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NEW $H^1(\Omega)$ CONFORMING FINITE ELEMENTS ON HEXAHEDRA

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Abstract: In this paper, we introduce new scalar finite element spaces on hexahedron. We prove the unisolvence of degrees of freedom and analyze our spaces using the discrete de Rham diagram.

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Key Words: finite element methods, H^1 conforming elements, curl conforming elements, de Rham diagram

1. Introduction

Suppose Ω is a bounded domain with two disjoint connected boundaries Γ and Σ. We seek to compute the time-harmonic electric field $\mathbf{E}$ corresponding to a given current density $\mathbf{F}$ by solving the time-harmonic electric field equation subject to the perfect conducting boundary condition and the impedance boundary condition as follows:

$$\nabla \times (\mu_r^{-1} \nabla \times \mathbf{E}) - \kappa^2 \varepsilon_r \mathbf{E} = \mathbf{F}, \quad \text{in } \Omega, \tag{1}$$

$$\mathbf{n} \times \mathbf{E} = 0, \quad \text{on } \Gamma, \tag{2}$$

$$\mu_r^{-1}(\nabla \times \mathbf{E}) \times \mathbf{n} - i\kappa\lambda \mathbf{E}_T = \mathbf{g}, \quad \text{on } \Sigma, \tag{3}$$

where $\mathbf{E}_T = (\mathbf{n} \times \mathbf{E}|_\Sigma) \times \mathbf{n}$ and $\mathbf{g}$ is a given tangential vector field on Σ. Using

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the Galerkin method, we can find a variational formulation. Taking the dot product of (1) by the complex conjugate of a smooth vector function ϕ and integrating over Ω, and then using the integration by parts we obtain

$$\int_\Omega [(\mu_r^{-1}\ \nabla \times \mathbf{E}) \cdot \nabla \times \bar{\phi}\ -\ \kappa^2(\varepsilon_r \mathbf{E}) \cdot \bar{\phi}]\ dV - i\kappa \int_\Sigma \lambda \mathbf{E}_T \cdot \bar{\phi}_T\ dA = \int_\Omega \mathbf{F} \cdot \bar{\phi}\ dV + \int_\Sigma \mathbf{g} \cdot \bar{\phi}_T\ dA, \quad (4)$$

where $\bar{\phi}_T = (\mathbf{n} \times \phi) \times \mathbf{n}$ on Ω. Under appropriate assumptions on the coefficients and domain, problem (4) has a unique solution[1], [2]. In performing the analysis of this problem, the $H(\mathrm{curl}\,, \Omega)$ conforming finite element space and the $H^1(\Omega)$ conforming finite element space play an important role. In this paper, we introduce a new family of scalar finite elements and show that the relevant function spaces are related by the famous de Rham diagram[3], [4].

2. New $H^1(\Omega)$ Conforming Finite Element Spaces

In this section, we will describe a new family of scalar finite elements on parallelepipeds with edges parallel to the coordinate axes. First, we generate a regular finite element mesh $\tau_h = \{K\}$ covering the domain Ω such that
(1) $\overline{\Omega} = \bigcup_{K \in \tau_h} \overline{K}$, where $\overline{\Omega}$ denotes the closure of Ω;
(2) for each $K \in \tau_h$, K is an open set with positive volume;
(3) if K_1 and K_2 are distinct elements in τ_h then $K_1 \bigcap K_2 = \varnothing$;
(4) each $K \in \tau_h$ is a Lipschitz domain.
For each element K, we define the parameters h_K and ρ_K such that

$$h_K = \text{diameter of the smallest sphere containing } \overline{K},$$

$$\rho_K = \text{diameter of the largest sphere contained in } \overline{K},$$

then $h = \max_{K \in \tau_h} h_K$ so that the index h denote the maximum diameter of the elements $K \in \tau_h$.

Any $K \in \tau_h$ can be obtained by mapping the reference element $\widehat{K}$, typically unit cube, using an affine map $F_K : \widehat{K} \to K$ such that $F_K(\widehat{K}) = K$ and $F_K(\hat{\mathbf{x}}) = B_K \hat{\mathbf{x}} + b_K$, where B_K is a nonsingular 3×3 matrix and b_K is a vector. Because of the simplicity of the mapping, although we define the elements on the reference domain $\widehat{K}$, the same definition can be used on a target element K in the mesh[5]. In order to define finite elements on parallelepipeds, we need the following polynomial space:

$$Q_{\ell,m,n}(\widehat{K}) = \{\text{polynomials of maximum degree } \ell \text{ in } \hat{x},\ \ m \text{ in } \hat{y} \text{ and } n \text{ in } \hat{z}\}.$$

Definition 1. Let $k \geq 1$. On the reference element, the gradient conforming element is defined as follows:

$$\widehat{U}(\widehat{K}) = Q^*_{k+1,\ k+1,\ k+1}(\widehat{K}),$$

where $Q^*_{k+1,\ k+1,\ k+1}(\widehat{K})$ is $Q_{k+1,\ k+1,\ k+1}(\widehat{K})$ space except constant multiple of the term $\hat{x}^{k+1}\hat{y}^{k+1}\hat{z}^{\ell}$, $\hat{x}^{k+1}\hat{y}^{\ell}\hat{z}^{k+1}$, $\hat{x}^{\ell}\hat{y}^{k+1}\hat{z}^{k+1}$ and $\hat{x}^{k+1}\hat{y}^{k+1}\hat{z}^{k+1}$ for $\ell = 0 \dots k$.

Then we see the dimension of $\widehat{U}(\widehat{K})$ is $(k+2)^3 - 3(k+1) - 1 = k^3 + 6k^2 + 9k + 4$.

Definition 2. Let $\hat{e}$ be a general edge of $\widehat{K}$ and $\hat{f}$ a general face. Let $\hat{p} \in H^{3/2+\delta}(\widehat{K})$ for some $\delta > 0$. We define the following degrees of freedom:

$$\hat{p}\ (\hat{\mathbf{a}}), \qquad \text{for the eight vertices } \hat{\mathbf{a}} \text{ of } \widehat{K}, \tag{5}$$

$$\int_{\hat{e}} \hat{p}\ \hat{q}\ d\hat{s}, \qquad \text{for each edges } \hat{e} \text{ of } \widehat{K}, \quad \hat{q} \in P_{k-1}(\hat{e}), \tag{6}$$

$$\int_{\hat{f}} \hat{p}\ \hat{q}\ d\hat{A}, \qquad \text{for each faces } \hat{f} \text{ of } \widehat{K}, \quad \hat{q} \in Q^*_{k-1,\ k-1}(\hat{f}), \tag{7}$$

$$\int_{\widehat{K}} \hat{p}\ \hat{q}\ d\hat{A}, \qquad \hat{q} \in Q^*_{k-1,\ k-1,\ k-1}(\widehat{K}), \tag{8}$$

where $Q^*_{k-1,\ k-1}(\hat{f})$ is $Q_{k-1,\ k-1}(\hat{f})$ space except constant multiple of the term $\hat{x}^{k-1}\hat{y}^{k-1}$, $\hat{y}^{k-1}\hat{z}^{k-1}$, $\hat{z}^{k-1}\hat{x}^{k-1}$ and $Q^*_{k-1,\ k-1,\ k-1}(\widehat{K})$ is $Q_{k-1,\ k-1,\ k-1}(\widehat{K})$ space except constant multiple of the term $\hat{x}^{k-1}\hat{y}^{k-1}\hat{z}^{\ell}$, $\hat{x}^{k-1}\hat{y}^{\ell}\hat{z}^{k-1}$, $\hat{x}^{\ell}\hat{y}^{k-1}\hat{z}^{k-1}$ and $\hat{x}^{k-1}\hat{y}^{k-1}\hat{z}^{k-1}$ for $\ell = 0 \dots k-2$.

First, we prove that the new element space is $H^1(\Omega)$ conforming.

Theorem 3. *If all the degrees of freedom of a function $\hat{p} \in \widehat{U}(\widehat{K})$ associated with a face $\hat{f}$ of $\widehat{K}$ vanish including vertices and edges of the face then $\hat{p} = 0$ on $\hat{f}$.*

Proof. We use the fact that the vertex degrees of freedom vanish on each edge $\hat{e}$ of $\hat{f}$. For example, on the edge $\hat{x} = \hat{y} = 0$ we have

$$\hat{p} = \hat{z}(1 - \hat{z})r,$$

for some $r \in P_{k-1}(\hat{e})$. Choosing $\hat{q} = r$ in the degrees of freedom (6) for this edge shows that $r = 0$.

Now using the fact that $\hat{p} = 0$ on each edge $\hat{e}$ of $\hat{f}$, which we assume to be the face $\hat{z} = 0$, we have

$$\hat{p} = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})r,$$

for some $r \in Q^*_{k-1,\ k-1}(\hat{f})$. Choosing $\hat{q} = r$ in the degrees of freedom (7) shows that $r = 0$ and hence $\hat{p} = 0$ on $\hat{f}$, as required. ∎

Next, we prove unisolvence of the element. The number of degrees of freedom and the dimension of $\widehat{U}(\widehat{K})$ are both k^3+6k^2+9k+4 and thus it suffices to show the following result.

Theorem 4. *If* $\hat{p} \in \widehat{U}(\widehat{K})$ *and all the degrees of freedom* (5) – (8) *of* $\hat{p}$ *vanish, then* $\hat{p}=0$.

Proof. From the theorem 3, we know that $\hat{p}=0$ on $\partial\widehat{K}$. Hence

$$\hat{p} = \hat{x}(1-\hat{x})\hat{y}(1-\hat{y})\hat{z}(1-\hat{z})r,$$

for some $Q^{*}_{k-1,\ k-1,\ k-1}(\widehat{K})$. Choosing $\hat{q}=r$ in the degrees of freedom (8) proves $r=0$ and hence $\hat{p}=0$, as required. □

The finite element space on a general element K can be obtained by using the diagonal affine map F_K via

$$U(K) = \{\hat{p} \circ F_K^{-1} \mid \hat{p} \in \widehat{U}(\widehat{K})\},$$

in the usual way. Then we have the following space:

$$U_h = \{p \in H^1(\Omega) \mid \ p|_K \in U(K) \text{ for all } K \in \tau_h\}. \tag{9}$$

Using the degrees of freedom (5) – (8) transformed on K, we can define an interpolant

$$\pi_K : H^{\frac{3}{2}+\delta}(K) \to U(K)$$

by requiring the degrees of freedom of $\pi_K p - p$ vanish. Then the global interpolant π_h is defined element-wise by

$$(\pi_h p)|_K = \pi_K(p|_K)$$

for all $K \in \tau_h$.

3. $\boldsymbol{H}(\mathbf{curl}\,,\boldsymbol{\Omega})$ Conforming Finite Element Spaces

In this section we will present the edge elements due to Kim-Kwak[6]. First, we consider the following vectors:

$$\hat{\alpha}_{11} = \{(0,\ \hat{x}^{k+1}\hat{y}^{k}\hat{z}^{\ell},\ 0)\}_{\ell=0}^{k}, \qquad \hat{\alpha}_{12} = \{(\hat{x}^{k}\hat{y}^{k+1}\hat{z}^{\ell},\ 0,\ 0)\}_{\ell=0}^{k},$$
$$\hat{\alpha}_{21} = \{(0,\ 0,\ \hat{x}^{\ell}\hat{y}^{k+1}\hat{z}^{k})\}_{\ell=0}^{k}, \qquad \hat{\alpha}_{22} = \{(0,\ \hat{x}^{\ell}\hat{y}^{k}\hat{z}^{k+1},\ 0)\}_{\ell=0}^{k},$$
$$\hat{\alpha}_{31} = \{(0,\ 0,\ \hat{x}^{k+1}\hat{y}^{\ell}\hat{z}^{k})\}_{\ell=0}^{k}, \qquad \hat{\alpha}_{32} = \{(\hat{x}^{k}\hat{y}^{\ell}\hat{z}^{k+1},\ 0,\ 0)\}_{\ell=0}^{k},$$

$$\begin{aligned}
&\hat{\beta}_1 = \{(\hat{x}^k \hat{y}^{k+1} \hat{z}^\ell,\ \hat{x}^{k+1} \hat{y}^k \hat{z}^\ell,\ 0)\}_{\ell=0}^k,\\
&\hat{\beta}_2 = \{(0,\ \hat{x}^\ell \hat{y}^k \hat{z}^{k+1},\ \hat{x}^\ell \hat{y}^{k+1} \hat{z}^k)\}_{\ell=0}^k,\\
&\hat{\beta}_3 = \{(\hat{x}^k \hat{y}^\ell \hat{z}^{k+1},\ 0,\ \hat{x}^{k+1} \hat{y}^\ell \hat{z}^k)\}_{\ell=0}^k,\\
&\hat{\gamma}_1 = \{(\hat{x}^k \hat{y}^{k+1} \hat{z}^{k+1},\ 0,\ 0)\},\\
&\hat{\gamma}_2 = \{(0,\ \hat{x}^{k+1} \hat{y}^k \hat{z}^{k+1},\ 0)\},\\
&\hat{\gamma}_3 = \{(0,\ 0,\ \hat{x}^{k+1} \hat{y}^{k+1} \hat{z}^k)\},\\
&\hat{\delta} = \{(\hat{x}^k \hat{y}^{k+1} \hat{z}^{k+1},\ \hat{x}^{k+1} \hat{y}^k \hat{z}^{k+1},\ \hat{x}^{k+1} \hat{y}^{k+1} \hat{z}^k)\}.
\end{aligned}$$

Definition 5. Let $k \geq 1$. On the reference element, the curl conforming element is defined as follows:

$$\widehat{\mathbf{V}}(\widehat{K}) = Q_{k,\ k+1,\ k+1}(\widehat{K}) \times Q_{k+1,\ k,\ k+1}(\widehat{K}) \times Q_{k+1,\ k+1,\ k}(\widehat{K}),$$

where the elements $\{\hat{\alpha}_{i,\ j}\}_{j=1,\ 2}$ for $i = 1, 2, 3$ are replaced by the elements $\hat{\beta}_i$, and the three elements $\hat{\gamma}_i$ are replaced by the single element $\hat{\delta}$. In other words, $\hat{\alpha}_{11},\ \hat{\alpha}_{12} \to \hat{\beta}_1,\ \ \hat{\alpha}_{21},\ \hat{\alpha}_{22} \to \hat{\beta}_2,\ \hat{\alpha}_{31},\ \hat{\alpha}_{32} \to \hat{\beta}_3$ and $\hat{\gamma}_1,\ \hat{\gamma}_2,\ \hat{\gamma}_3 \to \hat{\delta}$.

In order to define the degrees of freedom, we introduce two spaces. First, we define $\widehat{\mathbf{\Phi}}_k^*(\hat{x}, \hat{y})$ to be the subspace of $Q_{k-1,\ k}(\hat{x}, \hat{y}) \times Q_{k,\ k-1}(\hat{x}, \hat{y})$ where the two elements $(\hat{x}^{k-1}\hat{y}^k, 0)$ and $(0, \hat{x}^k\hat{y}^{k-1})$ are replaced by the single element $(\hat{x}^{k-1}\hat{y}^k, \hat{x}^k\hat{y}^{k-1})$. For the second space, we consider the following vectors:

$$\begin{aligned}
&\hat{\boldsymbol{\phi}}_{11} = \{(0,\ \hat{x}^{k-1} \hat{y}^k \hat{z}^\ell,\ 0)\}_{\ell=0}^{k-2}, \qquad \hat{\boldsymbol{\phi}}_{12} = \{(\hat{x}^k \hat{y}^{k-1} \hat{z}^\ell,\ 0,\ 0)\}_{\ell=0}^{k-2},\\
&\hat{\boldsymbol{\phi}}_{21} = \{(0,\ 0,\ \hat{x}^\ell \hat{y}^{k-1} \hat{z}^k)\}_{\ell=0}^{k-2}, \qquad \hat{\boldsymbol{\phi}}_{22} = \{(0,\ \hat{x}^\ell \hat{y}^k \hat{z}^{k-1},\ 0)\}_{\ell=0}^{k-2},\\
&\hat{\boldsymbol{\phi}}_{31} = \{(0,\ 0,\ \hat{x}^{k-1} \hat{y}^\ell \hat{z}^k)\}_{\ell=0}^{k-2}, \qquad \hat{\boldsymbol{\phi}}_{32} = \{(\hat{x}^k \hat{y}^\ell \hat{z}^{k-1},\ 0,\ 0)\}_{\ell=0}^{k-2},\\
&\hat{\boldsymbol{\psi}}_1 = \{(\hat{x}^k \hat{y}^{k-1} \hat{z}^\ell,\ \hat{x}^{k-1} \hat{y}^k \hat{z}^\ell,\ 0)\}_{\ell=0}^{k-2},\\
&\hat{\boldsymbol{\psi}}_2 = \{(0,\ \hat{x}^\ell \hat{y}^k \hat{z}^{k-1},\ \hat{x}^\ell \hat{y}^{k-1} \hat{z}^k)\}_{\ell=0}^{k-2},\\
&\hat{\boldsymbol{\psi}}_3 = \{(\hat{x}^k \hat{y}^\ell \hat{z}^{k-1},\ 0,\ \hat{x}^{k-1} \hat{y}^\ell \hat{z}^k)\}_{\ell=0}^{k-2},\\
&\hat{\boldsymbol{\xi}}_1 = \{(\hat{x}^k \hat{y}^{k-1} \hat{z}^{k-1},\ 0,\ 0)\},\\
&\hat{\boldsymbol{\xi}}_2 = \{(0,\ \hat{x}^{k-1} \hat{y}^k \hat{z}^{k-1},\ 0)\},\\
&\hat{\boldsymbol{\xi}}_3 = \{(0,\ 0,\ \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^k)\},\\
&\hat{\boldsymbol{\zeta}} = \{(\hat{x}^k \hat{y}^{k-1} \hat{z}^{k-1},\ \hat{x}^{k-1} \hat{y}^k \hat{z}^{k-1},\ \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^k)\}.
\end{aligned}$$

We now define $\widehat{\mathbf{\Psi}}_k^*(\widehat{K})$ to be the subspace of

$$Q_{k,\ k-1,\ k-1}(\widehat{K}) \times Q_{k-1,\ k,\ k-1}(\widehat{K}) \times Q_{k-1,\ k-1,\ k}(\widehat{K})$$

where for $i = 1, 2, 3$ the elements $\{\hat{\boldsymbol{\phi}}_{ij}\}_{j=1}^2$ are replaced by the elements $\hat{\boldsymbol{\psi}}_i$ and the three elements $\hat{\boldsymbol{\xi}}_i$ are replaced by the single element $\hat{\boldsymbol{\zeta}}$.

Definition 6. Let $\hat{\mathbf{u}} \in H^{/2+\delta}(\widehat{K})$, $\delta > 0$ such that $\widehat{\nabla} \times \hat{\mathbf{u}} \in (L^p(\widehat{K}))^3$ for some $p > 2$. The degrees of freedom are given on edge $\hat{e}$ with unit tangent $\hat{\mathbf{t}}$, on faces $\hat{f}$ with unit normal $\hat{\mathbf{n}}$ and in the interior of $\widehat{K}$ as follows:

$$\int_{\hat{e}} \hat{\mathbf{u}} \cdot \hat{\mathbf{t}}\, \hat{q}\, d\hat{s}, \qquad \text{for each edges } \hat{e} \text{ of } \widehat{K}, \quad \hat{q} \in P_k(\hat{e}), \tag{10}$$

$$\int_{\hat{f}} (\hat{\mathbf{u}} \times \hat{\mathbf{n}}) \cdot\ \hat{\mathbf{q}}\, d\hat{A}, \qquad \text{for each faces } \hat{f} \text{ of } \widehat{K}, \quad \hat{\mathbf{q}} \in \widehat{\mathbf{\Phi}}^*_k(\hat{f}), \tag{11}$$

$$\int_{\widehat{K}} \hat{\mathbf{u}} \cdot \hat{\mathbf{q}}\, d\hat{\mathbf{x}}, \qquad \hat{\mathbf{q}} \in \widehat{\mathbf{\Psi}}^*_k(\widehat{K}). \tag{12}$$

Theorem 7. *A vector function* $\hat{\mathbf{u}} \in \widehat{\mathbf{V}}(\widehat{K})$ *is uniquely determined by the degrees of freedom* (10)−(12)*. And the finite element space* $\widehat{\mathbf{V}}(\widehat{K})$ *is conforming in* $H(\mathrm{curl}\,, \Omega)$*.*

Proof. Since the number of conditions equals the dimension of $\widehat{\mathbf{V}}(\widehat{K})$, it suffices to show that if all the conditions are zero, then $\hat{\mathbf{u}} = 0$. First, we consider the face $\hat{z} = 0$. Then the tangential component of $\hat{\mathbf{u}}$ on this face is $(\hat{u}_1,\ \hat{u}_2) \in Q_{k,\ k+1}(\hat{x},\ \hat{y}) \times Q_{k+1,\ k}(\hat{x},\ \hat{y})$. On each edge of this face, the tangential component is polynomial of degree k. From the degrees of freedom in (10), we see that $\hat{\mathbf{u}} \cdot \hat{\mathbf{t}} = 0$ on each edge. This implies that on this face, we have

$$\hat{u}_1 = \hat{y}(1 - \hat{y})\hat{v}_1, \quad \hat{u}_2 = \hat{x}(1 - \hat{x})\hat{v}_2,$$

where $(\hat{v}_1,\ \hat{v}_2) \in \widehat{\mathbf{\Phi}}^*_k(\hat{x},\ \hat{y})$. Then by choosing $\hat{q}_1 = \hat{v}_2$ and $\hat{q}_2 = -\hat{v}_1$ in the degrees of freedom (11), we see $\hat{v}_1 = \hat{v}_2 = 0$. Hence $\hat{\mathbf{u}} \times \hat{\mathbf{n}} = 0$ on this face. By the same reason, we see that $\hat{\mathbf{u}} \times \hat{\mathbf{n}} = 0$ on all faces. This proves the conformity in $H(\mathrm{curl}\,)$ space. And we have

$$\hat{u}_1 = \hat{y}(1 - \hat{y})\hat{z}(1 - \hat{z})\hat{w}_1,$$

$$\hat{u}_2 = \hat{x}(1 - \hat{x})\hat{z}(1 - \hat{z})\hat{w}_2,$$

$$\hat{u}_3 = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})\hat{w}_3,$$

where $(\hat{w}_1,\ \hat{w}_2,\ \hat{w}_3) \in \widehat{\mathbf{\Psi}}^*_k(\widehat{K})$. Choosing $\hat{\mathbf{q}} = (\hat{w}_1,\ \hat{w}_2,\ \hat{w}_3)$ in degrees of freedom (12), we know that $\hat{\mathbf{u}} = 0$. □

For a generic element K, we define the finite element space on K as

$$\mathbf{V}(K) = \{(B_K^T)^{-1}\hat{\mathbf{v}} \circ F_K^{-1} \mid \hat{\mathbf{v}} \in \widehat{\mathbf{V}}(\widehat{K})\},$$

and set

$$\mathbf{V}_h = \{\mathbf{v} \in H(\mathrm{curl}\,, \Omega) \mid \mathbf{v}|_K \in \mathbf{V}(K) \text{ for all } K \in \tau_h\}. \tag{13}$$

Using the degrees of freedom (10) – (12) transformed on K, we can define the corresponding projection

$$\mathbf{r}_K : \mathbf{H}^{k+1}(K) \to \mathbf{V}(K)$$

for an arbitrary element K. Then the global projection operator $\mathbf{r}_h$ is defined piecewise by

$$(\mathbf{r}_h \mathbf{v})|_K = \mathbf{r}_K(\mathbf{v}|_K)$$

for all $K \in \tau_h$.

4. Analysis for the New Finite Element Spaces

In this section, using interpolation operators π_h and $\mathbf{r}_h$, we show that the scalar space U_h and the curl conforming space $\mathbf{V}_h$ are connected in an intimate way from the following de Rham diagram commutes[7], [8]:

$$\begin{array}{ccc} U & \xrightarrow{\;\nabla\;} & \mathbf{V} \\ {\scriptstyle \pi_h}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{r}_h} \\ U_h & \xrightarrow[\;\nabla\;]{} & \mathbf{V}_h \end{array}$$

Theorem 8. *If U_h is defined by* (9) *and $\mathbf{V}_h$ by* (13)*, then $\nabla U_h \subset \mathbf{V}_h$. In addition, if p is sufficiently smooth such that $\mathbf{r}_h \nabla p$ and $\pi_h p$ are defined, then we have $\nabla \pi_h p = \mathbf{r}_h \nabla p$.*

Proof. Clearly, if $p_h \in U_h$ then we see directly that $\nabla p_h \in \mathbf{V}_h$. Hence $\nabla U_h \subset \mathbf{V}_h$.

To prove the commuting property, we map to the reference element and show that all degrees of freedom (10) – (12) vanish for $\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}$. Then we conclude that $\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p} = 0$.

For the edge degrees of freedom (10), if $\hat{\boldsymbol{\tau}}$ is tangent to $\hat{e} = [\hat{\mathbf{a}}, \hat{\mathbf{b}}]$ and $\hat{q} \in P_k(\hat{e})$ then using (10) and integration by parts we have

$$\begin{aligned} &\int_{\hat{e}} (\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}) \cdot \hat{\boldsymbol{\tau}} \hat{q} \; d\hat{s} \\ &= \int_{\hat{e}} (\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \widehat{\nabla} \hat{p}) \cdot \hat{\boldsymbol{\tau}} \hat{q} \; d\hat{s} \end{aligned}$$

$$= \int_{\hat{e}} \frac{\partial}{\partial \hat{s}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \hat{q} \; d\hat{s}$$
$$= (\pi_{\widehat{K}} \hat{p} - \hat{p})(\hat{\mathbf{b}}) - (\pi_{\widehat{K}} \hat{p} - \hat{p})(\hat{\mathbf{a}}) - \int_{\hat{e}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \; \frac{\partial \hat{q}}{\partial \hat{s}} \; d\hat{s}.$$

Since $\frac{\partial \hat{q}}{\partial \hat{s}} \in P_{k-1}(\hat{e})$ and using the vertex interpolation property and the degrees of freedom (6) for $\pi_{\widehat{K}}$, we conclude that the right-hand side above vanishes.

For the face degrees of freedom, we use the degrees of freedom in (11) together with the divergence theorem in the plane containing $\hat{f}$ to show that if $\hat{\mathbf{q}} \in \widehat{\mathbf{\Phi}}_k^*(\hat{f})$ then

$$\int_{\hat{f}} (\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}) \times \hat{\mathbf{n}} \cdot \; \hat{\mathbf{q}} \; d\hat{A}$$
$$= \int_{\hat{f}} \widehat{\nabla}_{\hat{f}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \cdot \; \hat{\mathbf{q}} \; d\hat{A}$$
$$= \int_{\partial \hat{f}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \hat{\mathbf{n}}_{\hat{f}} \cdot \; \hat{\mathbf{q}} \; d\hat{s} - \int_{\hat{f}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \widehat{\nabla}_{\hat{f}} \cdot \; \hat{\mathbf{q}} \; d\hat{A},$$

where $\hat{\mathbf{n}}_{\hat{f}}$ is the outward normal to $\hat{f}$. Since $\hat{\mathbf{n}}_{\hat{f}} \cdot \; \hat{\mathbf{q}} \in P_{k-1}(\hat{e})$ and $\widehat{\nabla}_{\hat{f}} \cdot \; \hat{\mathbf{q}} \in Q^*_{k-1,\, k-1}(\hat{f})$, so that right-hand side vanishes using the edge and face degrees of freedom (6) and (7) for $\pi_{\widehat{K}}$. We have thus proved that the face degrees of freedom (11) for $\widehat{\nabla} \pi_{\widehat{K}} \hat{p}$ and $\mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}$ agree.

Finally, for the volume degrees of freedom, we use the degrees of freedom in (12) together with the integral identity to show that if $\hat{\mathbf{q}} \in \widehat{\mathbf{\Psi}}_k^*(\widehat{K})$ then

$$\int_{\widehat{K}} (\widehat{\nabla} \pi_{\widehat{K}} \hat{p} - \mathbf{r}_{\widehat{K}} \widehat{\nabla} \hat{p}) \cdot \hat{\mathbf{q}} \; d\hat{\mathbf{x}}$$
$$= \int_{\widehat{K}} \widehat{\nabla} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \cdot \hat{\mathbf{q}} \; d\hat{\mathbf{x}}$$
$$= \int_{\partial \widehat{K}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \hat{\mathbf{q}} \cdot \hat{\mathbf{n}} \; d\hat{A} - \int_{\widehat{K}} (\pi_{\widehat{K}} \hat{p} - \hat{p}) \widehat{\nabla} \cdot \hat{\mathbf{q}} \; d\hat{\mathbf{x}}.$$

Since $\hat{\mathbf{q}} \cdot \hat{\mathbf{n}} \in Q^*_{k-1,\, k-1}(\hat{f})$ for each face $\hat{f}$ and $\widehat{\nabla} \cdot \hat{\mathbf{q}} \in Q^*_{k-1,\, k-1,\, k-1}(\widehat{K})$, so the right-hand side vanishes, using the face and volume degrees of freedom for $\pi_{\widehat{K}}$. This completes the proof. $\square$

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