NEW $H^1(\Omega)$ CONFORMING FINITE ELEMENTS ON HEXAHEDRA

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Abstract: In this paper, we introduce new scalar finite element spaces on hexahedron. We prove the unisolvence of degrees of freedom and analyze our spaces using the discrete de Rham diagram.

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1. Introduction

Suppose $\Omega$ is a bounded domain with two disjoint connected boundaries $\Gamma$ and $\Sigma$. We seek to compute the time-harmonic electric field $E$ corresponding to a given current density $F$ by solving the time-harmonic electric field equation subject to the perfect conducting boundary condition and the impedance boundary condition as follows:

\[
\nabla \times (\mu_r^{-1} \nabla \times E) - \kappa^2 \varepsilon_r E = F, \quad \text{in } \Omega, \quad (1)
\]

\[
\mathbf{n} \times E = 0, \quad \text{on } \Gamma, \quad (2)
\]

\[
\mu_r^{-1} (\nabla \times E) \times \mathbf{n} - i \kappa \lambda E_T = g, \quad \text{on } \Sigma, \quad (3)
\]

where $E_T = (\mathbf{n} \times E|_{\Sigma}) \times \mathbf{n}$ and $g$ is a given tangential vector field on $\Sigma$. Using
the Galerkin method, we can find a variational formulation. Taking the dot product of (1) by the complex conjugate of a smooth vector function \( \phi \) and integrating over \( \Omega \), and then using the integration by parts we obtain

\[
\int_\Omega \left[ (\mu_r^{-1} \nabla \times \mathbf{E}) \cdot \nabla \times \bar{\phi} - \kappa^2 (\varepsilon_r \mathbf{E}) \cdot \bar{\phi} \right] dV - i\kappa \int_\Sigma \lambda \mathbf{E}_T \cdot \bar{\phi}_T dA
\]

\[
= \int_\Omega \mathbf{F} \cdot \bar{\phi} dV + \int_\Sigma \mathbf{g} \cdot \bar{\phi}_T dA,
\]

(4)

where \( \bar{\phi}_T = (\mathbf{n} \times \phi) \times \mathbf{n} \) on \( \Omega \). Under appropriate assumptions on the coefficients and domain, problem (4) has a unique solution[1], [2]. In performing the analysis of this problem, the \( H(\text{curl}, \Omega) \) conforming finite element space and the \( H^1(\Omega) \) conforming finite element space play an important role. In this paper, we introduce a new family of scalar finite elements and show that the relevant function spaces are related by the famous de Rham diagram[3], [4].

2. New \( H^1(\Omega) \) Conforming Finite Element Spaces

In this section, we will describe a new family of scalar finite elements on parallelepipeds with edges parallel to the coordinate axes. First, we generate a regular finite element mesh \( \tau_h = \{ K \} \) covering the domain \( \Omega \) such that

1. \( \overline{\Omega} = \bigcup_{K \in \tau_h} \overline{K} \), where \( \overline{\Omega} \) denotes the closure of \( \Omega \);
2. for each \( K \in \tau_h \), \( K \) is an open set with positive volume;
3. if \( K_1 \) and \( K_2 \) are distinct elements in \( \tau_h \) then \( K_1 \cap K_2 = \emptyset \);
4. each \( K \in \tau_h \) is a Lipschitz domain.

For each element \( K \), we define the parameters \( h_K \) and \( \rho_K \) such that

\[
h_K = \text{diameter of the smallest sphere containing } K,
\]

\[
\rho_K = \text{diameter of the largest sphere contained in } K,
\]

then \( h = \max_{K \in \tau_h} h_K \) so that the index \( h \) denote the maximum diameter of the elements \( K \in \tau_h \).

Any \( K \in \tau_h \) can be obtained by mapping the reference element \( \hat{K} \), typically unit cube, using an affine map \( F_K : \hat{K} \to K \) such that \( F_K(\hat{K}) = K \) and \( F_K(\hat{x}) = B_K \hat{x} + b_K \), where \( B_K \) is a nonsingular \( 3 \times 3 \) matrix and \( b_K \) is a vector.

Because of the simplicity of the mapping, although we define the elements on the reference domain \( \hat{K} \), the same definition can be used on a target element \( K \) in the mesh[5]. In order to define finite elements on parallelepipeds, we need the following polynomial space:

\[
Q_{\ell,m,n}(\hat{K}) = \{ \text{polynomials of maximum degree } \ell \text{ in } \hat{x}, \ m \text{ in } \hat{y} \text{ and } n \text{ in } \hat{z} \}.
\]
**Definition 1.** Let \( k \geq 1 \). On the reference element, the gradient conforming element is defined as follows:

\[
\hat{U}(\hat{K}) = Q_{k+1, k+1, k+1}(\hat{K}),
\]

where \( Q_{k+1, k+1, k+1}(\hat{K}) \) is \( Q_{k+1, k+1, k+1}(\hat{K}) \) space except constant multiple of the term \( \hat{x}^{k+1} \hat{y}^{k+1} \hat{z}^\ell \), \( \hat{x}^{k+1} \hat{y}^\ell \hat{z}^{k+1} \), \( \hat{x}^\ell \hat{y}^{k+1} \hat{z}^{k+1} \) and \( \hat{x}^{k+1} \hat{y}^{k+1} \hat{z}^{k+1} \) for \( \ell = 0 \ldots k \).

Then we see the dimension of \( \hat{U}(\hat{K}) \) is \( (k+2)^3 - 3(k+1) - 1 = k^3 + 6k^2 + 9k + 4 \).

**Definition 2.** Let \( \hat{e} \) be a general edge of \( \hat{K} \) and \( \hat{f} \) a general face. Let \( \hat{p} \in H^{3/2+\delta}(\hat{K}) \) for some \( \delta > 0 \). We define the following degrees of freedom:

\[
\hat{p}(\hat{a}), \quad \text{for the eight vertices } \hat{a} \text{ of } \hat{K}, \quad (5)
\]

\[
\int_{\hat{e}} \hat{p} \hat{q} \, d\hat{s}, \quad \text{for each edges } \hat{e} \text{ of } \hat{K}, \quad \hat{q} \in P_{k-1}(\hat{e}), \quad (6)
\]

\[
\int_{\hat{f}} \hat{p} \hat{q} \, d\hat{A}, \quad \text{for each faces } \hat{f} \text{ of } \hat{K}, \quad \hat{q} \in Q^*_{k-1, k-1}(\hat{f}), \quad (7)
\]

\[
\int_{\hat{K}} \hat{p} \hat{q} \, d\hat{A}, \quad \hat{q} \in Q^*_{k-1, k-1, k-1}(\hat{K}), \quad (8)
\]

where \( Q^*_{k-1, k-1, k-1}(\hat{f}) \) is \( Q_{k-1, k-1}(\hat{f}) \) space except constant multiple of the term \( \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1} \), \( \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1}, \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1}, \hat{z}^{k-1} \hat{x}^{k-1} \hat{z}^{k-1} \), \( \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1}, \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1}, \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1} \), \( \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1} \) and \( \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^{k-1} \) for \( \ell = 0 \ldots k - 2 \).

First, we prove that the new element space is \( H^1(\Omega) \) conforming.

**Theorem 3.** If all the degrees of freedom of a function \( \hat{p} \in \hat{U}(\hat{K}) \) associated with a face \( \hat{f} \) of \( \hat{K} \) vanish including vertices and edges of the face then \( \hat{p} = 0 \) on \( \hat{f} \).

**Proof.** We use the fact that the vertex degrees of freedom vanish on each edge \( \hat{e} \) of \( \hat{f} \). For example, on the edge \( \hat{x} = \hat{y} = 0 \) we have

\[
\hat{p} = \hat{z}(1 - \hat{z})r,
\]

for some \( r \in P_{k-1}(\hat{e}) \). Choosing \( \hat{q} = r \) in the degrees of freedom (6) for this edge shows that \( r = 0 \).

Now using the fact that \( \hat{p} = 0 \) on each edge \( \hat{e} \) of \( \hat{f} \), which we assume to be the face \( \hat{z} = 0 \), we have

\[
\hat{p} = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})r,
\]

for some \( r \in Q^*_{k-1, k-1}(\hat{f}) \). Choosing \( \hat{q} = r \) in the degrees of freedom (7) shows that \( r = 0 \) and hence \( \hat{p} = 0 \) on \( \hat{f} \), as required. \( \Box \)
Next, we prove unisolvence of the element. The number of degrees of freedom and the dimension of $\hat{U}(\hat{K})$ are both $k^3 + 6k^2 + 9k + 4$ and thus it suffices to show the following result.

**Theorem 4.** If $\hat{p} \in \hat{U}(\hat{K})$ and all the degrees of freedom $(5)-(8)$ of $\hat{p}$ vanish, then $\hat{p} = 0$.

**Proof.** From the theorem 3, we know that $\hat{p} = 0$ on $\partial\hat{K}$. Hence

$$\hat{p} = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})\hat{z}(1 - \hat{z})r,$$

for some $Q^*_{k-1, k-1, k-1}(\hat{K})$. Choosing $\hat{q} = r$ in the degrees of freedom (8) proves $r = 0$ and hence $\hat{p} = 0$, as required.

The finite element space on a general element $K$ can be obtained by using the diagonal affine map $F_K$ via

$$U(K) = \{ \hat{p} \circ F_{K}^{-1} | \hat{p} \in \hat{U}(\hat{K}) \},$$

in the usual way. Then we have the following space:

$$U_h = \{ p \in H^1(\Omega) | p|_{K} \in U(K) \text{ for all } K \in \tau_h \}. \quad (9)$$

Using the degrees of freedom $(5)-(8)$ transformed on $K$, we can define an interpolant

$$\pi_K : H^{1/2+\delta}(K) \rightarrow U(K)$$

by requiring the degrees of freedom of $\pi_K p - p$ vanish. Then the global interpolant $\pi_h$ is defined element-wise by

$$(\pi_h p)|_K = \pi_K(p|_K)$$

for all $K \in \tau_h$.

### 3. $H(\text{curl}, \Omega)$ Conforming Finite Element Spaces

In this section we will present the edge elements due to Kim-Kwak[6]. First, we consider the following vectors:

$$\hat{\alpha}_{11} = \{(0, \hat{x}^{k+1}\hat{y}\hat{z}\ell, 0)\}_{\ell=0}^k, \quad \hat{\alpha}_{12} = \{(\hat{x}^k\hat{y}^{k+1}\hat{z}\ell, 0, 0)\}_{\ell=0}^k, \quad \hat{\alpha}_{13} = \{(0, \hat{x}^{k+1}\hat{y}\ell\hat{z}\ell, 0)\}_{\ell=0}^k,$$

$$\hat{\alpha}_{21} = \{(0, \hat{x}\hat{y}^{k+1}\hat{z}\ell, 0)\}_{\ell=0}^k, \quad \hat{\alpha}_{22} = \{(0, \hat{x}^k\hat{y}\hat{z}^{k+1}, 0)\}_{\ell=0}^k, \quad \hat{\alpha}_{23} = \{(\hat{x}^k\hat{y}\ell\hat{z}^{k+1}, 0, 0)\}_{\ell=0}^k,$$

$$\hat{\alpha}_{31} = \{(0, \hat{x}^{k+1}\hat{y}\ell\hat{z}\ell, 0)\}_{\ell=0}^k, \quad \hat{\alpha}_{32} = \{(\hat{x}^k\hat{y}\hat{z}^{k+1}, 0, 0)\}_{\ell=0}^k,$$
Definition 5. Let \( k \geq 1 \). On the reference element, the curl conforming element is defined as follows:

\[
\hat{\mathbf{V}}(\hat{K}) = Q_{k-1, \hat{k}+1} \times Q_{k-1, \hat{k}+1}(\hat{K}) \times Q_{k+1, \hat{k}+1} \times Q_{k+1, \hat{k}+1}(\hat{K}),
\]

where the elements \( \{\hat{\alpha}_i, \hat{\gamma}_i\}_{i=1}^3 \) are replaced by the elements \( \hat{\beta}_i \), and the three elements \( \hat{\gamma}_i \) are replaced by the single element \( \hat{\delta} \). In other words, \( \hat{\alpha}_{11}, \hat{\alpha}_{12} \to \hat{\beta}_1 \), \( \hat{\alpha}_{21}, \hat{\alpha}_{22} \to \hat{\beta}_2 \), \( \hat{\alpha}_{31}, \hat{\alpha}_{32} \to \hat{\beta}_3 \) and \( \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3 \to \hat{\delta} \).

In order to define the degrees of freedom, we introduce two spaces. First, we define \( \hat{\mathbf{\Phi}}_k(\hat{x}, \hat{y}) \) to be the subspace of \( Q_{k-1, \hat{k}}(\hat{x}, \hat{y}) \times Q_{k-1, \hat{k}}(\hat{x}, \hat{y}) \) where the two elements \( (\hat{x}^{k-1}y^k, 0) \) and \( (0, \hat{x}^k y^{k-1}) \) are replaced by the single element \( (\hat{x}^{k-1}y^k, \hat{x}^k y^{k-1}) \). For the second space, we consider the following vectors:

\[
\begin{align*}
\hat{\phi}_{11} &= \{(0, \hat{x}^{k-1}y^k z^\ell, 0)\}_{\ell=0}^{k-2}, & \hat{\phi}_{12} &= \{(\hat{x}^k y^{k-1} z^\ell, 0, 0)\}_{\ell=0}^{k-2}, \\
\hat{\phi}_{21} &= \{(0, 0, \hat{x}^k y^{k-1} z^\ell)\}_{\ell=0}^{k-2}, & \hat{\phi}_{22} &= \{(0, \hat{x}^k y^{k-1} z^\ell, 0)\}_{\ell=0}^{k-2}, \\
\hat{\phi}_{31} &= \{(0, 0, \hat{x}^{k-1} y^k z^\ell)\}_{\ell=0}^{k-2}, & \hat{\phi}_{32} &= \{(\hat{x}^k y^{k-1} z^\ell, 0, 0)\}_{\ell=0}^{k-2}, \\
\hat{\psi}_1 &= \{(\hat{x}^k y^{k-1} z^\ell, \hat{x}^{k-1} y^k z^\ell, 0)\}_{\ell=0}^{k-2}, & \hat{\psi}_2 &= \{(0, \hat{x}^k y^{k-1} z^\ell, \hat{x}^k y^{k-1} z^\ell)\}_{\ell=0}^{k-2}, \\
\hat{\psi}_3 &= \{(\hat{x}^k y^{k-1} z^\ell, 0, \hat{x}^{k-1} y^k z^\ell)\}_{\ell=0}^{k-2}, & \hat{\xi}_1 &= \{(\hat{x}^k y^{k-1} z^\ell, 0, 0)\}, \\
\hat{\xi}_2 &= \{(0, \hat{x}^{k-1} y^k z^\ell, 0)\}, & \hat{\xi}_3 &= \{(0, 0, \hat{x}^{k-1} y^k z^\ell)\}, \\
\hat{\zeta} &= \{(\hat{x}^k y^{k-1} z^\ell, \hat{x}^{k-1} y^k z^\ell, \hat{x}^{k-1} y^k z^\ell)\}.
\end{align*}
\]

We now define \( \hat{\mathbf{\Phi}}_k^*(\hat{K}) \) to be the subspace of

\[
Q_{k-1, \hat{k}} \times Q_{k-1, \hat{k}} \times Q_{k-1, \hat{k}} \times Q_{k+1, \hat{k}} \times Q_{k+1, \hat{k}}.
\]

where for \( i = 1, 2, 3 \) the elements \( \{\phi_{ij}\}_{i=1}^3 \) are replaced by the elements \( \psi_i \) and the three elements \( \xi_i \) are replaced by the single element \( \zeta \).
**Definition 6.** Let \( \hat{\mathbf{u}} \in H^{2+\delta}(\hat{K}), \delta > 0 \) such that \( \hat{\nabla} \times \hat{\mathbf{u}} \in (L^p(\hat{K}))^3 \) for some \( p > 2 \). The degrees of freedom are given on edge \( \hat{e} \) with unit tangent \( \hat{t} \), on faces \( \hat{f} \) with unit normal \( \hat{n} \) and in the interior of \( \hat{K} \) as follows:

\[
\int_{\hat{e}} \hat{\mathbf{u}} \cdot \hat{t} \, \hat{q} \, d\hat{s}, \quad \text{for each edges } \hat{e} \text{ of } \hat{K}, \quad \hat{q} \in P_k(\hat{e}),
\]

(10)

\[
\int_{\hat{f}} (\hat{\mathbf{u}} \times \hat{\mathbf{n}}) \cdot \hat{q} \, d\hat{A}, \quad \text{for each faces } \hat{f} \text{ of } \hat{K}, \quad \hat{q} \in \hat{\Phi}_k^*(\hat{f}),
\]

(11)

\[
\int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{q} \, d\hat{x}, \quad \hat{q} \in \hat{\Psi}_k^*(\hat{K}).
\]

(12)

**Theorem 7.** A vector function \( \hat{\mathbf{u}} \in \hat{\mathcal{V}}(\hat{K}) \) is uniquely determined by the degrees of freedom (10)–(12). And the finite element space \( \hat{\mathcal{V}}(\hat{K}) \) is conforming in \( H(\text{curl}, \Omega) \).

**Proof.** Since the number of conditions equals the dimension of \( \hat{\mathcal{V}}(\hat{K}) \), it suffices to show that if all the conditions are zero, then \( \hat{\mathbf{u}} = 0 \). First, we consider the face \( \hat{z} = 0 \). Then the tangential component of \( \hat{\mathbf{u}} \) on this face is \( (\hat{u}_1, \hat{u}_2) \in Q_{k+1}(\hat{x}, \hat{y}) \times Q_{k+1}(\hat{x}, \hat{y}) \). On each edge of this face, the tangential component is polynomial of degree \( k \). From the degrees of freedom in (10), we see that \( \hat{\mathbf{u}} \cdot \hat{t} = 0 \) on each edge. This implies that on this face, we have

\[
\hat{u}_1 = \hat{y}(1 - \hat{y})\hat{v}_1, \quad \hat{u}_2 = \hat{x}(1 - \hat{x})\hat{v}_2,
\]

where \( (\hat{v}_1, \hat{v}_2) \in \hat{\Phi}_k^*(\hat{x}, \hat{y}) \). Then by choosing \( \hat{q}_1 = \hat{v}_2 \) and \( \hat{q}_2 = -\hat{v}_1 \) in the degrees of freedom (11), we see \( \hat{v}_1 = \hat{v}_2 = 0 \). Hence \( \hat{\mathbf{u}} \times \hat{\mathbf{n}} = 0 \) on this face. By the same reason, we see that \( \hat{\mathbf{u}} \times \hat{\mathbf{n}} = 0 \) on all faces. This proves the conformity in \( H(\text{curl}) \) space. And we have

\[
\hat{u}_1 = \hat{y}(1 - \hat{y})\hat{z}(1 - \hat{z})\hat{w}_1,
\]

\[
\hat{u}_2 = \hat{x}(1 - \hat{x})\hat{z}(1 - \hat{z})\hat{w}_2,
\]

\[
\hat{u}_3 = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})\hat{w}_3,
\]

where \( (\hat{w}_1, \hat{w}_2, \hat{w}_3) \in \hat{\Psi}_k^*(\hat{K}) \). Choosing \( \hat{q} = (\hat{w}_1, \hat{w}_2, \hat{w}_3) \) in degrees of freedom (12), we know that \( \hat{\mathbf{u}} = 0 \). \( \square \)

For a generic element \( K \), we define the finite element space on \( K \) as

\[
\mathcal{V}(K) = \{(B_K^T)^{-1}\hat{v} \circ F_K^{-1} \mid \hat{v} \in \hat{\mathcal{V}}(\hat{K})\},
\]
and set
\[ \mathbf{V}_h = \{ \mathbf{v} \in H(\text{curl}, \Omega) \mid \mathbf{v}|_K \in \mathbf{V}(K) \text{ for all } K \in \tau_h \}. \]  
(13)

Using the degrees of freedom (10) – (12) transformed on \( K \), we can define the corresponding projection
\[ \mathbf{r}_K : \mathbf{H}^{k+1}(K) \rightarrow \mathbf{V}(K) \]
for an arbitrary element \( K \). Then the global projection operator \( \mathbf{r}_h \) is defined piecewise by
\[ (\mathbf{r}_h \mathbf{v})|_K = \mathbf{r}_K (\mathbf{v}|_K) \]
for all \( K \in \tau_h \).

4. Analysis for the New Finite Element Spaces

In this section, using interpolation operators \( \pi_h \) and \( \mathbf{r}_h \), we show that the scalar space \( U_h \) and the curl conforming space \( \mathbf{V}_h \) are connected in an intimate way from the following de Rham diagram commutes[7], [8]:

\[ \begin{array}{ccc}
U & \xrightarrow{\nabla} & \mathbf{V} \\
\pi_h \downarrow & & \mathbf{r}_h \\
U_h & \xrightarrow{\nabla} & \mathbf{V}_h 
\end{array} \]

**Theorem 8.** If \( U_h \) is defined by (9) and \( \mathbf{V}_h \) by (13), then \( \nabla U_h \subset \mathbf{V}_h \). In addition, if \( p \) is sufficiently smooth such that \( \mathbf{r}_h \nabla p \) and \( \pi_h p \) are defined, then we have \( \nabla \pi_h p = \mathbf{r}_h \nabla p \).

**Proof.** Clearly, if \( p_h \in U_h \) then we see directly that \( \nabla p_h \in \mathbf{V}_h \). Hence \( \nabla U_h \subset \mathbf{V}_h \).

To prove the commuting property, we map to the reference element and show that all degrees of freedom (10) – (12) vanish for \( \mathbf{r}_h \nabla \hat{p} - \pi_h \nabla \hat{p} \). Then we conclude that \( \mathbf{r}_h \nabla \hat{p} - \pi_h \nabla \hat{p} = 0 \).

For the edge degrees of freedom (10), if \( \hat{\tau} \) is tangent to \( \hat{e} = [\hat{a}, \hat{b}] \) and \( \hat{q} \in P_k(\hat{e}) \) then using (10) and integration by parts we have
\[
\int_{\hat{e}} (\mathbf{r}_h \nabla \hat{p} - \pi_h \nabla \hat{p}) \cdot \hat{\tau} \hat{q} \, d\hat{s} = \int_{\hat{e}} (\nabla \hat{p} - \hat{\nabla} \hat{p}) \cdot \hat{\tau} \hat{q} \, d\hat{s}
\]
\[
\begin{align*}
&= \int_{\hat{e}} \frac{\partial}{\partial s}(\pi_{\hat{K}}\hat{p} - \hat{p}) \hat{q} \, d\hat{s} \\
&= (\pi_{\hat{K}}\hat{p} - \hat{p})(\hat{b}) - (\pi_{\hat{K}}\hat{p} - \hat{p})(\hat{a}) - \int_{\hat{e}} (\pi_{\hat{K}}\hat{p} - \hat{p}) \frac{\partial \hat{q}}{\partial \hat{s}} \, d\hat{s}.
\end{align*}
\]

Since \( \frac{\partial \hat{q}}{\partial \hat{s}} \in P_{k-1}(\hat{e}) \) and using the vertex interpolation property and the degrees of freedom (6) for \( \pi_{\hat{K}} \), we conclude that the right-hand side above vanishes.

For the face degrees of freedom, we use the degrees of freedom in (11) together with the divergence theorem in the plane containing \( \hat{f} \) to show that if \( \hat{q} \in \hat{\Phi}^+_k(\hat{f}) \) then

\[
\begin{align*}
&= \int_{\hat{f}} (\hat{\nabla}_{\pi_{\hat{K}}}\hat{p} - r_{\hat{K}}\hat{\nabla}\hat{p}) \times \hat{n} \cdot \hat{q} \, d\hat{A} \\
&= \int_{\hat{f}} \hat{\nabla}_{\hat{f}}(\pi_{\hat{K}}\hat{p} - \hat{p}) \cdot \hat{q} \, d\hat{A} \\
&= \int_{\partial f} (\pi_{\hat{K}}\hat{p} - \hat{p})\hat{n}_f \cdot \hat{q} \, d\hat{s} - \int_{\hat{f}} (\pi_{\hat{K}}\hat{p} - \hat{p})\hat{\nabla}_{\hat{f}} \cdot \hat{q} \, d\hat{A},
\end{align*}
\]

where \( \hat{n}_f \) is the outward normal to \( \hat{f} \). Since \( \hat{n}_f \cdot \hat{q} \in P_{k-1}(\hat{e}) \) and \( \hat{\nabla}_{\hat{f}} \cdot \hat{q} \in Q^*_{k-1, k-1}(\hat{f}) \), so that right-hand side vanishes using the edge and face degrees of freedom (6) and (7) for \( \pi_{\hat{K}} \). We have thus proved that the face degrees of freedom (11) for \( \hat{\nabla}_{\pi_{\hat{K}}}\hat{p} \) and \( r_{\hat{K}}\hat{\nabla}\hat{p} \) agree.

Finally, for the volume degrees of freedom, we use the degrees of freedom in (12) together with the integral identity to show that if \( \hat{q} \in \hat{\Psi}^+_k(\hat{K}) \) then

\[
\begin{align*}
&= \int_{\hat{K}} (\hat{\nabla}_{\pi_{\hat{K}}}\hat{p} - r_{\hat{K}}\hat{\nabla}\hat{p}) \cdot \hat{q} \, d\hat{x} \\
&= \int_{\hat{K}} \hat{\nabla}(\pi_{\hat{K}}\hat{p} - \hat{p}) \cdot \hat{q} \, d\hat{x} \\
&= \int_{\partial\hat{K}} (\pi_{\hat{K}}\hat{p} - \hat{p})\hat{q} \cdot \hat{n} \, d\hat{A} - \int_{\hat{K}} (\pi_{\hat{K}}\hat{p} - \hat{p})\hat{\nabla} \cdot \hat{q} \, d\hat{x}.
\end{align*}
\]

Since \( \hat{q} \cdot \hat{n} \in Q^*_{k-1, k-1}(\hat{f}) \) for each face \( \hat{f} \) and \( \hat{\nabla} \cdot \hat{q} \in Q^*_{k-1, k-1, k-1}(\hat{K}) \), so the right-hand side vanishes, using the face and volume degrees of freedom for \( \pi_{\hat{K}} \). This completes the proof. \( \square \)

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