 GENERALIZED NOWHERE DENSE SETS IN CLUSTER TOPOLOGICAL SETTING

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Abstract: The aim of the article is to generalize the notion of nowhere dense set with respect to a cluster topological space which is defined as a triplet \((X, \tau, \mathcal{E})\) where \((X, \tau)\) is a topological space and \(\mathcal{E}\) is a nonempty family of nonempty subsets of \(X\). The notions of \(\mathcal{E}\)-nowhere dense and locally \(\mathcal{E}\)-scattered sets are introduced and the necessary and sufficient conditions under which the family of all \(\mathcal{E}\)-nowhere dense sets is an ideal are given.

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1. Introduction and Basic Definitions

Cluster topological spaces provide a general framework with the involvement of ideal topological spaces [1], [2], [3], [9]. They have a wider application and its progress can find in [6], [7], [10]. The paper can be considered as a continuation of [5] where some cluster topological notions were introduced and it corresponds with the efforts to generalize the Baire classification of sets and the Baire category theorem [4], [11].

In [5] one can find an open problem to discover a necessary and sufficient condition under which the family of all \(\mathcal{E}\)-nowhere dense sets forms an ideal. In the first part we recall the basic notions and results of [5], a few counter examples are given and Section 3 is devoted to our main goal.
In the sequel, \((X, \tau)\) is a nonempty topological space. By \(\overline{A}\), \(A^o\), we denote the closure, the interior of \(A\) in \(X\), respectively. By \(A^\circ\) we denote the interior of \(A\).

**Definition 1.1.** (see [5]) Any nonempty system \(E \subset 2^X \setminus \{\emptyset\}\) will be called a cluster system in \(X\). If \(G\) is a nonempty open set and any nonempty open subset of \(G\) contains a set from \(E\), then \(E\) is called a \(\pi\)-network in \(G\). For a cluster system \(E\) and a subset \(A\) of \(X\), we define the set \(E(A)\) of all points \(x \in X\) such that for any neighborhood \(U\) of \(x\), the intersection \(U \cap A\) contains a set from \(E\). A triplet \((X, \tau, E)\) is called a cluster topological space. If \(\emptyset \neq Y \subset \emptyset\), then \((Y, \tau_Y, E_Y)\) where \(\tau_Y\) is the subspace topology is called a cluster topological subspace of \((X, \tau, E)\), provided \(E_Y \neq \emptyset\).

**Remark 1.1.** (see [1], [2], [3], [9]) Specially, if \(I\) is a proper ideal on \(X\), then a cluster system \(E_I = \{E \subset X : E \notin I\}\) leads to a local function of \(A\), i.e., \(E_I(A) = \{x \in X : \text{for any open set } U \text{ containing } x \text{ there is } E \in E_I \text{ such that } E \subset U \cap A\} = \{x \in X : U \cap A \notin I, \text{ for any open set } U \text{ containing } x \}\). Then, \(A^*(I, \tau)\) what is called the local function of \(A\) with respect to \(I\) and \(\tau\). Note, if \(E = 2^X \setminus \{\emptyset\}\), then \(E(A) = \overline{A}\).

The next definition introduces some basic notions derived from \(E\)-operator reminding the properties of local function which are known from an ideal topological space.

**Definition 1.2.** A set \(A\) is called \(E\)-scattered if \(A\) contains no set from \(E\). A set \(A\) is locally \(E\)-scattered at a point \(x \in X\) if there is an open set \(U\) containing \(x\) such that \(U \cap A\) is \(E\)-scattered (i.e., \(x \notin E(A)\)). A is locally \(E\)-scattered if \(A\) is locally \(E\)-scattered at any point from \(A\) (i.e., \(A \cap E(A) = \emptyset\)) and \(A\) is \(E\)-dense in itself if \(A \subset E(A)\). A set \(A\) is \(E\)-nowhere dense if for any nonempty open set \(U\) there is a nonempty open subset \(H\) of \(U\) such that \(H \cap A\) is \(E\)-scattered. The family of all \(E\)-nowhere dense sets, locally \(E\)-scattered sets, nowhere dense sets, is denoted by \(N_E, S_E, N\), respectively.

**Remark 1.2.** By Definition 1.1, \(E\) is a nonempty system of nonempty subsets of \(X\). A trivial case \(E = \emptyset\) (\(\emptyset \in E\)) leads to the trivial results, since \(E(A) = \emptyset\) and \(N_E = 2^X\) \((E(A) = X\) and \(N_E = \emptyset)\) for any \(A \subset X\).

**Definition 1.3.** Let \(E_1\) and \(E_2\) be two cluster systems. \(E_1 < E_2\) if for any \(E_1 \in E_1\) there is \(E_2 \in E_2\) such that \(E_2 \subset E_1\). \(E_1\) and \(E_2\) are equivalent, \(E_1 \sim E_2\), if \(E_1 < E_2\) and \(E_2 < E_1\).
2. Preliminary Results

The next properties of \( \mathcal{E} \)-operator are clear and the proof of the following lemma is omitted.

Lemma 2.1. (see [5])

(1) \( \mathcal{E}(\emptyset) = \emptyset \),
(2) \( \mathcal{E}(A) \) is closed,
(3) \( \mathcal{E}(A) \subset \overline{A} \),
(4) \( \mathcal{E}(\mathcal{E}(A)) \subset \mathcal{E}(A) \),
(5) \( \mathcal{E} \) is a \( \pi \)-network in an open set \( G \neq \emptyset \) if and only if \( \mathcal{E}(G) = \mathcal{E}(\overline{G}) = \overline{G} \).

Lemma 2.2.

(1) If \( \mathcal{E}_1 \subset \mathcal{E}_2 \), then \( \mathcal{E}_1 < \mathcal{E}_2 \),
(2) if \( \mathcal{E}_1 < \mathcal{E}_2 \), then \( \mathcal{E}_1(A) \subset \mathcal{E}_2(A) \) and \( \mathcal{N}_{\mathcal{E}_2} \subset \mathcal{N}_{\mathcal{E}_1} \),
(3) if \( \mathcal{E}_1 \sim \mathcal{E}_2 \), then \( \mathcal{E}_1(A) = \mathcal{E}_2(A) \) and \( \mathcal{N}_{\mathcal{E}_2} = \mathcal{N}_{\mathcal{E}_1} \),
(4) \( \mathcal{N}_{\mathcal{E}_1} \cup \mathcal{E}_2 \subset \mathcal{N}_{\mathcal{E}_i} , i = 1, 2 \),
(5) \( \mathcal{N}_{\mathcal{E}_i} \subset \mathcal{N}_{\mathcal{E}_i \cap \mathcal{E}_2} , i = 1, 2 \),
(6) if \( A_1 \subset A_2 \), then \( \mathcal{E}(A_1) \subset \mathcal{E}(A_2) \),
(7) let \( A \subset Y \subset X \). If \( A \) is \( \mathcal{E}_Y \)-nowhere dense, then \( A \in \mathcal{N}_{\mathcal{E}} \),
(8) if \( A_t \) is \( \mathcal{E} \)-dense in itself for any \( t \in T \), then \( \bigcup_{t \in T} A_t \) is so,
(9) if \( A \) is \( \mathcal{E} \)-dense in itself, then \( \overline{A} \) is so.

Proof. We will show only (7), (8) and (9). Other items are easy to prove.

(7) Let \( G \) be a nonempty open subset of \( X \). If \( G \cap Y = \emptyset \), there is noting to prove. Suppose \( G \cap Y \neq \emptyset \). Then \( G \cap Y \in \tau_Y \), so there is a nonempty open set \( H \in \tau \), such that \( \emptyset \neq H \cap Y \subset G \cap Y \) and \( H \cap Y \cap A \) contains no set from \( \mathcal{E}_Y \), hence \( H \cap Y \cap A = H \cap A \) contains no set from \( \mathcal{E} \). Since \( H \cap G \) is a nonempty open subset of \( G \) and \( H \cap G \cap A \) contains no set from \( \mathcal{E} \), \( A \in \mathcal{N}_\mathcal{E} \).

(8) It follows from \( \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \mathcal{E}(A_t) \subset \mathcal{E}(\bigcup_{t \in T} A_t) \).

(9) Since \( A \subset \mathcal{E}(A) \), \( \overline{A} \subset \mathcal{E}(\overline{A}) = \mathcal{E}(A) \subset \mathcal{E}(\overline{A}) \) by Lemma 2.1 (2) and Lemma 2.2 (6).

Lemma 2.3. (see [5]) The next conditions are equivalent:
Theorem 2.1. (see [5])

(1) \( N \subset N_\mathcal{E} \). Consequently, if \( A \in N \), then \( A \in N_\mathcal{E} \),

(2) if \( A \in N_\mathcal{E} \) and \( B \subset A \), then \( B \in N_\mathcal{E} \),

(3) \( A \setminus \mathcal{E}(A) \in N_\mathcal{E} \) and \( A \setminus \mathcal{E}(A) \) is locally \( \mathcal{E} \)-scattered,

(4) any \( \mathcal{E} \)-scattered set is locally \( \mathcal{E} \)-scattered and any locally \( \mathcal{E} \)-scattered set is from \( N_\mathcal{E} \),

(5) if \( A \in N \) and \( B \in N_\mathcal{E} \), then \( A \cup B \in N_\mathcal{E} \),

(6) if \( G_t \in N_\mathcal{E} \) and \( G_t \) is open for any \( t \in T \), then \( \bigcup_{t \in T} G_t \in N_\mathcal{E} \). Consequently, if \( A_t \subset Y \subset X \) is \( \tau_Y \)-open and \( A_t \in N_{\mathcal{E}Y} \), then \( \bigcup_{t \in T} A_t \in N_\mathcal{E} \),

(7) \( A \in N_\mathcal{E} \) if and only if \( A \) is a sum of a locally \( \mathcal{E} \)-scattered set and a set from \( N \),

(8) if \( A, B \in N_\mathcal{E} \) and one of them is closed, then \( A \cup B \in N_\mathcal{E} \).

Proof. (1): Let \( A \in N \). By Lemma 2.1 (3), \( \mathcal{E}(A) \subset \overline{A} \), so \( (\mathcal{E}(A))^\circ \subset \overline{A}^\circ = \emptyset \) and by Lemma 2.3, \( A \in N_\mathcal{E} \).

(2): It follows from the implications: \( B \subset A \Rightarrow \mathcal{E}(B) \subset \mathcal{E}(A) \Rightarrow (\mathcal{E}(B))^\circ \subset (\mathcal{E}(A))^\circ = \emptyset \) and Lemma 2.3.

(3): Let \( G \) be nonempty open. If \( G \cap (A \setminus \mathcal{E}(A)) \) is empty, there is nothing to prove. Let \( x \in G \cap (A \setminus \mathcal{E}(A)) \). Then there is an open subset \( H \) of \( G \) containing \( x \), such that \( H \cap A \) contains no set from \( \mathcal{E} \). Then \( H \cap (A \setminus \mathcal{E}(A)) \) is \( \mathcal{E} \)-scattered. That means \( A \setminus \mathcal{E}(A) \in N_\mathcal{E} \). The second part is clear.

(4): The first part is clear. Let \( A \) be locally \( \mathcal{E} \)-scattered. Since \( A \cap \mathcal{E}(A) = \emptyset \), \( A = A \setminus \mathcal{E}(A) \) is \( \mathcal{E} \)-nowhere dense by (3).

(5): Let \( G \) be nonempty open. Since \( A \) is nowhere dense and \( B \) is \( \mathcal{E} \)-nowhere dense, there are two nonempty open sets \( G_0 \subset G \) and \( H \subset G_0 \) such
that $A \cap G_0 = \emptyset$ and $B \cap H$ is $E$-scattered. Hence $(A \cup B) \cap H = B \cap H$ is $E$-scattered, so $A \cup B$ is $E$-nowhere dense.

(6): Let $\{H_s\}_{s \in S}$ be a maximal family of pairwise disjoint open sets such that any $H_s$ is a subset of some set from $\{G_t\}_{t \in T}$ and $A := \cup_{t \in T}G_t \setminus \cup_{s \in S}H_s$ is nowhere dense. It is clear that $B := \cup_{s \in S}H_s$ is $E$-nowhere dense. By item (5), $\cup_{t \in T}G_t = A \cup B$ is $E$-nowhere dense. The consequence follows from Lemma 2.2 (7).

(7): ” $\Rightarrow$ ” It follows from equation $A = (A \setminus \mathcal{E}(A)) \cup (A \cap \mathcal{E}(A))$, item (3) and Lemma 2.3. The opposite implication follows from the items (4) and (5).

(8): Suppose $A$ is closed. If $A^0 = \emptyset$, then $A$ is nowhere dense and by item (5), $A \cup B \in \mathcal{N}_E$.

Let $A^0 \neq \emptyset$ and $U$ be a nonempty open set. Suppose $U \cap A^0 = \emptyset$. Since $A \setminus A^0$ is nowhere dense, so there is a nonempty open set $H \subset U$ such that $H \cap (A \setminus A^0) = H \cap A = \emptyset$. Since $B \in \mathcal{N}_E$, there is a nonempty open set $H_0 \subset H$, such that $H_0 \cap B = (H_0 \cap A) \cup (H_0 \cap B) = H_0 \cap (A \cup B)$ contains no set from $\mathcal{E}$, hence $A \cup B \in \mathcal{N}_E$. Finally, suppose $U \cap A^0 \neq \emptyset$. Since $A \in \mathcal{N}_E$, there is a nonempty open set $H_1 \subset U \cap A^0 \subset A$, such that $H_1 \cap A = H_1$ contains no set from $\mathcal{E}$, consequently $(A \cup B) \cap H_1$ contains no set from $\mathcal{E}$. So $A \cup B \in \mathcal{N}_E$. \qed

If $\mathcal{E}_{II} = \{E : E$ is of second category in $(X, \tau)\}$, then $\mathcal{N}_{\mathcal{E}_{II}}$ is the family of all sets of first category. So, item (6) of Theorem 2.1 is a generalization of the Banach category theorem.

3. Main Results

Next theorem deals with a relationship between an $E$-nowhere dense set and a nowhere dense one and we will find some conditions under which $\mathcal{N}_E$ forms an ideal.

**Theorem 3.1.** Let $\mathcal{E}$ be a $\pi$-network in an open set $G_0$. If $A \subset \overline{G_0}$ is closed and $E$-nowhere dense, then $A$ is nowhere dense. Consequently, if $\mathcal{E}(X) = X$ and $A$ is a closed subset of $X$, then $A$ is nowhere dense if and only if $A$ is $\mathcal{E}$-nowhere dense.

**Proof.** Let $G_1$ be nonempty open. If $A \cap G_1 = \emptyset$, there is nothing to prove. Let $A \cap G_1 \neq \emptyset$ and $G := G_1 \cap G_0$. Since $\mathcal{E}$ is a $\pi$-network in $G_0$, $H := G \cap (X \setminus A) \neq \emptyset$ (if $G \subset A$, then there is a nonempty open subset $H_0$ of $G$ such that $H_0 \cap A = H_0$ contains no set from $\mathcal{E}$, contradiction with assumption that $\mathcal{E}$ is a $\pi$-network in $G_0$). So, $H$ is a nonempty open subset of $G$ and disjoint from $A$. \qed
Remark 3.1. No assumption in Theorem 3.1 can be omitted. Let \( X = \{a, b\} \), \( \tau = \{X, \emptyset\} \), \( \mathcal{E} = \{X\} \), \( A = \{a\} \). Then \( \mathcal{E} \) is a \( \pi \)-network in \( X \). The set \( A \) is \( \mathcal{E} \)-nowhere dense, \( A \) is not closed and \( A \) is not nowhere dense.

The assumption that \( \mathcal{E} \) is a \( \pi \)-network can not be omitted. Consider \( X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \) with the usual topology, \( \mathcal{E} = \{E : E \text{ is infinite}\} \). It is clear that \( \mathcal{E} \) is not a \( \pi \)-network in \( X \). Put \( A = \{1, \frac{1}{2}\} \). The set \( A \) is closed and \( \mathcal{E} \)-nowhere dense. But \( A \) is not nowhere dense.

It is known that \( A \) is nowhere dense iff \( \overline{A} \) is so. An analogous equivalence for the \( \mathcal{E} \)-nowhere dense sets leads to the fact that \( N_{\mathcal{E}} \) is an ideal.

Theorem 3.2. (see [5]) If \( \overline{A} \) is \( \mathcal{E} \)-nowhere dense whenever \( A \) is \( \mathcal{E} \)-nowhere dense, then \( N_{\mathcal{E}} \) is an ideal.

An obvious question is whether the assumption of Theorem 3.2 implies the equality \( N = N_{\mathcal{E}} \) and if the opposite implication holds. Next examples will give the negative answers.

Example 3.1. Let \( X = \{a, b\} \), \( \tau = 2^X \), \( \mathcal{E} = \{X\} \). Then \( N_{\mathcal{E}} = 2^X \) and \( N = \{\emptyset\} \neq N_{\mathcal{E}} \).

Example 3.2. Let \( X = \{a, b, c\} \), \( \tau = \emptyset, \{a, b\}, X \), \( \mathcal{E} = \{\{a\}\} \). Then \( N_{\mathcal{E}} = \emptyset, \{b\}, \{c\}, \{b, c\} \) is an ideal. The set \( \{b\} \in N_{\mathcal{E}} \), but \( \overline{\{b\}} = \{a, b, c\} \notin N_{\mathcal{E}} \).

In the case if \( \mathcal{E} \) is a \( \pi \)-network in \( X \), the opposite implication is valid but the assumption that \( \overline{A} \) is \( \mathcal{E} \)-nowhere dense whenever \( A \) is \( \mathcal{E} \)-nowhere dense seems to be too strong and it leads to the equation \( N = N_{\mathcal{E}} \).

Theorem 3.3. (see [5]) Let \( \mathcal{E} \) be a \( \pi \)-network in \( X \). Then the next conditions are equivalent:

1. \( \overline{A} \) is \( \mathcal{E} \)-nowhere dense if and only is \( A \) is \( \mathcal{E} \)-nowhere dense,
2. \( N = N_{\mathcal{E}} \).

In [5] it is recommended to investigate a condition under which \( N_{\mathcal{E}} \) is an ideal. In this section we introduce a notion of additive cluster system.

Theorem 3.4. \( N_{\mathcal{E}} \) is an ideal if and only if any sum of two locally \( \mathcal{E} \)-scattered sets is from \( N_{\mathcal{E}} \).

Proof. "\( \Rightarrow \)" By Theorem 2.1 (4), any locally \( \mathcal{E} \)-scattered set is from \( N_{\mathcal{E}} \), so the sum of two \( \mathcal{E} \)-scattered sets is from \( N_{\mathcal{E}} \).
“⇐” Let $A, B \in \mathcal{N}_\mathcal{E}$. By Theorem 2.1 (7), $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where $A_1, B_1$ are locally $\mathcal{E}$-scattered and $A_2, B_2 \in \mathcal{N}$. Then $A \cup B = (A_1 \cup B_1) \cup (A_2 \cup B_2)$ is a sum of a locally $\mathcal{E}$-scattered and a nowhere dense set, so $A \cup B \in \mathcal{N}_\mathcal{E}$, by Theorem 2.1 (7). \hfill \Box

**Corollary 3.1.** If $S_\mathcal{E}$ is an ideal, then $N_\mathcal{E}$ is so.

The opposite implication does not hold, as the next example shows.

**Example 3.3.** Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with the usual topology and $\mathcal{E} = \{E \subset X : X \setminus E$ is finite $\}$. Then $X_1 = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ and $X_2 = \{0\}$ are locally $\mathcal{E}$-scattered, but $X_1 \cup X_2 = X$ is not so. It is clear $N_\mathcal{E} = 2^X$ is an ideal.

**Definition 3.1.** A cluster system $\mathcal{E}$ is $\mathcal{N}$-additive if for any $A, B \subset X$ there is a nowhere dense set $R$, such that $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B) \cup R$.

**Theorem 3.5.** $N_\mathcal{E}$ is an ideal if and only if $\mathcal{E}$ is $\mathcal{N}$-additive.

*Proof.* "⇒" Let $A, B \in N_\mathcal{E}$. First, we will show $(\mathcal{E}(A \cup B))^o \subset \mathcal{E}(A) \cup \mathcal{E}(B)$. Let $x \in (\mathcal{E}(A \cup B))^o$ and $x \notin \mathcal{E}(A) \cup \mathcal{E}(B)$. Then there is an open subset $H$ of $(\mathcal{E}(A \cup B))^o$ containing $x$ and $H \cap A$ and $H \cap B$ contain no set from $\mathcal{E}$. So, $H \cap A$ and $H \cap B$ are $\mathcal{E}$-nowhere dense set. Since $N_\mathcal{E}$ is an ideal, there is a nonempty open subset $G$ of $H$, such that $G \cap (A \cup B)$ contains no set from $\mathcal{E}$. On the other hand, $x \in (\mathcal{E}(A \cup B))^o$, hence $G \cap (A \cup B)$ contains a set from $\mathcal{E}$, a contradiction.

Since $\mathcal{E}(A \cup B) \setminus (\mathcal{E}(A \cup B))^o$ in nowhere dense and $\mathcal{E}(A \cup B) = [\mathcal{E}(A \cup B) \setminus (\mathcal{E}(A \cup B))^o] \cup \mathcal{E}(A) \cup \mathcal{E}(B)$, $\mathcal{E}$ is $\mathcal{N}$-additive.

"⇐" Let $A, B \in N_\mathcal{E}$. Then $\mathcal{E}(A \cup B) = R \cup \mathcal{E}(A) \cup \mathcal{E}(B)$, where $R$ is a nowhere dense set. Since $\mathcal{E}(A), \mathcal{E}(B)$ are nowhere dense, $\mathcal{E}(A \cup B)$ is a nowhere dense set, so $A \cup B \in N_\mathcal{E}$, by Lemma 2.3. \hfill \Box

**4. Derived Cluster Systems**

It is well known that a set $A$ is of first category if and only if $D(A) = \emptyset$ where $D(A)$ is the set of all points in which $A$ is of first category, i.e., for any $x \in A$ there is an open set $U$ containing $x$ such that $A \cap U$ does not contain a set of second category. Question is if there is a similar characterization of $\mathcal{E}$-nowhere dense sets, namely $A \in N_\mathcal{E}$ iff $\mathcal{E}(A) = \emptyset$. Next example shows that similar characterization exists for the ideal $\mathcal{N}$ of all nowhere dense sets.
**Example 4.1.** Let \( \mathcal{E}_N = \{ E : E \not\subseteq \mathcal{N} \} \). It is clear that \( \mathcal{E}_N(A) = \overline{\mathcal{A}}^\circ \). By Lemma 2.3, \( A \in \mathcal{N}_{\mathcal{E}_N} \) iff \( \mathcal{E}_N(A) \) is nowhere dense iff \( \overline{\mathcal{A}}^\circ = \emptyset \) iff \( A \) is nowhere dense. Consequently, \( \mathcal{N}_{\mathcal{E}_N} = \mathcal{N} \). So, \( A \) is a nowhere dense set iff \( A \in \mathcal{N}_{\mathcal{E}_N} \) iff \( \mathcal{E}_N(A) = \overline{\mathcal{A}}^\circ = \emptyset \).

The next example shows that the equivalence \( A \in \mathcal{N}_\mathcal{E} \) if and only if \( \mathcal{E}(A) = \emptyset \) does not hold in general.

**Example 4.2.** Let \( X = \{ 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \} \) with the usual topology and \( \mathcal{E} = \{ \{ 1 \} \} \cup \{ E : E \text{ is infinite} \} \). Then \( \mathcal{E}(A) = \emptyset \) iff \( A \) is finite and \( 1 \not\in A \). It is clear \( \mathcal{N}_\mathcal{E} = \{ A : 1 \not\in A \} \).

**Definition 4.1.** Let \( \mathcal{E} \) be a cluster system. Put \( \mathcal{E}^* = \{ E : E \not\subseteq \mathcal{N}_\mathcal{E} \} = \{ E : (\mathcal{E}(E))^\circ \neq \emptyset \} \) (see Lemma 2.3). A set \( A \) is called \( \mathcal{E} \)-preopen if \( A \subset (\mathcal{E}(A))^\circ \). A cluster system of all nonempty \( \mathcal{E} \)-preopen sets is denoted by \( \mathcal{E}^{\text{po}} \).

Now we will study a connection among \( \mathcal{N}_\mathcal{E}, \mathcal{N}_{\mathcal{E}^*} \) and \( \mathcal{N}_{\mathcal{E}^{\text{po}}} \).

**Lemma 4.1.** Let \( (\mathcal{E}(A))^\circ \neq \emptyset \). If \( H \) is a nonempty open subset of \( (\mathcal{E}(A))^\circ \), then \( A \cap H \in \mathcal{E}^* \) and \( A \cap (\mathcal{E}(A))^\circ \) is \( \mathcal{E} \)-preopen.

**Proof.** Denote \( \mathcal{G} := (\mathcal{E}(A))^\circ \). First we prove \( H \subset \mathcal{E}(A \cap H) \). Let \( x \in H \) and \( U \subset H \subset G \) be an open set containing \( x \). Since \( x \in U \subset G \subset \mathcal{E}(A) \), \( A \cap U = A \cap H \cap U \) contains a set from \( \mathcal{E} \), so \( x \in \mathcal{E}(A \cap H) \). Since \( H \subset \mathcal{E}(A \cap H) \), \( (\mathcal{E}(A \cap H))^\circ \) is nonempty, so \( A \cap H \in \mathcal{E}^* \).

Let \( x \in (\mathcal{E}(A))^\circ \). Then for any open set \( U \) containing \( x \) and \( U \subset (\mathcal{E}(A))^\circ \subset \mathcal{E}(A) \) there is \( E \in \mathcal{E} \) such that \( E \subset U \subset A \subset U \cap A \subset (\mathcal{E}(A))^\circ \), hence \( x \in \mathcal{E}(A \cap (\mathcal{E}(A))^\circ) \). We have proved \( (\mathcal{E}(A))^\circ \subset \mathcal{E}(A \cap (\mathcal{E}(A))^\circ) \). That means \( A \cap (\mathcal{E}(A))^\circ \subset (\mathcal{E}(A))^\circ \subset [\mathcal{E}(A \cap (\mathcal{E}(A))^\circ)]^\circ \), so \( A \cap (\mathcal{E}(A))^\circ \) is \( \mathcal{E} \)-preopen. \( \Box \)

**Theorem 4.1.** Let \( \mathcal{E} \) be a cluster system. Then \( \mathcal{E}^{\text{po}} \sim \mathcal{E}^* < \mathcal{E} \) and \( \mathcal{N}_\mathcal{E} = \mathcal{N}_{\mathcal{E}^*} = \mathcal{N}_{\mathcal{E}^{\text{po}}} \).

**Proof.** Since \( \mathcal{E}^{\text{po}} \subset \mathcal{E}^* \), \( \mathcal{E}^{\text{po}} < \mathcal{E}^* \). Let \( A \in \mathcal{E}^* \). Then \( (\mathcal{E}(A))^\circ \neq \emptyset \) and by Lemma 4.1, \( A \subset (\mathcal{E}(A))^\circ \) is \( \mathcal{E} \)-preopen subset of \( A \), so \( \mathcal{E}^* < \mathcal{E}^{\text{po}} \). That means \( \mathcal{E}^* \sim \mathcal{E}^{\text{po}} \) and by Lemma 2.2 (3), \( \mathcal{N}_{\mathcal{E}^*} = \mathcal{N}_{\mathcal{E}^{\text{po}}} \).

The relation \( \mathcal{E}^* < \mathcal{E} \) is clear, so by Lemma 2.2 (2), \( \mathcal{N}_\mathcal{E} \subset \mathcal{N}_{\mathcal{E}^*} \). Let \( A \in \mathcal{N}_{\mathcal{E}^*} \) and \( A \not\in \mathcal{N}_\mathcal{E} \). Then \( G := (\mathcal{E}(A))^\circ \neq \emptyset \). Since \( A \in \mathcal{N}_{\mathcal{E}^*} \), there is a nonempty open set \( H \subset G \), such that \( A \cap H \) contains no set from \( \mathcal{E}^* \). By Lemma 4.1, \( A \cap H \in \mathcal{E}^* \), a contradiction. \( \Box \)

The following theorem gives a characterization of sets from \( \mathcal{N}_\mathcal{E} \) by the \( \mathcal{E}^* \)-operator.
Theorem 4.2. $A \in \mathcal{N}_\mathcal{E}$ if and only if $\mathcal{E}^*(A) = \emptyset$.

Proof. If $\mathcal{E}^*(A) = \emptyset$, then $A \in \mathcal{N}_\mathcal{E}$ and by Theorem 4.1, $A \in \mathcal{N}_\mathcal{E}$.

Let $A \in \mathcal{N}_\mathcal{E}$ and suppose $\mathcal{E}^*(A) \neq \emptyset$. Let $x \in \mathcal{E}^*(A)$. Then for any open set $G$ containing $x$ the intersection $G \cap A$ contains a set $B \in \mathcal{E}^*$ and by Definition 4.1, $(\mathcal{E}(B))^\circ \neq \emptyset$. Since $A \in \mathcal{N}_\mathcal{E}$, for $(\mathcal{E}(B))^\circ$ there is a nonempty open set $H \subset (\mathcal{E}(B))^\circ$ such that $A \cap H$ does not contain a set from $\mathcal{E}$. By Lemma 4.1 and Theorem 4.1, $A \cap H \in \mathcal{E}^* \prec \mathcal{E}$, so $A \cap H$ contains a set from $\mathcal{E}$, a contradiction. \qed

References
