(\beta\ {}^{\rho}I^{q}J^{\alpha}g_2(s)(\xi_i)-J^{q}g_2(s)(T)\Big).$$

Thus,

$$\begin{aligned}&|z_1(t)-z_2(t)|\\ \le\ & J^{q}|g_1(s)-g_2(s)|(t)\\ &+\frac{t}{|\Lambda|}\left(\beta\ {}^{\rho}I^{q}J^{\alpha}|g_1(s)-g_2(s)|(\xi_i)+J^{q}|g_1(s)-g_2(s)|(T)\right)\end{aligned}$$

$$\leq \left\{ J^q\ell(s)(T) + \frac{T}{|\Lambda|}\Big(|\beta| \ ^\rho I^q J^q \ell(s)(\xi_i) + J^q\ell(s)(T)(T)\Big)\right\}\|x-\bar{x}\|.$$

Hence,

$$\|z_1 - z_2\| \leq \left\{ J^q\ell(s)(T) + \frac{T}{|\Lambda|}\Big(|\beta| \ ^\rho I^q J^q \ell(s)(\xi_i) + J^q\ell(s)(T)(T)\Big)\right\}\|x-\bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\mathcal{A}(x), \mathcal{A}(\bar{x})) \leq \delta \|x - \bar{x}\|$$

where

$$\delta = J^q\ell(s)(T) + \frac{T}{|\Lambda|}\Big(|\beta| \ ^\rho I^q J^q \ell(s)(\xi_i) + J^q\ell(s)(T)(T)\Big).$$

So $\mathcal{A}$ is a contraction, since $\delta < 1$ by (17).

Step 2: *$\mathcal{A}$ is compact and upper semi-continuous.* This will be established in several claims.

Claim I: *$\mathcal{A}$ maps bounded sets into bounded sets in $\mathcal{C}$.* Let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a bounded set in $\mathcal{C}$. Then, for each $h \in \mathcal{A}(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = J^\alpha f(s)(t) + \frac{t}{\Lambda}\Big\{ \beta \ ^\rho I^q J^\alpha f(s)(\xi) - J^\alpha f(s)(T)\Big\}.$$

Then, for $t \in [0, T]$, we have

$$\begin{aligned}
|h(t)| &\leq J^\alpha |f(s)|(T) + \frac{T}{|\Lambda|}\Big\{ |\beta|^\rho I^q J^\alpha |f(s)|(\xi) - J^\alpha |f(s)|(T)\Big\} \\
&\leq \psi(\|x\|) J^\alpha p(s)(T) + \psi(\|x\|)\frac{|\beta| T}{|\Lambda|}^{\rho} I^q J^\alpha p(s)(\xi) + \psi(\|x\|)\frac{T}{|\Lambda|} J^\alpha p(s)(T) \\
&\leq \psi(\|x\|)\left\{ J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \ ^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T)\right\}.
\end{aligned}$$

Thus,

$$\|h\| \leq \psi(r)\left\{ J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \ ^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T)\right\}.$$

Claim II: *$\mathcal{B}$ maps bounded sets into equi-continuous sets.* Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then, for each $h \in \mathcal{A}(x)$, we obtain

$$\begin{aligned}
& |h(\tau_2) - h(\tau_1)| \\
\leq\ & |J^\alpha f(s)(\tau_2) - J^\alpha f(s)(\tau_1)| + \frac{|\tau_2 - \tau_1|}{|\Lambda|} J^\alpha |f(s)|(T) \\
& + \frac{|\beta||\tau_2 - \tau_1|}{|\Lambda|} |^\rho I^q J^\alpha |f(s)|(\xi) \\
\leq\ & \frac{\psi(r)}{\Gamma(q)} \left| \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] p(s) ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} p(s) ds \right| \\
& + \psi(r) \frac{|\tau_2 - \tau_1|}{|\Lambda|} \Big(J^q p(s)(T) + |\beta|\ {}^\rho I^q J^q p(s)(\xi) \Big).
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_2 - \tau_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{A} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is completely continuous.

Next we show that $\mathcal{B}$ is an upper semi-continuous multi-valued mapping. It is known by Lemma 4.2 that $\mathcal{B}$ will be upper semicontinuous if we establish that it has a closed graph, since it is already shown to be completely continuous. Thus we will prove that *$\mathcal{A}$ has a closed graph.*

Claim III: *$\mathcal{A}$ has a closed graph.* Let $x_n \to x_*, h_n \in \mathcal{A}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{A}(x_*)$. Associated with $h_n \in \mathcal{A}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$h(t) = J^\alpha f_n(s)(t) + \frac{t}{\Lambda} \Big\{ \beta\ {}^\rho I^q J^\alpha f_n(s)(\xi) - J^\alpha f_n(s)(T) \Big\}.$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = J^\alpha f_*(s)(t) + \frac{t}{\Lambda} \Big\{ \beta\ {}^\rho I^q J^\alpha f_*(s)(\xi) - J^\alpha f_*(s)(T) \Big\}.$$

Let us consider the linear operator $\Theta : L^1([0,T], \mathbb{R}) \to \mathcal{C}$ given by

$$f \mapsto \Theta(f)(t) = J^\alpha f(s)(t) + \frac{t}{\Lambda} \Big\{ \beta\ {}^\rho I^q J^\alpha f(s)(\xi) - J^\alpha f(s)(T) \Big\}.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \Big\| J^\alpha (f_n(s) - f_*(s))(t)$$

$$+\frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}(f_n(s)-f_*(s))(\xi)-J^{\alpha}(f_n(s)-f_*(s))(T)\Big\}\Big\|\to 0,$$

as $n\to\infty$. Thus, it follows by Lemma 4.3 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, we have that

$$h_*(t)=J^{\alpha}f_*(s)(t)+\frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}f_*(s)(\xi)-J^{\alpha}f_*(s)(T)\Big\},$$

for some $f_*\in S_{F,x_*}$. Hence $\mathcal{A}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{A}$ is compact valued and upper semi-continuous.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.4 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x\in\lambda\mathcal{A}(x)+\lambda\mathcal{B}(x)$ for $\lambda\in(0,1)$, then there exist $f\in S_{F,x}$ and $g\in S_{G,x}$ such that

$$\begin{aligned}x(t)\ &=\ J^{\alpha}f(s)(t)+\frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}f(s)(\xi)-J^{\alpha}f(s)(T)\Big\}\\&\quad+J^{\alpha}g(s)(t)+\frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q}J^{\alpha}g(s)(\xi)-J^{\alpha}g(s)(T)\Big\},\quad t\in[0,T].\end{aligned}$$

By our assumptions, we can obtain

$$\begin{aligned}|x(t)|\ &\le\ \psi(\|x\|)\left\{J^{\alpha}p(s)(T)+\frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi)+\frac{T}{|\Lambda|}J^{\alpha}p(s)(T)\right\}\\&\quad+J^{\alpha}M(s)(T)+\frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}M(s)(\xi)+\frac{T}{|\Lambda|}J^{\alpha}M(s)(T).\end{aligned}$$

Thus

$$\begin{aligned}\|x\|\ &\le\psi(\|x\|)\left\{J^{\alpha}p(s)(T)+\frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi)+\frac{T}{|\Lambda|}J^{\alpha}p(s)(T)\right\}\\&\quad+J^{\alpha}M(s)(T)+\frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}M(s)(\xi)+\frac{T}{|\Lambda|}J^{\alpha}M(s)(T).\end{aligned}\tag{21}$$

If condition (ii) of Theorem 4.4 holds, then there exists $\lambda\in(0,1)$ and $x\in\partial B_M$ with $x=\lambda\mathcal{N}(x)$. Then, x is a solution of (2) with $\|x\|=M$. Now, by the inequality (21), we get

$$\frac{M}{\psi(M)\Psi_1+\Psi_2}\le 1,$$

which contradicts (18). Hence, $\mathcal{N}$ has a fixed point in $[0,T]$ by Theorem 4.4, and consequently the problem (2) has a solution. This completes the proof. □

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