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ON THE SOLUTION SET OF MULTIVALUED FUZZY FRACTIONAL SYSTEMS

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Abstract: In this paper we investigate fuzzy fractional integral inclusions under compactness type conditions. We prove the existence of solutions when the right-hand side is almost upper semicontinuous. We also show that the solution set is connected. Finally, an application to fuzzy fractional differential inclusions is given.

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In Memory of Professor Drumi Bainov
July 2, 1933 – July 1, 2011.

1. Introduction

The modeling of the real problems often requires some kinds of uncertainty and it is one of the main reasons to investigate the fuzzy systems and multivalued differential equations. Starting with the work of Kaleva [19], the theory of fuzzy differential and integral equations was rapidly developed. We refer the reader to [19, 22, 23, 24, 26, 29]. However, the differential inclusions (multivalued differential equations) give more adequate models of the real processes and they are extensively used in optimal control (see, e.g. [11, 18]).

In the last years several directions of the theory of fractional differential equations and inclusions has rapidly developed due to their applications in physics, biological sciences, engineering, chemistry etc. (see, e.g. [7, 17]). The main properties of the fractional differential equations are studied in [13, 20, 21, 25]. Fractional differential inclusions are considered, among others, in [2, 15, 28].

Recently, the interest for fractional differential and integral equations with uncertainty has increased (see [1, 3, 5, 6, 28] and the references therein). To the authors knowledge, there are a few papers devoted to fuzzy fractional differential and integral inclusions (see, e.g. [4]).

Let $\mathbb{E}$ be the space of fuzzy sets and $I = [t_0, T]$. Let $F : I \times \mathbb{E} \rightrightarrows \mathbb{E}$ be a given multifunction. Consider the following fuzzy fractional integral inclusion

$$x(t) \in x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s)) ds, \ t \in I, \tag{1.1}$$

where $0 < q < 1$. As usual, $x(\cdot)$ is a solution of (1.1) if there exists a strongly measurable selection $f(s) \in F(s, x(s))$ such that

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds. \tag{1.2}$$

Our main target is to investigate the properties of the solution set of (1.1) under compactness type conditions. We prove a variant of Kneser's theorem with the aid of the locally Lipschitz approximations. This technique is commonly used to derive some topological properties of the solution set. We refer the reader to [10, 11, 18], where the results are restricted to reflexive Banach spaces.

It is suggested in [5] that the solutions of the following fuzzy fractional differential inclusion

$$D_c^q x(t) \in F(t, x(t)),\ t \in I = [t_0, T],\ x(t_0) = x_0, \tag{1.3}$$

where D_c^q is the Caputo derivative of order q, can be defined as the solutions of (1.1). Hence, our results are applicable also to (1.3).

2. Preliminaries

In this section we recall some definitions and results that we shall use in the paper. For the theory of ordinary and evolution differential inclusions we refer the reader to [11].

Let $\mathbb{E} = \{x : \mathbb{R}^n \to [0,1];\ x \text{ satisfies } (1) - (4)\}$ be the space of fuzzy numbers, where:

(1) x is normal, i.e. there exists $v_0 \in \mathbb{R}^n$ such that $x(v_0) = 1$;

(2) x is fuzzy convex, i.e. $x(\lambda v + (1-\lambda)w) \geq \min\{x(v), x(w)\}$ whenever $v, w \in \mathbb{R}^n$ and $\lambda \in [0,1]$;

(3) x is upper semicontinuous;

(4) the closure of the set $\{v \in \mathbb{R}^n;\ x(v) > 0\}$ is compact.

For $x \in \mathbb{E}$ and $\alpha \in (0,1]$, the set $[x]^\alpha = \{v \in \mathbb{R}^n;\ x(v) \geq \alpha\}$ is called α-level set of x. It follows from (1) – (4) that the α-level sets $[x]^\alpha$ are nonempty, convex and compact subsets of $\mathbb{R}^n$ for all $\alpha \in (0,1]$. The fuzzy zero is defined by $\hat{0}(v) = \begin{cases} 0 \text{ for } v \neq 0, \\ 1 \text{ for } v = 0. \end{cases}$

The set $\mathbb{E}$ is a semilinear space with the following operations:

$$(x + y)(v) = \sup_{v_1 + v_2 = v} \min\{x(v_1), y(v_2)\},$$

$$\lambda(v) = \begin{cases} x(v/\lambda) \text{ at } \lambda \neq 0 \\ \chi_0(v) \text{ at } \lambda = 0, \end{cases}$$

where $x, y \in \mathbb{E}$ and $\lambda \in \mathbb{R}$. The metric in $\mathbb{E}$ is

$$D(x,y) = \sup_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha),$$

where $D_H(A,B) = \max\{\max_{a\in A}\min_{b\in B}|a-b|, \max_{b\in B}\min_{a\in A}|a-b|\}$ is the Hausdorff distance between the convex compact subsets of $\mathbb{R}^n$. Then, $\mathbb{E}$ is a complete semilinear metric space with respect to the metric D (see [26]). This space is not locally compact and it is nonseparable. The metric D satisfies the following properties:

(i) $D(x+y, z+y) = D(x,z)$,

(ii) $D(\lambda x, \lambda y) = \lambda D(x,y)$,

(iii) $D(x+\overline{x}, y+\overline{y}) \le D(x,y) + D(\overline{x}, \overline{y})$, for any $\lambda \ge 0$ and $x, y, z, \overline{x}, \overline{y} \in \mathbb{E}$.

The distance from $x \in \mathbb{E}$ to the closed bounded set $A \subset \mathbb{E}$ is defined by

$$dist(x,A) = \inf_{y\in A} D(x,y)$$

and the Hausdorff distance between two closed and bounded subsets A, B of $\mathbb{E}$ is defined by

$$d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} D(x,y), \sup_{y\in B}\inf_{x\in A} D(x,y)\}.$$

The function $f : I \to \mathbb{E}$ is said to be continuous if it is continuous with respect to the metric D. The map $f : I \to \mathbb{E}$ is said to be strongly measurable if there exists a sequence $\{f_m\}_{m=1}^{\infty}$ of step functions $f_m : I \to \mathbb{E}$ such that $\lim_{m\to\infty} D(f_m(t), f(t)) = 0$ for a.a. $t \in I$. Clearly, $[f]^\alpha(\cdot)$ are strongly measurable for every $\alpha \in (0,1]$ if $f(\cdot)$ is strongly measurable. The converse does not necessarily holds (see [19]).

If $f : I \to \mathbb{E}$ is strongly measurable and $D(f(t), \hat{0}) \le \lambda(t)$, where $\lambda(\cdot)$ is Lebesgue integrable real valued function, then f is Bochner integrable and

$$\int_{t_0}^{t} f(s)ds = \lim_{m\to\infty} \int_{t_0}^{t} f_m(s)ds,$$

where $f_m(\cdot)$ are the step functions with $f_m(t) \to f(t)$ for a.a. $t \in I$.

In the fuzzy set literature starting from [26], the integral of fuzzy functions is defined levelwise, i.e. there exists $g(t) \in \mathbb{E}$ such that $[g]^\alpha(t) = \int_{t_0}^{t} [f]^\alpha(s)ds$. As it is shown in [19], there are levelwise integrable functions which are not almost everywhere separably valued, i.e. not Bochner integrable.

The function $g : I \to \mathbb{E}$ is called absolutely continuous if there exists a strongly measurable function $f : I \to \mathbb{E}$ such that $g(t) = \int\limits_{t_0}^{t} f(s)ds$.

The space $\mathbb{E}$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$ (see Theorem 2.1 of [19]). The embedding map $j : \mathbb{E} \to \mathbb{X}$ is isometry and isomorphism. Hence, $f : I \to \mathbb{E}$ is continuous if and only if $j(f)(\cdot)$ is continuous. Furthermore, $j(\cdot)$ preserves differentiation and integration. Namely, if $\dot{f}(t)$ exists, then $\frac{d}{dt}j(f)(t)$ also exists and $j\left(\dot{f}\right)(t) = \frac{d}{dt}j(f)(t)$, where $\frac{d}{dt}$ is the usual differential operator. Now, if $g : I \to \mathbb{E}$ is strongly measurable and integrable, then $j(g)(\cdot)$ is strongly measurable and Bochner integrable, and

$$j\left(\int\limits_{t_0}^{t} g(s)ds\right) = \int\limits_{t_0}^{t} j\left(g\right)(s)ds \text{ for all } t \in I. \tag{2.1}$$

The multifunction F is said to be almost USC when for every $\delta > 0$ there exists a compact set $I_\delta \subset I$ with Lebesgue measure $meas(I \setminus I_\delta) < \delta$ such that $F|_{I_\delta \times \mathbb{E}}$ is USC.

In the next section we recall some properties of the measure of noncompactness used in the paper. In the third section we focus our discussions on the differential inclusion (1.1).

3. Measures of Noncompactness

Let Y be a complete metric space and denote by $\mathcal{B}(Y)$ the family of all bounded subsets of Y. We recall that the Hausdorff measure of noncompactness $\beta_Y : \mathcal{B}(Y) \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\beta_Y(A) := \inf\{d > 0; A \text{ can be covered by finite number of balls with radius } d \text{ and centers in } Y\}.$$

The Kuratowski measure of noncompactness $\rho_Y : \mathcal{B}(Y) \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\rho(A) := \inf\{d > 0; A \text{ can be covered by finite number of sets with diameter } \leq d\}.$$

For any bounded set $A \subset Y$ we denote $\operatorname{diam}(A) = \sup_{a,b \in A} \varrho_Y(a,b)$, where $\varrho_Y(\cdot,\cdot)$ is the distance in Y. Then, it is easy to see that A is a subset of a ball with radius equal to $\operatorname{diam}(A)$.

Remark 3.1. It is known that if $A \subset L \subset Y$ then $\beta_Y(A) \leq \beta_L(A) \leq 2\beta_Y(A)$ and $\rho(A)$ does not depend on the subspace Y. Furthermore, $\rho(A) \leq 2\beta(A) \leq 2\rho(A)$.

Let $\gamma(\cdot)$ represents both $\rho(\cdot)$ and $\beta(\cdot)$. Then some properties of $\gamma(\cdot)$ are listed below. Let $A, B \in \mathcal{B}(Y)$.

(i) $\gamma(A) = 0$ if and only if $\bar{A}$ is compact.

(ii) $\gamma(\overline{co}\ A) = \gamma(A)$.

(iii) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$.

(iii) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(iv) $\gamma(A \bigcup B) = \max\{\gamma(A), \gamma(B)\}$.

(v) $\gamma(\cdot)$ is continuous with respect to the Hausdorff distance.

The following result regarding the imbedding map $j : \mathbb{E} \to \mathbb{X}$ will be used later in the paper.

Theorem 3.2. *Let A be a bounded subset in $\mathbb{E}$. Then $\beta(j(A)) \leq \beta(A) \leq 2\beta(j(A))$.*

Proof. We know that $j(\cdot)$ is an isometry and isomorphism, which implies that $\beta_{\mathbb{E}}(A) = \beta_{j(\mathbb{E})}(j(A))$. Since $j(\mathbb{E}) \subset \mathbb{X}$ then $\beta_{j(\mathbb{E})}(j(A)) \geq \beta_{\mathbb{X}}(j(A))$. Consequently, $\beta_{\mathbb{E}}(A) \geq \beta_{\mathbb{X}}(j(A))$. The latter can be written as $\beta(A) \geq \beta(j(A))$.

Now, we will prove the other part of the inequality. Since $j(\cdot)$ preserves the diameter, one has that $\rho(j(A)) = \rho(A)$. Using $\rho(j(A)) \leq 2\beta(j(A))$ (see [12] page 42) we have $\rho(A) \leq 2\beta(j(A))$. But $\beta(A) \leq \rho(A)$. Therefore $\beta(A) \leq 2\beta(j(A))$. □

Remark 3.3. From Theorem 3.2 we also get that $\beta(A) \leq 2\beta(j(A)) \leq 2\beta(A)$.

The following result is interesting itself because it proves the most important property of $\beta(\cdot)$ in the case of fuzzy sets. For the proof see [15].

Theorem 3.4. *Let $\{f_m(\cdot)\}_{m=1}^{\infty}$ be an integrally bounded sequence of strongly measurable fuzzy functions from I to $\mathbb{E}$. Then the function $t \to \beta(\{f_m(t), m \geq 1\})$ is measurable and*

$$\beta\left(\int_t^{t+h}\left\{\bigcup_{m=1}^{\infty} j(f_m(s))\right\} ds\right) \leq 2\int_t^{t+h} \beta\left\{\bigcup_{m=1}^{\infty} f_m(s)\right\} ds. \tag{3.1}$$

Definition 3.5. The multifunction $G : I \times \mathbb{E} \rightrightarrows \mathbb{E}$ is said to satisfy compactness type condition (CTC) if there exists a Perron function $w(\cdot,\cdot)$ such that $\beta(G(t,A)) \leq \frac{1}{2}w(t,\beta(A))$ for any bounded set $A \subset \mathbb{E}$.

We recall the definition of a Perron function. The Carathéodory function $g : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be Perron function if for every $r > 0$ there exists $m_r \in L^{\frac{1}{q_2}}(I, \mathbb{R}^+)$, $q_2 \in [0,q), q \in (0,1)$, such that $|g(t,w)| \leq m_r(t)$ for all $w \in r\mathbb{B}$, $g(t,\cdot)$ is monotone nondecreasing, $g(t,0) = 0$ for a.a. $t \in I$ and $u(t) \equiv 0$ is the unique solution of $u(t) \leq \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_0}^{t}(t-s)^{q-1}g(s,u(s))ds$, with $u(0) = 0$.

4. Fuzzy Fractional Integral Inclusions

In this section we will study the properties of the solution set of the fuzzy integral inclusion (1.1). To this aim, we need the following result regarding the fuzzy fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))ds,\ t \in (t_0,T]. \tag{4.1}$$

Theorem 4.1. *Assume that there exists a constant $\eta > 0$ such that $D(f(t,x),\hat{0}) \leq \eta(1 + D(x,\hat{0}))$, that $f(\cdot,x)$ is strongly measurable and $f(t,\cdot)$ is locally Lipschitz, i.e. for every $x \in \mathbb{E}$ there exists a neighborhood $U_x \ni x$ and a constant L_x such that $D(f(t,y),f(t,z)) \leq L_x D(y,z)$ for every $y,z \in U_x$. Then the integral equation (4.1) admits an unique solution on the interval I, which depends continuously on x_0.*

Proof. Suppose first that the needed unique solution $x(\cdot)$ exists on some subinterval $[t_0, \tau] \subset I$. By hypothesis, there exists $\delta > 0$ such that $f(t, \cdot)$ is Lipschitz with a constant $L > 0$ on $x(\tau) + \delta\mathbb{B}$. Suppose that $\tau < T$ and let $y, z \in C(I, \mathbb{E})$ with $y(t) = z(t) = x(t)$ on $[t_0, \tau]$. Since y and z are continuous in τ, there exists $\mu > 0$ such that $y(t)$ and $z(t)$ belong to $x(\tau) + \delta\mathbb{B}$ for every $t \in [\tau, \tau + \mu]$. For $w \in C(I, \mathbb{E})$ we define the operator

$$\mathbf{L}(w(t)) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, w(s)) ds.$$

Applying $\mathbf{L}$ to the functions $y(\cdot)$ and $z(\cdot)$, for $t \in [\tau, \tau + \mu]$ we have

$$\begin{aligned} D(\mathbf{L}(y(t)), \mathbf{L}(z(t))) =& \frac{1}{\Gamma(q)} D\left(\int_{t_0}^{t} (t-s)^{q-1} f(s, y(s)) ds, \right. \\ & \left. \int_{t_0}^{t} (t-s)^{q-1} f(s, z(s)) ds. \right) \\ \leq & \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} D(f(s, y(s)), f(s, z(s))) ds \\ \leq & \frac{L}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} D(y(s), z(s)) ds \\ \leq & \frac{L\mu^q}{\Gamma(q+1)} \max_{t \in [\tau, \tau+\mu]} D(y(s), z(s)). \end{aligned}$$

If μ is sufficiently small, then $\mathbf{L}$ is a contraction and hence the unique solution $x(\cdot)$ of (4.1) can be extended to $[t_0, \tau + \mu]$. A standard application of Zorn's lemma proves the existence and uniqueness on the whole interval I. Furthermore, $D(y(x_0, t), y(y_0, t)) \leq r(t)$, where $D_c^q r(t) = Lr(t)$ and $r(t_0) = D(x_0, y_0)$. □

Proposition 4.2. *Assume that there exists a positive constant* $\lambda > 0$ *such that*

$$\max_{v \in F(t,x)} D(v, \hat{0}) \leq \lambda(1 + D(x, \hat{0})). \tag{4.2}$$

Then the solution set of

$$D_c^q x(t) \in \overline{co}\ F(t, x(t) + \mathbb{B}),\ x(t_0) = x_0 \tag{4.3}$$

is Hölderian of degree q (if nonempty).

Proof. Suppose that there exists a solution $x(\cdot)$ of (4.3). Then,

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s) ds,$$

where the function g is a strongly measurable selection $g(s) \in \overline{co}\ F(s, x(s)+\mathbb{B})$. First, we will prove that $x(\cdot)$ is bounded. For this purpose observe that it follows from (4.2) that

$$\max_{v \in \overline{co}\ F(t,x+\mathbb{B})} D(v, \hat{0}) \leq \lambda(1 + D(x+\mathbb{B}, \hat{0})) \leq \lambda(2 + D(x, \hat{0}))$$

and we obtain that

$$D(x(t), \widehat{0}) \leq D(x_0, \widehat{0}) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \lambda(2 + D(x\left(s\right), \widehat{0})) ds.$$

Due to Lemma 7.1.1 in [16], and taking into account (4.2), we conclude that there exists a constant $M_1 > 0$ such that $D(x(t), \widehat{0}) \leq M_1$. Using again (4.2), we derive that

$$\max_{v \in F(t,x(t)+\mathbb{B})} D(v, \hat{0}) \leq M,$$

where $M = \lambda(2 + M_1)$.

Let $t_0 \leq \tau < t \leq T$. Then one has that

$$\begin{aligned}
D(y(t), y(\tau)) =& \frac{1}{\Gamma(q)} D\left(\int_{\tau}^{t} (t-s)^{q-1} g_y(s) ds, \right. \\
& \left. \int_{t_0}^{\tau} \left((\tau-s)^{q-1} - (t-s)^{q-1}\right) g_y(s) ds \right) \\
\leq& \frac{M}{\Gamma(q)} \left\{ \left| \int_{\tau}^{t} (t-s)^{q-1} ds \right| + \left| \int_{t_0}^{\tau} \left((t-s)^{q-1} - (\tau-s)^{q-1}\right) ds \right| \right\} \\
=& \frac{M}{\Gamma(q+1)} \left(|(t-s)^q|_{\tau}^{t}| + |\left((t-s)^q - (\tau-s)^q\right)|_{t_0}^{\tau}| \right) \\
=& \frac{M}{\Gamma(q+1)} \left(2(t-\tau)^q - (t-t_0)^q + (\tau-t_0)^q\right) \\
\leq& \frac{2M}{\Gamma(q+1)} (t-\tau)^q.
\end{aligned}$$

Therefore the solution set of (4.3) is Hölderian of degree q with a constant less than $\dfrac{2N}{\Gamma(q+1)}$. □

We consider the following **standing hypotheses**:

H1. $F(\cdot,\cdot)$ is almost USC with nonempty convex compact values and there exists $\lambda > 0$ such that (4.2) holds.

H2. $F(\cdot,\cdot)$ satisfies (CTC).

The following topological definitions are used in Theorem 4.4.

Definition 4.3. a) The set $Y \subset \mathbb{E}$ is said to be connected if for any two open disjoint sets $\mathbb{O}_1$, $\mathbb{O}_2$ such that $Y \subset \mathbb{O}_1 \bigcup \mathbb{O}_2$ either $Y \bigcap \mathbb{O}_1$ or $Y \bigcap \mathbb{O}_2$ is empty.

b) The set $Y \subset \mathbb{E}$ is said to be contractible (contractible to a point) if there exist a point $a \in Y$ and a continuous function $H : [0,1] \times Y \to Y$ such that $H(0,x) = x$, and $H(1,x) = a$ for all $x \in Y$.

Now we give the main result of the present paper.

Theorem 4.4. *Under the standing hypotheses, the solution set* $\mathbb{S}$ *of* (1.1) *is nonempty, compact and connected.*

Proof. **Step 1.** We prove that the solution set $\mathbb{S}$ is nonempty and compact.

To this aim, we shall use the locally Lipschitz approximations of F (see Section 2.4 of [11]), defined by

$$F_k(t,x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(x) C_\lambda^k(t) \text{ with } C_\lambda^k(t) = \overline{co}\ F(t, x_\lambda + 2r_k\mathbb{B}).$$

Recall that $(\varphi_\lambda)_{\lambda\in\Lambda}$ is a locally Lipschitz partition of unity subordinate to some locally finite refinement $(U_\lambda)_{\lambda\in\Lambda}$ of $\{x + r_k\mathbb{B};\ x \in \mathbb{E}\}$ with $r_k = 3^{-k}$ and $x_\lambda \in U_\lambda \subset x_\lambda + r_k\mathbb{B}$. We then have

$$F(t,x) \subset F_{k+1}(t,x) \subset F_k(t,x) \subset \overline{co}\ F(t, x + 3r_k\mathbb{B}) \tag{4.4}$$

on $I \times \mathbb{E}$. Since $F(\cdot,\cdot)$ is almost USC, one has that for every fixed $\bar{x} \in \mathbb{E}$ there exists a strongly measurable selection $f_{\bar{x}}(\cdot)$ of $F(\cdot,\bar{x})$ (see [11] page 29). Thus, there exists a strongly measurable selection $\varsigma_\lambda(\cdot)$ of $F(\cdot, x_\lambda)$, hence $\varsigma_\lambda(t) \in C_\lambda^k(t)$ for a.a. $t \in I$. Define $f^k : I \times \mathbb{E} \to \mathbb{E}$ by

$$f^k(t,x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x)\varsigma_\lambda(t) \in F_k(t,x). \tag{4.5}$$

Since $(U_\lambda)_{\lambda\in\Lambda}$ is a locally finite refinement, one has that $f^k(\cdot,x)$ is strongly measurable and $f^k(t,\cdot)$ is locally Lipschitz.

Consider equation (4.1) with f replaced by f^k. Due to Theorem 4.1, it admits an unique solution $x^k(\cdot)$ given by

$$x^k(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f^k(s, x^k(s)) ds. \tag{4.6}$$

Denote $M(t) = \bigcup_{m=1}^{\infty} \{x^m(t)\}$. Clearly, $\beta(M(t)) = \beta\left(\bigcup_{m=k}^{\infty} \{x^m(t)\}\right)$ for any $k \geq 1$. Using the properties of β, Theorem 3.4 and the hypothesis (CTC), we have that

$$\begin{aligned} \beta(M(t)) =& \frac{1}{\Gamma(q)} \beta\left(\int_{t_0}^{t} (t-s)^{q-1} \left(\bigcup_{m=k}^{\infty} (f^m(s, x^m(s))\right) ds\right) \\ \leq& \frac{2}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \beta(F_k(s, M(s))) ds \\ \leq& \frac{2}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \beta(\overline{co}\ F(s, M(s) + 3r_k\mathbb{B})) ds \\ =& \frac{2}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \beta(F(s, M(s) + 3r_k\mathbb{B})) ds \\ \leq& \frac{2}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \frac{1}{2} w(s, \beta(M(s)) + 3r_k) ds. \end{aligned}$$

Furthermore, $\lim_{k\to\infty} w(s, \beta(M(s)) + 3r_k) = w(s, \beta(M(s))$ for a.a. $s \in I$. Due to Lebesgue dominate convergence theorem,

$$\beta(M(t)) \leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} w(s, \beta(M(s))) ds.$$

Since $w(\cdot, \cdot)$ is a Perron function, we get that $\beta(M(t)) = 0$ and hence $M(t)$ is relatively compact. Moreover, due to Proposition 4.2, the set $\{x^k(\cdot)\}_k$ is equicontinuous. Then, by Arzela-Ascoli's theorem, passing to subsequences, we obtain that $\lim_{k\to\infty} x^k(t) = x(t)$ uniformly on I.

Let us remark that, by (4.4), for any $n \geq 1$ and any $k \geq n$ we have that

$$f^k(t, x) \in \overline{co}\ F(t, x + 3r_n\mathbb{B})$$

on $I \times \mathbb{E}$. It follows that

$$\{f^k(t, x^k(t))\}_{k\geq n} \subseteq \overline{co}\ F(t, M(t) + 3r_n\mathbb{B}) \tag{4.7}$$

for any natural n. Then, using (4.7) and (CTC), we obtain that

$$\beta(\bigcup_{k=1}^{\infty} f^k(t,x^k(t))) = \beta(\bigcup_{k=n}^{\infty} f^k(t,x^k(t))) \leq \beta(\overline{co}\ F(t,M(t)+3r_n\mathbb{B})$$

$$\leq \frac{1}{2}w(t,\beta(M(t)+3r_n\mathbb{B})) \leq \frac{1}{2}w(t,\beta(M(t))+3r_n) = \frac{1}{2}w(t,3r_n)$$

for any natural n. Since $r_n \to 0$ and $w(\cdot,\cdot)$ is a Perron function, we get that

$$\beta(\bigcup_{k=1}^{\infty} f^k(t,x^k(t))) = 0.$$

Furthermore, $\mathbb{E}$ can be embedded as closed convex cone $j(\mathbb{E})$ in a Banach space $\mathbb{X}$. Then, using Theorem 3.2,

$$\beta(j(\bigcup_{k=1}^{\infty} f^k(t,x^k(t)))) \leq \beta(\bigcup_{k=1}^{\infty} f^k(t,x^k(t))) = 0,$$

so $j(\bigcup_{k=1}^{\infty} f^k(t,x^k(t)))$ is relatively compact in $\mathbb{X}$. Due to Diestel theorem (see Proposition 1.4 in [27]) we get that the sequence $\{j(f^k(\cdot,x^k(\cdot)))\}_k$ is weakly relatively compact in $L_1(I,\mathbb{X})$. Therefore, $\{j(f^k(\cdot,x^k(\cdot)))\}$ is weakly convergent (on a subsequence) in $L_1(I,\mathbb{X})$ to $j(f(\cdot))$. Then it is standard to prove with the help of Mazur's lemma that $x(\cdot)$ is a solution of (1.1), so $\mathbb{S}$ is nonempty.

Denote by $\mathbb{S}_n$ the solution set of (1.1) with F_n instead of F. Clearly, $\mathbb{S} \subset \bigcap_{n\geq 1} \mathbb{S}_n$. Using the same arguments as in the first part of the proof, we can show that if $x_n \in \mathbb{S}_n$ for any $n \geq 1$ then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ that converges uniformly to some $x \in \mathbb{S}$. We get that $\mathbb{S} = \bigcap_{n\geq 1} \bar{\mathbb{S}}_n$ is $C(I,\mathbb{E})$ compact and $\lim_{n\to\infty} \beta(\mathbb{S}_n) = 0$. Moreover, $\lim_{n\to\infty} D_H(\mathbb{S}_n,\mathbb{S}) = 0$.

Step 2. We prove that the solution set $\mathbb{S}$ is connected.

First, we prove that $\mathbb{S}_k$ are contractible. To this end, we take $\tau \in [0,1]$ and denote $a_\tau = t_0 + \tau(T-t_0)$. Let $u \in \mathbb{S}_k$, i.e.

$$u(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} g_u(s)ds,$$

where $g_u(s) \in F_k(s,u(s))$ is strongly measurable.

Next we define the map $H : [0,1] \times \mathbb{S}_k \to \mathbb{S}_k$ as follows

$$H(\tau,u)(t) = \begin{cases} u(t) & \text{on } [t_0, a_\tau]\,, \\ \tilde{x}(t) & \text{on } [a_\tau, T]\,, \end{cases} \tag{4.8}$$

where

$$\tilde{x}(t) = x_0 + \frac{1}{\Gamma(q)}\left[\int_{t_0}^{a_\tau}(t-s)^{q-1}g_u(s)ds + \int_{a_\tau}^{t}(t-s)^{q-1}f^k(s,\tilde{x}(s))ds\right].$$

Clearly $H(0,u)(t) = x^k(t)$, where $x^k(\cdot)$ is given by (4.6), and $H(1,u)(t) = u(t)$.

Since $\mathbb{S}_k$ is equicontinuous, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $D(u(a_\tau), u(a_\tau + \Delta)) < \dfrac{\varepsilon}{35}$, and $D((\tilde{x}(a_\tau), \tilde{x}(a_\tau + \Delta)) < \dfrac{\varepsilon}{35}$ for every $\Delta < \delta$. Furthermore,

$$\frac{1}{\Gamma(q)}|\int_{t_0}^{a_\tau}[(a_\tau - s)^{q-1} - (a_\tau + \Delta - s)^{q-1}]g_u(s)ds|$$
$$\leq \frac{1}{\Gamma(q)}\int_{t_0}^{a_\tau}[(a_\tau - s)^{q-1} - (a_\tau + \Delta - s)^{q-1}]|g_u(s)|ds$$
$$\leq \frac{M_1}{\Gamma(q+1)}[(a_\tau + \Delta - t_0)^q - (a_\tau - t_0)^q].$$

Consequently, $D(H(\tau,u)(t), H(\tau+\Delta,u)(t)) \leq r(t)$, where

$$r(t) = \frac{2\varepsilon}{35} + \frac{M_1}{\Gamma(q+1)}[(a_\tau + \Delta - t_0)^q - (a_\tau - t_0)^q] + \frac{L}{\Gamma(q)}\int_{a_\tau+\Delta}^{t}(t-s)^{q-1}r(s)ds,$$

because of $D(f^k(t,x), f^k(t,y)) \leq LD(x,y)$. From Lemma 6.11 of [14] we know that

$$r(t) < \left(\frac{2\varepsilon}{35} + \frac{M_1}{\Gamma(q+1)}[(a_\tau + \Delta - t_0)^q - (a_\tau - t_0)^q]\right)E_q(L(t - a_\tau - \Delta)),$$

where $E_q(\cdot)$ is the Mittag–Leffler function. The last fact together with Theorem 4.1 implies that $H(\cdot,\cdot)$ is continuous and hence $\mathbb{S}_k$ are contractible. The latter implies that $\mathbb{S}_k$ are connected.

Suppose that $\mathbb{S}$ is not connected. Then there exist two disjoint open sets A, B such that $\mathbf{A} = \mathbb{S}\bigcap A \neq \emptyset$, $\mathbf{B} = \mathbb{S}\bigcap B \neq \emptyset$ and $\mathbb{S} = \mathbf{A}\bigcup \mathbf{B}$. We consider first the case when $d(\mathbf{A},\mathbf{B}) := \min\limits_{a\in\mathbf{A}, b\in\mathbf{B}} |a-b| = 0$. Consequently, there exists $\hat{x} \in \mathbb{S}$ which is point of density of $\mathbf{A}$ and $\mathbf{B}$ simultaneously, which is a contradiction. If $d(\mathbf{A},\mathbf{B}) > 0$, then using the fact that $\lim\limits_{n\to\infty} D_H(\mathbb{S}_n, \mathbb{S}) = 0$, we obtain a contradiction with the connectedness of $\mathbb{S}_n$. Consequently, the set $\mathbb{S}$ is connected. □

We recall that a set $A \subset \mathbb{E}$ is said to be compact R_δ if there exists a decreasing sequence of compact contractible sets A_k such that $A = \bigcap_{k=1}^{\infty} A_k$ (see e.g. [16]). The lack of the semigroup property of the solutions of (1.1) and the fact that we have to know the Caputo derivative from t_0 to a_τ in (4.8) doesn't permit us to prove that the solution set of (1.1) is R_δ, in contrast with the ordinary fuzzy differential equations (see e.g. [15]).

The obstacle to show that the solution set of (1.1) is compact R_δ is the fact that the solution set of

$$D_c^q x(t) \in F_k(t,x),\ x(t_0) = x_0 \tag{4.9}$$

is not closed and not precompact in general. Moreover we can not prove that its closure is contractible. To overcome that difficulty one can assume that $F(t,\cdot)$ maps bounded sets into relatively compact sets, and, clearly, in this case, the solution set of (4.9) is compact. The following theorem is then valid:

Theorem 4.5. *Let F map bounded sets into relatively compact sets. Under the standing hypotheses, the solution set $\mathbb{S}$ of* (1.1) *is nonempty compact R_δ.*

Proof. Clearly in this case the solution set of (4.9) is compact. Indeed f $\{x_n(\cdot)\}_n$ is a sequence of solutions of (4.9), then $\beta\left(\bigcup_{k=1}^{\infty}\right) = 0$. Due to Arzela Ascoli theorem passing to subsequences if necessary $x_n(\cdot) \to x(\cdot)$ uniformly on I. Using Diestel criterion $\{\dot{x}_n(\cdot)\}_n$ is weakly L_1 precompact. Then we can show as it the proof of Theorem 4.4 that $x(\cdot)$ is a solution of (4.9) and hence the solution set of (4.9) is nonempty compact and contractible. Therefore the solution set of (1.1) is compact $R\delta$. □

5. Fuzzy Fractional Differential Inclusions

In this section we apply the previous result to the fuzzy fractional differential inclusion (1.3).

Let $u, v \in \mathbb{E}$. If there exists $w \in \mathbb{E}$ such that $u = v \oplus w$ then w is denoted by $u \ominus v$ and it is called the H-difference between u and v.

Let $f : I \to \mathbb{E}$ and $t_0 \in I$. We say that f is differentiable (H-differentiable) at τ if there exists an element $f'(\tau) \in \mathbb{E}$ such that for all $h > 0$ sufficiently small, there are $f(\tau + h) \ominus f(\tau)$, $f(\tau) \ominus f(\tau - h)$, and

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0). \tag{5.1}$$

Let $u : [a, b] \to \mathbb{E}$ be Bochner integrable. The Riemann-Liouville fractional integral $I^q u(\cdot)$ of order $q > 0$ is defined by

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t - s)^{q-1} u(s) ds, \ a < t < b.$$

The Riemann-Liouville fractional derivative $D^q u$ of order $0 < q < 1$ of u is defined by

$$D^q u(t) = \frac{d}{dt} I^{1-q} u(t) = \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_a^t (t - s)^{-q} u(s) ds, \ a < t < b,$$

provided that the expression on right-hand side is defined.

The Caputo fractional derivative $D_c^q u$ of order $0 < q < 1$ of u is defined by

$$D_c^q u(t) = \frac{1}{\Gamma(1 - q)} \int_a^t (t - s)^{-q} \dot{u}(s) ds, \ a < t < b,$$

provided that the expression on right-hand side is defined. For example, $u(\cdot)$ should be absolutely continuous with Bochner integrable derivative.

Theorem 4.4 can be reformulated as follows.

Theorem 5.1. *Under the standing hypotheses, the solution set* $\mathbb{S}$ *of* (1.3) *is nonempty compact and connected.*

The given definition of (Hukuhara) derivative has some bad properties. To avoid these bad properties, B. Bede and his coauthors defined generalized derivative (see [8, 9]).

We say that f is Bede differentiable (B-differentialble) at τ if there is an element $f'_B(\tau) \in \mathbb{E}$, such that for all $h < 0$, sufficiently nearby 0, there are $f(\tau + h) \ominus f(\tau)$, $f(\tau) \ominus f(\tau - h)$, and limits

$$\lim_{h \to 0^-} \frac{f(\tau + h) \ominus f(\tau)}{h} \text{ and } \lim_{h \to 0^-} \frac{f(\tau) \ominus f(-h)}{h},$$

which are equal to $f'_B(\tau)$.

Recall that the embedding map $j : \mathbb{B} \to \mathbb{X}$ is isometry and isomorphism. Clearly $j(f'_H(t)) = \frac{d}{dt} j(f(t)) \neq j(f'_B(t))$, when $f'_H(t) \neq f'_B(t)$.

The function f is said to have generalized derivative at τ if it is H-differentiable or B-differentiable at τ.

Notice that if $f'_H(\tau) = f'_B(\tau)$ then $f(\tau)$ is crisp.

The generalized derivative has many applications in the case of one dimensional fuzzy numbers. However, in the case of multidimensional fuzzy numbers (fuzzy vectors), Bede's derivative poses some problems.

Consider now the fuzzy fractional differential inclusion (with respect to Bede derivative)

$$D_c^{q_B} x(t) \in F(t, x(t)), \; x(t_0) = x_0, \; t \in I = [t_0, T], \tag{5.2}$$

where $D_c^{q_B} x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x'_B(s) ds$. The function $y : I \to \mathbb{E}$ will be a solution of (5.2) if there exists a strongly measurable selection $f(t) \in F(t, x(t))$, such that

$$y(t) = y_0 \ominus \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds.$$

The solution will exist only if the difference $\ominus$ exists.

Clearly, if $f(\cdot,\cdot)$ is a Carathéodory single valued fuzzy function, Lipschitz on the state variable and there exists a solution with respect to B-derivative, then the solution of the fractional differential equation

$$D_c^q x(t) = f(t, x(t)), \; x(t_0) = x_0, \; t \in [t_0, T] \tag{5.3}$$

does not have unique solution. In this case we are not able to prove that the solution set of (1.1) is connected.

6. Conclusions

In this paper we study the main properties of the solution set of a class of fuzzy fractional differential inclusions with Caputo derivative. We show that the solution set is nonempty, compact and connected under compactness type assumptions. That is we extend the classical Kneser's theorem to the case of fuzzy fractional systems. Notice that all the results of the paper can be proved if Hausdorff measure of noncompactness β is replaced by the Kuratowski one ρ.

We prove that if F maps bounded sets into relative compact then the solution set of (1.1) is a compact R_δ. The last result can be used to prove existence of solutions to fuzzy differential inclusions with nonlocal condition.

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