CONTROLLING CHAOTIC DYNAMICS OF A CONTINUOUS ECOLOGICAL MODEL

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Abstract: In this paper a technique of controlling chaotic behavior of a three species food chain model with Sokol-Howell functional response by involving intraspecific competition is discussed. The model is proved to be dissipative in the positive octant. Controlling conditions of chaotic behavior of the system are carried out. Stability analysis of the equilibrium points is analyzed. Lyapunov exponents and bifurcation diagrams are used to study the complex dynamics of the model. The impact of the intraspecific competition in the dynamics of the food chain model with Sokol-Howell response is investigated theoretically as well as numerically.

AMS Subject Classification: 37D45, 37D50, 37C75

Key Words: bifurcation diagram, chaotic, period doubling, Sokol-Howell response

1. Introduction

Controlling chaos and chaotic behavior has been studied by many authors in

Received: March 2, 2016

Published: August 31, 2016

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url: www.acadpubl.eu

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the laboratory and theoretically. Since the work of May [22] in the last century, researchers in ecology, biology and applied engineering reported the existence of chaos for different range of parameter value of the system, see [2,8,19,20,29]. The principle of competitive exclusion is one of the best known result in theoretical ecology. Lotka [5] and Volterra [30] work asserts that two or more consumer species cannot coexist indefinitely on single limiting resources. Ayala [10] proved experimentally that two species of *Drosophila* could coexist upon a single limit resource. Ayala's experiment leads to a large body of work, which continuous to this day, see [1,2,6,8,15-17,19-25]. Upadhyay [20] studied Rosenzweig-McArthur [18] model to investigate the existence of complex dynamical behavior in a food chain model involving Holling type IV functional response of the form $\frac{wx}{d+x+x^2/i}$ and the effects of the intraspecific competition in the first level of the model is also studied. Sokol and Howell [28] suggested a simplified Holling type IV function of the form $\frac{wx}{d+x^2}$ and found that it is simpler and better than the original function of Holling type IV, about how these functional responses obtained see [7,13,21,22] and [26-28]. Lyapunov exponents and bifurcation diagrams are tools have been used to study the complex dynamics of the food chain models, see [3,11,12].

The main purpose of this paper is to study the effect of intraspecific competition on the dynamics of three species food chain model. A three species food chain model with Sokol-Howell functional response is proposed and analyzed analytically as well as numerically, in case of existence and non-existence of the intraspecific competition in all levels of the model. Bifurcation diagrams are drawn depending on the growth rate parameter of the prey x and to the death rate of the middle predator y to show the impact of the intraspecific competition on the dynamics of the model.

2. The Mathematical Model

Consider the three species food chain model consisting of the prey whose population density at time t denoted by $x(t)$, the intermediate predator whose population density at time t denoted by $y(t)$ and the top predator whose population density at time t denoted by $z(t)$. The intermediate predator y preys on its sole food x at the lower level according to Sokol-Howell functional response, and the top predator z preys on y at the second level according to the same functional response. The dynamics of the above food chain model can be

represented by the following set of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= x(a_1 - b_1x) - \frac{v_0xy}{h_1 + x^2} : x(0) \geq 0, \\ \frac{dy}{dt} &= \frac{v_1xy}{h_1 + x^2} - d_1y - b_2y^2 - \frac{v_2yz}{h_2 + y^2} : y(0) \geq 0, \\ \frac{dz}{dt} &= \frac{v_3yz}{h_2 + y^2} - d_2z - b_3z^2 : z(0) \geq 0.\end{aligned}\tag{1}$$

Here the positive constants a_1, b_k, h_i, d_i ($i = 1, 2$) and v_j ($j = 0, 1, 2, 3$), denote to: a_1 is the growth rate of the prey x , b_k ($k = 1, 2, 3$) represents the intraspecific competition among individuals, h_i 's are the half-saturation constant, d_i 's are the predator's death rate, v_0 is the maximum value which per capita reduction rate can x attain, v_1 has similar meaning to v_0 , v_2 are the maximum value which per capita reduction rate can y attain, v_3 has similar meaning to v_2 . Then the system of the three species food chain model given by (1) can be written as follows:

$$\begin{aligned}\frac{dx}{dt} &= x \left(a_1 - b_1x - \frac{v_0y}{h_1 + x^2} \right) = F_1(x, y, z), \\ \frac{dy}{dt} &= y \left(\frac{v_1x}{h_1 + x^2} - d_1 - b_2y - \frac{v_2z}{h_2 + y^2} \right) = F_2(x, y, z), \\ \frac{dz}{dt} &= z \left(\frac{v_3y}{h_2 + y^2} - d_2 - b_3z \right) = F_3(x, y, z),\end{aligned}\tag{2}$$

with $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. Obviously the functions F_i ($i = 1, 2, 3$) of system (2) are continuous and have continuous partial derivatives on the positive octant $R_+^3 = \{(x, y, z) \in R_+^3 : x \geq 0, y \geq 0, z \geq 0\}$. Therefore these functions are Lipschitzian on R_+^3 , and hence the solution of the system (2) exists and is unique.

3. Dissipativeness of the Model

In this section, the dissipativeness of the solution of the system (2) in R_+^3 is established in the next theorem. The ecological model system is said to be dissipative if all population initiating in R_+^3 are uniformly limited by their environment, see [9].

Theorem 1. *All the solutions of the three species food chain system (2) are uniformly bounded, and let Ω be the set defined by*

$$\Omega = \left\{ (x, y, z) \in R_+^3 : 0 \leq x \leq \frac{a_1}{b_1}, 0 \leq M_1 \leq \frac{v_1 a_1}{v_0 b_1 d_1} (a_1 + d_1), \right.$$

$$0 \leq M_2 \leq \frac{v_1 a_1}{v_0 b_1 B} (a_1 + B) \Big\}$$

where the values of $M_1 = \frac{v_1}{v_0}x + y$, $M_2 = \frac{v_1}{v_0}x + y + \frac{v_2}{v_3}z$ and $B = \min\{d_1, d_2\}$. Moreover, the system (2) is dissipative in R_+^3 .

Proof. Let $(x(t), y(t), z(t))$ be any solution of the system with non-negative initial condition. The first equation of system (2) implies that

$$\frac{dx}{dt} \leq x(a_1 - b_1 x), \quad (3)$$

by applying the comparison theorem [14], we obtain that

$$\lim_{t \rightarrow \infty} \sup x(t) \leq \frac{a_1}{b_1} \quad (4)$$

implies that $x(t) \leq \frac{a_1}{b_1}$, $\forall t \geq 0$. Now as $t \rightarrow \infty$.

$$\Rightarrow 0 \leq x(t) \leq \frac{a_1}{b_1}.$$

Consider now $M_2(t) = \frac{v_1}{v_0}x + y + \frac{v_2}{v_3}z$. Then

$$\begin{aligned} \frac{dM_2(t)}{dt} &= \frac{v_1}{v_0}a_1x - b_1x^2 - d_1y - b_2y^2 - \frac{v_2}{v_3}d_2z - b_3\frac{v_2}{v_3}z^2, \\ &\leq \frac{v_1 a_1}{v_0 b_1} (a_1 + B) - B \left(\frac{v_1}{v_0}x + y + \frac{v_2}{v_3}z \right), \end{aligned}$$

where $B = \min\{d_1, d_2\}$. Accordingly, we obtain the following differential inequality

$$\frac{dM_2(t)}{dt} + BM_2 \leq \frac{v_1 a_1}{v_0 b_1} (a_1 + B).$$

Again, by applying the comparison theorem [14] we obtain that for $t \rightarrow \infty$,

$$0 \leq M_2 \leq \frac{v_1 a_1}{v_0 b_1 B} (a_1 + B), \quad \forall t \geq 0. \quad (5)$$

Therefore, every solution initiating in non-negative octant are attracted in a bounded set Ω defined above.

Further, the calculation of the supremum of $x(t)$, $M_1(t) = \frac{v_1}{v_0}x + y$ and $M_2(t) = \frac{v_1}{v_0}x + y + \frac{v_2}{v_3}z$ as $t \rightarrow \infty$ are given:

The sup of $x(t)$ already proved in inequality (4). Now, It is easy to prove the following inequalities, see [6,16]:

$$\begin{aligned}\lim_{t \rightarrow \infty} \sup M_1(t); M_1(t) &= \frac{v_1}{v_0}x + y \leq \frac{v_1 a_1}{v_0 b_1 d_1}(a_1 + d_1) \\ \lim_{t \rightarrow \infty} \sup M_2(t); M_2(t) &= \frac{v_1}{v_0}x + y + \frac{v_2}{v_3}z \leq \frac{v_1 a_1}{v_0 b_1 B}(a_1 + B),\end{aligned}$$

Therefore, system (2) is dissipative.

4. Stability of the Model

In this section, the local stability analysis of the non-negative equilibrium points of system (2) are discussed. There are at most four non-negative equilibrium points E_0, E_1, E_2 and the interior equilibrium point E_3 which is left aside, see [15,23]. Now the existence and stability conditions of the above equilibrium points are given below:

- The trivial equilibrium point $E_0 = (0, 0, 0)$ always exists.
- The equilibrium point $E_1 = (\frac{a_1}{b_1}, 0, 0)$ always exists on the boundary of the first octant.
- The middle predator can exists and survive depending on its prey. therefore, the equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$ exists uniquely in the positive quadrant of $x - y$ plane if there is a positive solution to the following equations

$$\begin{aligned}f_1 &= a_1 - b_1 x - \frac{v_0 y}{h_1 + x^2} = 0, \\ f_2 &= \frac{v_1 x}{h_1 + x^2} - d_1 - b_2 y = 0,\end{aligned}\tag{6}$$

while $f_3 = \frac{v_3 y}{h_2 + y^2} - d_2 = 0$, since $z = 0$. Now provided $0 < x < \frac{a_1}{b_1}$ the first equation of (6) implies that

$$\bar{y} = \frac{1}{v_0}(a_1 - b_1 \bar{x})(h_1 + \bar{x}^2).\tag{7}$$

By substituting (7) in the second equation of (6) we obtained that

$$f(\bar{x}) = S_5 \bar{x}^5 + S_4 \bar{x}^4 + S_3 \bar{x}^3 + S_2 \bar{x}^2 + S_1 \bar{x}^1 + S_0,\tag{8}$$

where

$$\begin{aligned} S_5 &= \frac{b_1 b_2}{v_0} > 0, S_4 = -\frac{a_1 b_2}{v_0} < 0, \\ S_3 &= \frac{2b_1 b_2 h_1}{v_0} > 0, S_2 = -\left(d_1 + \frac{2a_1 b_2 h_1}{v_0}\right) < 0, \\ S_1 &= \left(v_1 + \frac{b_1 b_2 h_1^2}{v_0}\right) > 0, S_0 = -h_1 \left(d_1 + \frac{a_1 b_2 h_1}{v_0}\right) < 0. \end{aligned}$$

According to Descartes's rule of signs, equation (8) has at least one positive root. Now the positive equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$, where $\bar{x}$ is the positive root of equation (8) and $\bar{y}$ given by equation (7) exists uniquely in InR_+^3 .

- In the absence of prey x , then both y and z can not survive, so there is no equilibrium point in the $y - z$ plane.

Now in order to investigate the dynamical behavior of the three species food chain system (2) near the above equilibrium points. The variational matrix V of the system (2) at (x, y, z) is computed as:

$$V(x, y, z) = \begin{bmatrix} x \frac{\partial f_1}{\partial x} + f_1 & x \frac{\partial f_1}{\partial y} & x \frac{\partial f_1}{\partial z} \\ y \frac{\partial f_2}{\partial x} & y \frac{\partial f_2}{\partial y} + f_2 & y \frac{\partial f_2}{\partial z} \\ z \frac{\partial f_3}{\partial x} & z \frac{\partial f_3}{\partial y} & z \frac{\partial f_3}{\partial z} + f_3 \end{bmatrix},$$

where

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= -b_1 + \frac{2v_0 xy}{(h_1 + x^2)^2}, \quad \frac{\partial f_1}{\partial y} = -\frac{v_0}{(h_1 + x^2)}, \quad \frac{\partial f_1}{\partial z} = 0, \\ \frac{\partial f_2}{\partial x} &= \frac{v_1(h_1 - x^2)}{(h_1 + x^2)^2}, \quad \frac{\partial f_2}{\partial y} = \frac{2v_2 yz}{(h_2 + y^2)^2}, \quad \frac{\partial f_2}{\partial z} = -\frac{v_2}{(h_2 + y^2)}, \\ \frac{\partial f_3}{\partial x} &= 0, \quad \frac{\partial f_3}{\partial y} = \frac{v_3(h_2 - y^2)}{(h_2 + y^2)^2}, \quad \frac{\partial f_3}{\partial z} = 0. \end{aligned}$$

Further, the stability analysis of system (2) is carried out and according to the variational matrix $V_i; i = 0, 1, 2$ of $E_i; i = 0, 1, 2$ respectively the following results are obtained:

$$V_0 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -a_1 & -\frac{v_0 x}{h_1 + x^2} & 0 \\ 0 & \frac{v_1 x}{h_1 + x^2} - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} \bar{x} \left(-b_1 + \frac{2\bar{x}(a_1 - b_1\bar{x})}{h_1 + \bar{x}^2} \right) & -\frac{v_0\bar{x}}{(h_1 + \bar{x}^2)} & 0 \\ \frac{v_1\bar{y}(h_1 - \bar{x}^2)}{(h_1 + \bar{x}^2)^2} & \frac{v_1\bar{x}}{(h_1 + \bar{x}^2)} - d_1 - 2b_2\bar{y} & \frac{-v_2\bar{y}}{h_2 + \bar{y}^2} \\ 0 & 0 & \frac{v_3\bar{y}}{h_2 + \bar{y}^2} - d_2 \end{bmatrix}.$$

Here the trivial equilibrium point E_0 is saddle-node, having an unstable manifold along x -direction, while E_0 having stable manifold along y and z -direction. The equilibrium point E_1 is a saddle point having stable manifold along x and z -direction if the following condition holds

$$\frac{v_1x}{h_1 + x^2} > d_1, \quad (9)$$

while equilibrium point E_1 having stable manifold along x , y and z -direction if

$$\frac{v_1x}{h_1 + x^2} < d_1. \quad (10)$$

The equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$ having stable manifold along x and y -direction if the following conditions hold.

$$\frac{2\bar{x}^2(a_1 - b_1\bar{x})}{h_1 + \bar{x}^2} < b_1, \quad (11)$$

$$\frac{1}{2\bar{y}} \left(\frac{v_1\bar{x}}{h_1 + \bar{x}^2} - d_1 \right) < b_2, \quad (12)$$

while E_2 is unstable saddle if the opposite of condition (11) and or condition (12) hold.

It is easy to verify that the root of this equation satisfy the following relations:

$$\begin{aligned} \lambda_1 + \lambda_2 &= \bar{x} \left(-b_1 + \frac{2\bar{x}(a_1 - b_1\bar{x})}{h_1 + \bar{x}^2} \right) + \left[\frac{v_1\bar{x}}{(h_1 + \bar{x}^2)} - d_1 - 2b_2\bar{y} \right], \\ \lambda_1\lambda_2 &= \bar{x} \left(-b_1 + \frac{2\bar{x}(a_1 - b_1\bar{x})}{h_1 + \bar{x}^2} \right) \left[\frac{v_1\bar{x}}{(h_1 + \bar{x}^2)} - d_1 - 2b_2\bar{y} \right], \\ \lambda_3 &= \frac{v_3\bar{y}}{h_2 + \bar{y}^2} - d_2. \end{aligned}$$

Notice that $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$ (i.e condition (11) and (12) hold), hence E_2 is locally asymptotically stable in the $IntR_+^2$ of the $x - y$ plane. In addition, since the eigenvalue λ_3 describes the dynamics in the z -direction orthogonal on the $x - y$ plane. Therefore, if

$$\frac{v_3\bar{y}}{h_2 + \bar{y}^2} < d_2 \quad (13)$$

holds, then E_2 is locally asymptotically stable in the $IntR_+^3$. Moreover, if $\frac{v_3\bar{y}}{h_2+\bar{y}^2} > d_2$ holds, then E_2 is a saddle point in the $IntR_+^3$ with locally stable manifold in $x - y$ plane and with locally unstable manifold in z -direction.

5. Numerical Simulation

The global dynamics of system (2) is studied numerically. The food chain system is solved numerically using six order Runge-Kutta method [12]. There are many cases of system (2) and we expect the rich dynamics since the system is proved to be dissipative. For the following set of fixed parameters values

$$\begin{aligned} a_1 = 0.25, \quad b_1 = 0.3, \quad b_2 = b_3 = 0.001, \quad d_1 = 0.017, \quad d_2 = 0.025, \\ h_1 = 1.0, \quad h_2 = 2.0, \quad v_0 = 0.1, \quad v_1 = 0.27, \quad v_2 = 0.27, \quad v_3 = 0.095. \end{aligned} \quad (14)$$

The attractor of system (2) are drawn with their time series for the initial condition (1.2, 1.2, 1.2). The main objective is to explore the possibility of chaotic behavior of system (2) by depending on the controlling parameter and keeping other parameters of (14) fixed.

Figure 1: Bifurcation diagram of system (2) for data in (14) the successive maxima of y as function of a_1 with $b_2 = b_3 = 0.0$ and $a_1 \in (0.2, 0.34)$.

The bifurcation diagram with respect to sensitive parameters is shown in Fig.1. A various sequence of period doubling bifurcation leading to route of chaos in system (2) dynamics with respect to successive maxima of y and the intrinsic growth rate of the prey $a_1 \in (0.2, 0.34)$ for data set (14) with the values

Figure 2: Bifurcation diagram of system (2) for data set (14) the successive maxima of y as function of a_1 and $a_1 \in (0.2, 0.34)$.

Figure 3: Largest Lyapunov exponent of system (2) with data set (14) for a_1 with $b_2 = b_3 = 0.0$ and $a_1 \in (0.2, 0.35)$.

of the intraspecific competition of the middle predator b_2 and the top predator b_3 are ignored (*i.e.* $b_2 = b_3 = 0$). The largest positive Lyapunov exponent for data set (14) and a_1 is shown in Fig.(3) and this positive exponent conforms the chaotic behavior of system (2) for $a_1 \in (0.2, 0.35)$.

Figure 4: 3D of system (2) chaotic attractor for data (14) with $b_2 = b_3 = 0.0$.

Figure 5: (A) 3D of system (2) chaotic attractor for data in (14). (B) 2D $x - z$ plane of fig.5(A).

The second bifurcation diagram is generated in Fig.2 according to successive maxima of the middle predator y and $a_1 \in (0.2, 0.34)$ for data set (14) where the values of the intraspecific competition of the predators y and z are increased for the typical values $b_2 = b_3 = 0.001$. It is observed that system (2) exhibits chaotic dynamics and a period doubling bifurcation is also included. The change in the dynamics of system (2) is observed in the range $[0.23, 0.32]$

Figure 6: (A) 2D y - z plane of system (2) chaotic attractor for data set (14) with $b_3 = 0.005$. (B) limit cycle attractor for data in (14) with $b_3 = 0.007$. (C) stable point for data set (14) with $b_3 = 0.0072$ (D) time series of Fig.6(C) $x = 0.41, y = 1.47, z = 1.18$.

due to increasing the values of the intraspecific competition b_2 and b_3 of the middle predator y and the top predator z respectively.

Now the first case is by ignoring the intraspecific competition of the middle predator y and the top predator z , it is observed that system (2) dynamics is chaotic as we shown in Fig.4 and bifurcation diagram in Fig.(1). Increasing the intraspecific competition in y and z for the data set (14) change the dynamics behavior as we shown in Fig.5 and bifurcation diagram Fig.2.

The second case by fixing the intraspecific competition $b_2 = 0.001$ and varying the value of b_3 in the range 0.005-0.072. If the value of $b_3=0.005$, then system (2) is less chaotic than Fig.(4) and Fig.(5) as shown in Fig.6(A), while if $b_3=0.007$, then the model approach to limit cycle attractor as plotted in Fig.6(B). Further increasing the value of b_3 more than 0.071 change the dynamics of system (2) to stable point (0.414, 1.474, 1.182) as shown in Fig.6(C)

Figure 7: (A) 3D of system (2) period doubling with period-8 for $b_2 = 0.005$. (B) period doubling with period-4 for $b_2 = 0.007$. (C) period doubling with period-2 for $b_2 = 0.008$. (D) 3D of system (2) for data set (14) limit cycle with period-1 for $b_2 = 0.012$. (E) stable point for $b_2 = 0.045$. (F) time series of Fig.7(E) $x = 0.683, y = 0.659, z = 0.713$.

and Fig.6(D) for the typical value of $b_3 = 0.072$.

The third case is by fixing the intraspecific competition of the top predator $b_3 = 0.001$ and varying the value of b_2 in the range 0.005-0.045 with data set (14). For the value of $b_2 = 0.005$ the system dynamics is period doubling with period-8, while increasing the value of b_2 to 0.007 then the model is period doubling with period-4. Further increasing b_2 to 0.012 then the system is limit cycle with period-1 and finally for $b_2 = 0.045$ the model approach to stable at the point (0.683, 0.659, 0.713), see Fig.7.

6. Results

The food chain model system (2) is studied theoretically and the numerical simulation as plotted in Figs.(1-7) with data set (14) the following results are obtained:

- 1) The intraspecific competition of the top predator b_3 is a sensitive parameter of system (2). Increasing the value of b_3 change the dynamics of system (2) from chaotic to periodic and then to asymptotically stable, so increasing b_3 has a stabilizing effect and also prove that a small change in b_3 will leads to major change in the dynamics of system (2).
- 2) In Fig.4 system (2) has a chaotic dynamics, but if we increasing the values of the intraspecific competition of the middle predator b_2 , then system (2) approaches to an asymptotically stable as in Fig.7, so increasing b_2 can force system (2) to stay-out of chaotic area and has a stabilizing effect on the dynamics of the model.
- 3) System (2) with fixing the growth rate a_1 of the prey population x in data set (14) depend on the controlling parameters the intraspecific competition b_2 , b_3 and the death rate d_2 of the top predator population z to control the chaotic dynamics of a food chain model (2).
- 4) The bifurcation diagrams and the largest Lyapunun exponent indicate the existence of rich dynamics chaotic, limit cycle with period doubling. In addition to that, the bifurcations diagrams show the effect of the intraspecific competition in a continuous ecological model system (2).

7. Conclusions

A predator-prey food chain system (2) with Sokol-Howell functional response is proposed and investigated. The dissipativeness of the system is proved. Stability analysis of the model and the existence of the equilibrium points are carried out. The period doubling route to chaos is observed easily from the bifurcations diagrams and Fig.7. The numerical simulation shows the rich dynamics of the model. The bifurcation diagrams show that small change in the values of the intraspecific competition b_2 and b_3 changes the dynamics of the model completely and these parameters controlling the chaotic behavior of continuous ecological model system (2).

Acknowledgements

The authors acknowledged the financial support received from Ministry of Higher Education, Malaysia in the form of an FRGS research grant and the first author acknowledged the support of INSPERM in UPM University, Malaysia.

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