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STUDY ON i-v FUZZY TRANSLATION AND MULTIPLICATION OF i-v FUZZY $\beta-$SUBALGEBRA

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Abstract: In this paper, we discuss the notion of an Interval valued fuzzy translation of i-v fuzzy $\beta-$subalgebra and investigate some of their basic properties.

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1. Introduction

After the notion of fuzzy sets, Zadeh in [9,10] made an extension of a fuzzy set by an interval valued fuzzy set (ie. a fuzzy set with an interval valued membership function). This interval valued fuzzy set is referred as an i-v fuzzy set and applied in various algebraic structures.

Iseki et al. [6] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. During 2002, Neggers et al. [7] discussed $\beta-$algebras. In 2013 Chandramouleeswaran et al. [3] dealt Fuzzy Translation and Fuzzy Mul-

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tiplication in BF/BG-algebras. In 2014 [1] Aub Ayub Ansari et al. applied the Fuzzy Translation on Fuzzy $\beta-$ideals of $\beta-$algebra. Motivated by these in [4,5], we introduced an interval valued fuzzy $\beta-$ sub-algebras of $\beta-$algebra and product on i-v fuzzy $\beta-$subalgebra. In [2], Barbhuiya focused the Fuzzy Translations and fuzzy multiplications of interval valued fuzzy BG-algebra. Recently Sujatha et al. [8] introduced the notion of intuitionistic fuzzy $\alpha-$translation on $\beta-$algebras. With all these ideas, in this paper, we discuss the notion on i-v fuzzy translation of i-v fuzzy $\beta-$subalgebras.

2. Preliminares

In this section we recall some basic definitions needed for our work.

Definition 2.1. [7] A β-algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

1. $x - 0 = x$
2. $(0 - x) + x = 0$
3. $(x - y) - z = x - (z - y)\ \forall\ x, y, z\ \in\ X.$

Example 2.2. Let $X = \{0, a, b, c\}$ be a set with constant 0 and binary operation $+$ and $-$are defined on X by the following Cayley's table

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

$-$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X, +, -, 0)$ is a $\beta-$algebra.

Definition 2.3. A non empty subset A of a $\beta-$algebra $(X, +, -, 0)$ is called a β-subalgebra of X, if $\forall\ x, y \in X$

1. $x + y \in A$
2. $x - y \in A$

Example 2.4. In the above example of the $\beta-$algebra X, the subset $\{0, b\}$,$\{0, a\}$,$\{0, c\}$ are $\beta-$subalgebra of X. But the subset $A = \{0, a, b\}$is not a $\beta-$subalgebra of X, since $(a + b = c \notin A)$

Definition 2.5. [1] Let μ be a fuzzy set in a $\beta-$algebra X. Then μ is called a fuzzy β-subalgebra of X, if $\forall\ x, y \in X$

1. $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
2. $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$

Definition 2.6. [10] An interval valued fuzzy set (briefly i-v fuzzy set) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}\ \ \forall\ x \in X$$

(briefly denoted by $A = [\mu_A^L, \mu_A^U]$), where μ_A^L and μ_A^U are two fuzzy sets in X such that $\mu_A^L(x) \leq \mu_A^U(x)\ \ \forall\ x \in X$.

Let $\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]\ \ \forall\ x \in X$ and let $D[0,1]$ denotes the family of all closed subintervals of $[0,1]$. If $\mu_A^L(x) = \mu_A^U(x) = c$, say, where $0 \leq c \leq 1$, then we have $\overline{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\overline{\mu}_A(x) \in D[0,1]\ \ \forall\ x \in X$, and therefore the i-v fuzzy set A is given by

$$A = \{(x, \overline{\mu}_A(x))\}\ \ \forall\ x \in X,$$

where $\overline{\mu}_A : X \to D[0,1]$.

Now let us define what is known as *refined mimimum*(briefly *rmim*) of two elements in $D[0,1]$. We also define the symbols " $\geq$ " , " $\leq$ ", and " $=$ " in case of two elements in $D[0,1]$.

Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$.

Then we have:

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}];$$

$D_1 \geq D_2$ if and only if $a_1 \geq a_2,\ b_1 \geq b_2$.

Similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

Remark 2.7. Let $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$.Then

1. $D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2\ \ \&\ \ b_1 \leq b_2$
2. $D_1 = D_2 \Leftrightarrow a_1 = a_2\ \ \&\ \ b_1 = b_2$
3. $D_1 + D_2 = [a_1 + a_2, b_1 + b_2]$ whenever $a_1 + a_2 \leq 1$ and $b_1 + b_2 \leq 1$
4. $D_1 - D_2 = [a_1 - a_2, b_1 - b_2]$ whenever $a_1 - a_2 \leq 1$ and $b_1 - b_2 \leq 1$

Definition 2.8. [4] Let $\overline{\mu}_A$ be an i-v fuzzy subset in X. Then $\overline{\mu}_A$ is said to be interval valued fuzzy(i-v-fuzzy) $\beta-$subalgebra of X, if $\forall\ x, y \in X$

1. $\overline{\mu}_A(x+y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$
2. $\overline{\mu}_A(x-y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$

Example 2.9. Consider the $\beta-$algebra $X = \{0, a, b, c\}$ in example 2.2. Define an i-v fuzzy subset $\overline{\mu}$ of X defined by

$$\overline{\mu}(x) = \begin{cases} [0.3, 0.7] : & x = 0 \\ [0.1, 0.5] : & x = a, c \\ [0.2, 0.6] : & x = b \end{cases}$$

Then $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Definition 2.10. [1] Let μ be a fuzzy set of a $\beta-$algebra X and $\alpha \in [0, T]$ where $T = 1 - sup\{\mu(x)/x \in X\}$. Then the fuzzy set $\mu_\alpha^T : X \to D[0,1]$ is called a fuzzy $\alpha-$translation of μ if $\mu_\alpha^T(x) = \mu(x) + \alpha, \forall x \in X$.

3. Interval Valued Fuzzy Translations of β-Subalgebra

This section, deals with the notion of Interval valued fuzzy translation of $\beta-$subalgebra. In what follows, X denotes a $\beta-$algebra and for any i-v fuzzy set $\overline{\mu}$ of X, we denote $\overline{T} = [1,1] - rsup\{\overline{\mu}(x)/x \in X\}$ unless otherwise specified.we start with,

Definition 3.1. Let $\overline{\mu}$ be an i-v fuzzy set of X and $\overline{\alpha} \in [\overline{0}, \overline{T}]$, where $\overline{\alpha} = [\alpha^L, \alpha^U]$ with $\alpha^L \in [0, T^L]$ & $\alpha^U \in [0, T^U]$ and $\overline{0} = [0,0]$. A mapping $\overline{\mu}_{\overline{\alpha}}^{\overline{T}} : X \to D[0,1]$ is said to be an i-v fuzzy $\overline{\alpha}-$translation of $\overline{\mu}$ if it satisfies $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x) = \overline{\mu}(x) + \overline{\alpha}, \forall x \in X$.

Example 3.2. Consider the $\beta-$algebra $X = \{0, a, b, c\}$in example 2.2. Define an interval valued fuzzy subset $\overline{\mu}$ of X by

$$\overline{\mu}(x) = \begin{cases} [0.3, 0.7] : & x = 0 \\ [0.1, 0.5] : & x = a, c \\ [0.2, 0.6] : & x = b \end{cases}$$

Then $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X. Here $\overline{T} = [1,1] - rsup\{\overline{\mu}(x)/x \in X\} = [1,1] - [0.3, 0.7] = [0.7, 0.3]$. choose $\overline{\alpha} = [0.04, 0.08] \in [\overline{0}, \overline{T}]$. Then the i-v fuzzy set $\overline{\mu}_{\overline{\alpha}}^{\overline{T}} : X \to D[0,1]$ is given by $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(0) = [0.34, 0.78]$, $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(a) = \overline{\mu}_{\overline{\alpha}}^{\overline{T}}(c) = [0.14, 0.58]$ *and* $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(b) = [0.24, 0.68]$ is a i-v fuzzy $\overline{\alpha}-$ Translation of $\overline{\mu}$.

Theorem 3.3. For any i-v fuzzy $\beta-$subalgebra $\overline{\mu}$ of X and $\overline{\alpha} \in [\overline{0}, \overline{T}]$, the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x)$ of $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Proof. Let $x, y \in X$ and $\overline{\alpha} \in [\overline{0}, \overline{T}]$, Then:

$$\overline{\mu}(x+y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$$

and

$$\overline{\mu}(x-y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}.$$

Now

$$\begin{aligned}\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x+y) &= \overline{\mu}(x+y) + \overline{\alpha} \\ &\geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\} + \overline{\alpha} \\ &= rmin\{\overline{\mu}(x) + \overline{\alpha}, \overline{\mu}(y) + \overline{\alpha}\} \\ &= rmin\{\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x), \overline{\mu}_{\overline{\alpha}}^{\overline{T}}(y)\}\end{aligned}$$

Similarly, $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x-y) \geq rmin\{\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x), \overline{\mu}_{\overline{\alpha}}^{\overline{T}}(y)\}$
Hence $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}$ of $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

The following is the converse of the above theorem.

Theorem 3.4. For any i-v fuzzy subset $\overline{\mu}$ of X and $\overline{\alpha} \in [\overline{0}, \overline{T}]$. If the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}$ of $\overline{\mu}$ is also an i-v fuzzy $\beta-$subalgebra of X, then so is $\overline{\mu}$.

Proof. Let $x, y \in X$
Assume that $\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x)$ of $\overline{\mu}$ is a i-v fuzzy $\beta-$subalgebra of X for some $\overline{\alpha} \in [\overline{0}, \overline{T}]$.

Then:

$$\begin{aligned}\overline{\mu}(x+y) + \overline{\alpha} &= \overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x+y) \\ &\geq rmin\{\overline{\mu}_{\overline{\alpha}}^{\overline{T}}(x), \overline{\mu}_{\overline{\alpha}}^{\overline{T}}(y)\} \\ &= rmin\{\overline{\mu}(x) + \overline{\alpha}, \overline{\mu}(y) + \overline{\alpha}\} \\ &= rmin\{\overline{\mu}(x), \overline{\mu}(y)\} + \overline{\alpha}\end{aligned}$$

$\Rightarrow \overline{\mu}(x+y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Similarly,$\overline{\mu}(x-y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Hence $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Remark 3.5. In general for any i-v fuzzy set $\overline{\mu}$ of X, the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(\overline{\alpha} \in [\overline{0}, \overline{T}])$ of $\overline{\mu}$ need not be an i-v fuzzy $\beta-$subalgebra of X, as shown by the following example.

Let X be the $\beta-$algebra given in Example 3.2. Consider the i-v fuzzy set $\overline{\mu}$

$$\overline{\mu}(x) = \begin{cases} [0.4, 0.6]: & x = 0 \\ [0.3, 0.5]: & x = a \\ [0.2, 0.4]: & x = b \\ [0.1, 0.3]: & x = c \end{cases}$$

Let $\overline{\alpha} = [0.02, 0.03]$. Then the corresponding $\overline{\alpha}-$translation is

$$\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(0) = [0.42, 0.63], \quad \overline{\mu}^{\overline{T}}_{\overline{\alpha}}(a) = [0.32, 0.53],$$

$$\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(b) = [0.22, 0.43] \text{ and } \overline{\mu}^{\overline{T}}_{\overline{\alpha}}(c) = [0.12, 0.33].$$

Now $\overline{\mu}(a+b) = \overline{\mu}(c) = [0.1, 0.3] \ngeq [0.2, 0.4] = rmin\{\overline{\mu}(a), \overline{\mu}(b)\}$ and $\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(a+b) = \overline{\mu}^{\overline{T}}_{\overline{\alpha}}(c) = [0.12, 0.33] \ngeq [0.22, 0.43] = rmin\{\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(a), \overline{\mu}^{\overline{T}}_{\overline{\alpha}}(b)\}$.

Hence $\overline{\mu}$ and $\overline{\mu}^{\overline{T}}_{\overline{\alpha}}$ are not i-v fuzzy $\beta-$subalgebra of X.

Corollary 3.6. Let $\overline{\mu}$ be an i-v fuzzy set of X. If $\overline{\alpha} = \overline{0}$ then the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}^{\overline{T}}_{\overline{\alpha}}$ of $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Theorem 3.7. Let $\overline{\mu}$ be given an i-v fuzzy $\beta-$subalgebra of X. Then for $\overline{\alpha}, \overline{\alpha'} \in [\overline{0}, \overline{T}], (\overline{\mu}^{\overline{T}}_{\overline{\alpha}} \cap \overline{\mu}^{\overline{T}}_{\overline{\alpha'}})$ and $(\overline{\mu}^{\overline{T}}_{\overline{\alpha}} \cup \overline{\mu}^{\overline{T}}_{\overline{\alpha'}})$ are also an i-v fuzzy $\beta-$subalgebra of X.

Proof. Let $\overline{\mu}^{\overline{T}}_{\overline{\alpha}}$ and $\overline{\mu}^{\overline{T}}_{\overline{\alpha'}}$ be two i-v fuzzy translation of an i-v fuzzy β-subalgebra $\overline{\mu}$ of X, where $\overline{\alpha}, \overline{\alpha'} \in [\overline{0}, \overline{T}]$

Assume that $\overline{\alpha} \leq \overline{\alpha'}$ by theorem 3.3 $\overline{\mu}^{\overline{T}}_{\overline{\alpha}}$ and $\overline{\mu}^{\overline{T}}_{\overline{\alpha'}}$ be two i-v fuzzy translation of $\beta-$subalgebra of X. Now

$$\begin{aligned} (\overline{\mu}^{\overline{T}}_{\overline{\alpha}} \cap \overline{\mu}^{\overline{T}}_{\overline{\alpha'}})(x) &= rmin\{\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(x), \overline{\mu}^{\overline{T}}_{\overline{\alpha'}}(x)\} \\ &= rmin\{\overline{\mu}(x) + \overline{\alpha}, \overline{\mu}(x) + \overline{\alpha'}\} \\ &= \overline{\mu}(x) + \overline{\alpha} \\ &= \overline{\mu}^{\overline{T}}_{\overline{\alpha}}(x). \end{aligned}$$

Also

$$(\overline{\mu}^{\overline{T}}_{\overline{\alpha}} \cup \overline{\mu}^{\overline{T}}_{\overline{\alpha'}})(x) = rmax\{\overline{\mu}^{\overline{T}}_{\overline{\alpha}}(x), \overline{\mu}^{\overline{T}}_{\overline{\alpha'}}(x)\}$$

$$= rmax\{\overline{\mu}(x) + \overline{\alpha}, \overline{\mu}(x) + \overline{\alpha'}\}$$
$$= \overline{\mu}(x) + \overline{\alpha'}$$
$$= \overline{\mu}^{\overline{T}}_{\overline{\alpha'}}(x)$$

$(\overline{\mu}^{\overline{T}}_{\overline{\alpha}} \cap \overline{\mu}^{\overline{T}}_{\overline{\alpha'}})$ and $(\overline{\mu}^{\overline{T}}_{\overline{\alpha}} \cup \overline{\mu}^{\overline{T}}_{\overline{\alpha'}})$ is an i-v fuzzy $\beta-$subalgebra of X.

Theorem 3.8. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two i-v fuzzy $\beta-$subalgebras of X. Let $\overline{T} = rmin\{\overline{T}_{\overline{\mu}_1}, \overline{T}_{\overline{\mu}_2}\}$ where $\overline{T}_{\overline{\mu}_1} = [1, 1] - rsup\{\overline{\mu}_1(x) : x \in X\}$ and $\overline{T}_{\overline{\mu}_2} = [1, 1] - rsup\{\overline{\mu}_2(x) : x \in X\}$. Then the intersection of $\overline{\alpha}-$translation of $\overline{\mu}_1$ and $\overline{\alpha'}-$translation of $\overline{\mu}_2$ for some $\overline{\alpha}, \overline{\alpha'} \in [\overline{0}, \overline{T}]$ is an i-v fuzzy $\beta-$subalgebra of X.

Proof. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two i-v fuzzy $\beta-$subalgebra of X.
Then by theorem 3.3 $\overline{\mu}_{1\overline{\alpha}}^{\overline{T}}$ and $\overline{\mu}_{2\overline{\alpha'}}^{\overline{T}}$ are i-v fuzzy $\beta-$subalgebra of X.
For $x, y \in X$,

$$\begin{aligned}(\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})(x+y) &= rmin\{\overline{\mu}_{1\overline{\alpha}}^{\overline{T}}(x+y), \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}}(x+y)\} \\ &\geq rmin\{rmin\{\overline{\mu}_{1\overline{\alpha}}^{\overline{T}}(x), \overline{\mu}_{1\overline{\alpha}}^{\overline{T}}(y)\}, rmin\{\overline{\mu}_{2\overline{\alpha'}}^{\overline{T}}(x), \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}}(y)\}\} \\ &= rmin\{rmin\{\overline{\mu}_{1\overline{\alpha}}^{\overline{T}}(x), \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}}(x)\}, rmin\{\overline{\mu}_{1\overline{\alpha}}^{\overline{T}}(y), \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}}(y)\}\} \\ &= rmin\{(\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})(x), (\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})(y)\}.\end{aligned}$$

Similarly

$$(\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})(x-y) \geq rmin\{(\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})(x), (\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})(y)\}.$$

Therefore

$$(\overline{\mu}_{1\overline{\alpha}}^{\overline{T}} \cap \overline{\mu}_{2\overline{\alpha'}}^{\overline{T}})$$

is an i-v fuzzy $\beta-$subalgebra of X.

Definition 3.9. Let $f : X \to Y$ be a function. Let $\overline{\mu}_X$ and $\overline{\mu}_Y$ be an i-v fuzzy $\overline{\alpha}-$translation on X and Y respectively. Then inverse image of $\overline{\mu}_Y$ under f is defined by $f^{-1}(\overline{\mu}_Y) = \{f^{-1}(\overline{\mu}_Y)^{\overline{T}}_{\overline{\alpha}}(x) : x \in X\}$ such that $f^{-1}(\overline{\mu}_Y)^{\overline{T}}_{\overline{\alpha}}(x) = \overline{\mu}_Y(f(x) + \overline{\alpha})$

Theorem 3.10. Let X and Y be two $\beta-$algebras and $f : X \to Y$ be a homomorphism. If the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}_Y$ of Y is an i-v fuzzy $\beta-$subalgebra of Y, then $f^{-1}(\overline{\mu}_Y)$ is an i-v fuzzy $\beta-$subalgebra of X.

Proof. Let the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}_Y$ of Y be an i-v fuzzy $\beta-$subalgebra of Y.

Take $x, y \in Y$. Then

$$\begin{aligned} f^{-1}(\overline{\mu}_Y\tfrac{\overline{T}}{\overline{\alpha}})(x+y) &= f^{-1}(\overline{\mu}_Y)(x+y)+\overline{\alpha} \\ &= \overline{\mu}_Y(f(x+y)+\overline{\alpha}) \\ &= \overline{\mu}_Y(f(x)+f(y))+\overline{\alpha} \\ &\geq rmin\{\overline{\mu}_Y(f(x)+\overline{\alpha}),\overline{\mu}_Y(f(y)+\overline{\alpha})\} \\ &= rmin\{f^{-1}(\overline{\mu}_Y\tfrac{\overline{T}}{\overline{\alpha}})(x), f^{-1}(\overline{\mu}_Y\tfrac{\overline{T}}{\overline{\alpha}})(y)\} \end{aligned}$$

Similarly, $f^{-1}(\overline{\mu}_Y\frac{\overline{T}}{\overline{\alpha}})(x-y) \geq rmin\{f^{-1}(\overline{\mu}_Y\frac{\overline{T}}{\overline{\alpha}})(x), f^{-1}(\overline{\mu}_Y\frac{\overline{T}}{\overline{\alpha}})(y)\}$.

Hence $f^{-1}(\overline{\mu}_Y)$ is an i-v fuzzy $\beta-$subalgebra of X.

Theorem 3.11. Let X and Y be two $\beta-$algebras and $f : X \to Y$ be a epimorphism. If the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}_X$ of X is an i-v fuzzy $\beta-$subalgebra of X, then $f(\overline{\mu}_X)$ is an i-v fuzzy $\beta-$subalgebra of Y.

Proof. Let the i-v fuzzy $\overline{\alpha}-$translation $\overline{\mu}_X$ of X is an i-v fuzzy $\beta-$subalgebra of X.

Take $x, y \in Y$. Then

$$\begin{aligned} f(\overline{\mu}_X\tfrac{\overline{T}}{\overline{\alpha}})(x+y) &= f(\overline{\mu}_X)(x+y)+\overline{\alpha} \\ &= \overline{\mu}_X(f(x+y)+\overline{\alpha}) \\ &= \overline{\mu}_X(f(x)+f(y))+\overline{\alpha} \\ &\geq rmin\{\overline{\mu}_X(f(x)+\overline{\alpha}),\overline{\mu}_X(f(y)+\overline{\alpha})\} \\ &= rmin\{f(\overline{\mu}_X\tfrac{\overline{T}}{\overline{\alpha}})(x), f(\overline{\mu}_X\tfrac{\overline{T}}{\overline{\alpha}})(y)\} \end{aligned}$$

similarly,$f(\overline{\mu}_X\frac{\overline{T}}{\overline{\alpha}})(x-y) \geq rmin\{f(\overline{\mu}\frac{\overline{T}}{\overline{\alpha}}(x)), f(\overline{\mu}\frac{\overline{T}}{\overline{\alpha}}(y))\}$.

Hence $f(\overline{\mu}_X\frac{\overline{T}}{\overline{\alpha}})$ is an i-v fuzzy $\beta-$subalgebra of Y.

Theorem 3.12. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two i-v fuzzy $\beta-$subalgebras of X. Let $\overline{T} = rmin\{\overline{T}_{\overline{\mu}_1}, \overline{T}_{\overline{\mu}_2}\}$ where $\overline{T}_{\overline{\mu}_1} = [1,1] - rsup\{\overline{\mu}_1(x) : x \in X\}$ and $\overline{T}_{\overline{\mu}_2} = [1,1]-rsup\{\overline{\mu}_2(x) : x \in X\}$. Let $\overline{\alpha} \in [\overline{0}, \overline{T}]$. Then the $\overline{\alpha}-$translation of cartesian product $\overline{\mu}_1 \times \overline{\mu}_2$ of $\overline{\mu}_1$ and $\overline{\mu}_2$ is an i-v fuzzy $\beta-$subalgebra of $X \times X$.

Proof. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be an i-v fuzzy $\beta-$subalgebra of a $\beta-$algebra X and $\overline{\alpha} \in [\overline{0}, \overline{T}]$.

Now by theorem 3.3 $\overline{\mu}_1\frac{\overline{T}}{\overline{\alpha}}$ and $\overline{\mu}_2\frac{\overline{T}}{\overline{\alpha}}$ are i-v fuzzy $\beta-$subalgebra of X.

Clearly $\overline{\mu}_{1\frac{T}{\overline{\alpha}}} \times \overline{\mu}_{2\frac{T}{\overline{\alpha}}}$ is an i-v fuzzy $\beta-$subalgebra of $X \times X$. Also

$$\begin{aligned}(\overline{\mu}_1 \times \overline{\mu}_2)_{\frac{T}{\overline{\alpha}}}(a,b) &= (\overline{\mu}_1 \times \overline{\mu}_2)(a,b) + \overline{\alpha} \\ &= rmin\{\overline{\mu}_1(a), \overline{\mu}_2(b)\} + \overline{\alpha} \\ &= rmin\{\overline{\mu}_1(a) + \overline{\alpha}, \overline{\mu}_2(b) + \overline{\alpha}\} \\ &= rmin\{\overline{\mu}_{1\frac{T}{\overline{\alpha}}}(a), \overline{\mu}_{2\frac{T}{\overline{\alpha}}}(b)\} \\ &= (\overline{\mu}_{1\frac{T}{\overline{\alpha}}} \times \overline{\mu}_{2\frac{T}{\overline{\alpha}}})(a,b).\end{aligned}$$

Hence $(\overline{\mu}_1 \times \overline{\mu}_2)_{\frac{T}{\overline{\alpha}}}$ is an i-v fuzzy $\beta-$subalgebra of $X \times X$

4. Interval Valued Fuzzy Multiplication of β-Subalgebra

In this section, we introduce the notion of interval valued fuzzy $\overline{\phi}-$multiplication. To illustrate the concept, we discuss some examples. Also we prove some simple results.

Definition 4.1. Let $\overline{\mu}$ be an i-v fuzzy subset of X and $\overline{\phi} \in D[0,1]$. A mapping $\overline{\mu}^{M}_{\overline{\phi}} : X \to D[0,1]$ is said to be an i-v fuzzy $\overline{\phi}-$multiplication of $\overline{\mu}$ if it satisfies $\overline{\mu}^{M}_{\overline{\phi}}(x) = \overline{\phi}.\overline{\mu}(x) \ \ \forall\, x \in X$

Example 4.2. Consider the above example 3.2. Let $\overline{\phi} = [0.2, 0.3]$. Then the $\overline{\phi}-$multiplication of i-v fuzzy set $\overline{\mu}$ is given by

$$\overline{\mu}^{M}_{\overline{\phi}}(0) = [0.06, 0.21], \quad \overline{\mu}^{M}_{\overline{\phi}}(a) = \overline{\mu}^{M}_{\overline{\phi}}(c) = [0.02, 0.15] \ \textit{and} \ \overline{\mu}^{M}_{\overline{\phi}}(b) = [0.04, 0.18].$$

Theorem 4.3. For any i-v fuzzy $\beta-$subalgebra $\overline{\mu}$ of X and $\overline{\phi} \in D[0,1]$, the i-v fuzzy $\overline{\phi}-$multiplication $\overline{\mu}^{M}_{\overline{\phi}}(x)$ of $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Proof. Let $x, y \in X$ and $\overline{\phi} \in D[0,1]$, Then

$$\overline{\mu}(x+y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$$

and

$$\overline{\mu}(x-y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}.$$

Now

$$\begin{aligned}\overline{\mu}^{M}_{\overline{\phi}}(x+y) &= \overline{\phi}.\overline{\mu}(x+y) \\ &\geq \overline{\phi}.rmin\{\overline{\mu}(x), \overline{\mu}(y)\}\end{aligned}$$

$$= rmin\{\overline{\phi}.\overline{\mu}(x), \overline{\phi}.\overline{\mu}(y)\}$$
$$= rmin\{\overline{\mu}^M_{\overline{\phi}}(x), \overline{\mu}^M_{\overline{\phi}}(y)\}$$

Similarly, $\overline{\mu}^M_{\overline{\phi}}(x-y) \geq rmin\{\overline{\mu}^M_{\overline{\phi}}(x), \overline{\mu}^M_{\overline{\phi}}(y)\}$.

Hence $\overline{\mu}^M_{\overline{\phi}}$ of $\overline{\mu}$ is a i-v fuzzy $\beta-$subalgebra of X.

The following is the converse of the above theorem.

Theorem 4.4. For any i-v fuzzy subset $\overline{\mu}$ of X and $\overline{\phi} \in D[0,1]$. If the i-v fuzzy $\overline{\phi}-$multiplication $\overline{\mu}^M_{\overline{\phi}}$ of $\overline{\mu}$ is also an i-v fuzzy $\beta-$subalgebra of X, then so is $\overline{\mu}$.

Proof. Let $x, y \in X$. Assume that $\overline{\mu}^M_{\overline{\phi}}(x)$ of $\overline{\mu}$ is a i-v fuzzy $\beta-$subalgebra of X for some $\overline{\phi} \in D[0,1]$.

Then

$$\begin{aligned} \overline{\phi}.\overline{\mu}(x+y) &= \overline{\mu}^M_{\overline{\phi}}(x+y) \\ &\geq rmin\{\overline{\mu}^M_{\overline{\phi}}(x), \overline{\mu}^M_{\overline{\phi}}(y)\} \\ &= rmin\{\overline{\phi}.\overline{\mu}(x), \overline{\phi}.\overline{\mu}(y)\} \\ &= \overline{\phi}.rmin\{\overline{\mu}(x), \overline{\mu}(y)\} \end{aligned}$$

$\Rightarrow \overline{\mu}(x+y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Similarly,$\overline{\mu}(x-y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Hence $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Definition 4.5. Let $\overline{\mu}$ be an i-v fuzzy subset of X, $\overline{\phi} \in D[0,1]$ and $\overline{\alpha} \in [\overline{0}, \overline{T}]$. A mapping $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}} : X \to D[0,1]$ is said to be an i-v fuzzy magnified $-\overline{\phi}\overline{\alpha}-$translation of $\overline{\mu}$ if it satisfies $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x) = \overline{\phi}.\overline{\mu}(x) + \overline{\alpha} \;\; \forall\, x \in X$.

Example 4.6. Consider the $\beta-$algebra $X = \{0, a, b, c\}$in example 2.2. Define an interval valued fuzzy subset $\overline{\mu}$ of X by

$$\overline{\mu}(x) = \begin{cases} [0.3, 0.7] : & x = 0 \\ [0.1, 0.5] : & x = a, c \\ [0.2, 0.6] : & x = b \end{cases}$$

Then $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X. Here $\overline{T} = [1,1] - rsup\{\overline{\mu}(x)/x \in X\} = [1,1] - [0.3, 0.7] = [0.7, 0.3]$. choose $\overline{\alpha} = [0.04, 0.08] \in [[0,0], [0.7, 0.3]$ and $\overline{\phi}. = [0.1, 0.3] \in D[0,1]$.

Then the i-v fuzzy set $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}} : X \to D[0,1]$ is given by

$$\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(0) = [0.07, 0..29], \quad \overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(a) = \overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(c) = [0.05, 0.23]$$

and

$$\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(b) = [0.06, 0.26].$$

Theorem 4.7. Let $\overline{\mu}$ be an i-v fuzzy subset of X, $\overline{\phi} \in D[0,1]$ and $\overline{\alpha} \in [\overline{0}, \overline{T}]$. A mapping $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}} : X \to D[0,1]$ is an i-v fuzzy magnified-$\overline{\phi}\overline{\alpha}$-translation of $\overline{\mu}$. Then $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X if and only if $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}$ is an i-v fuzzy $\beta-$subalgebra of X.

Proof. Let $\overline{\mu}$ be an i-v fuzzy subset of X, $\overline{\phi} \in D[0,1]$ and $\overline{\alpha} \in [\overline{0}, \overline{T}]$. A mapping $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}} : X \to D[0,1]$ is said to be an i-v fuzzy magnified-$\overline{\phi}\overline{\alpha}$-translation of $\overline{\mu}$.

Assume that $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Then $\overline{\mu}(x+y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$ and $\overline{\mu}(x-y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Now

$$\begin{aligned}
\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x+y) &= \overline{\phi}.\overline{\mu}(x+y) + \overline{\alpha} \\
&\geq \overline{\phi}.rmin\{\overline{\mu}(x), \overline{\mu}(y)\} + \overline{\alpha} \\
&= rmin\{\overline{\phi}.\overline{\mu}(x) + \overline{\alpha}, \overline{\phi}.\overline{\mu}(y) + \overline{\alpha}\} \\
&= rmin\{\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x), \overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(y)\}
\end{aligned}$$

Similarly, $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x-y) \geq rmin\{\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x), \overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(y)\}$.

Hence $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}$ of $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

Assume that $\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x)$ of $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X

Then:

$$\begin{aligned}
\overline{\phi}.\overline{\mu}(x+y) + \overline{\alpha} &= \overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x+y) \\
&\geq rmin\{\overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(x), \overline{\mu}^{M\overline{T}}_{\overline{\phi}\overline{\alpha}}(y)\} \\
&= rmin\{\overline{\phi}.\overline{\mu}(x) + \overline{\alpha}, \overline{\phi}.\overline{\mu}(y) + \overline{\alpha}\} \\
&= \overline{\phi}.rmin\{\overline{\mu}(x), \overline{\mu}(y)\} + \overline{\alpha}
\end{aligned}$$

$\Rightarrow \overline{\mu}(x+y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Similarly,$\overline{\mu}(x-y) \geq rmin\{\overline{\mu}(x), \overline{\mu}(y)\}$.

Hence $\overline{\mu}$ is an i-v fuzzy $\beta-$subalgebra of X.

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