FIXED POINTS FOR COMPATIBLE MAPPINGS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we proved the common fixed point result for compatible mappings in multiplicative metric spaces.

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1. Introduction and Preliminaries

It is well known that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [1] introduced the concept of multiplicative metric spaces as follows:
Definition 1.1. Let $X$ be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping $d$ together with $X$, that is, $(X, d)$ is a multiplicative metric space.

Example 1.2. ([3]) Let $\mathbb{R}^n_+$ be the collection of all $n$-tuples of positive real numbers. Let $d^* : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left| \frac{x_1}{y_1} \right|^* \left| \frac{x_2}{y_2} \right|^* \cdots \left| \frac{x_n}{y_n} \right|^*,$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $| \cdot |^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}^n_+, d^*)$ is a multiplicative metric space.

One can refer to [3] for detailed a multiplicative metric topology.

Definition 1.3. Let $(X, d)$ be a multiplicative metric space. Then a sequence $\{x_n\}$ in $X$ said to be

(1) a multiplicative convergent to $x$ if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}, \epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a multiplicative Cauchy sequence if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

(3) We call a multiplicative metric space complete if every multiplicative Cauchy sequence convergent to $x \in X$.

In 2012, Özavsar and Çevikel [3] gave the concept of multiplicative contractive mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.

Definition 1.4. Let $f$ be a mapping of a multiplicative metric space $(X, d)$ into itself. Then $f$ is said to be a multiplicative contraction if there exists a real number $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq \lambda d(x, y) \quad \text{for all } x, y \in X.$$
In 2015, Kang et al. [2] introduced the notion of compatible mappings as follows:

**Definition 1.5.** Let $f$ and $g$ be mappings of a multiplicative metric space $(X, d)$ Then $f$ and $g$ are called **compatible** if

$$\lim_{n \to \infty} d(fgx_n, gfx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$.

Now we give some properties related to compatible mappings and its variants in a multiplicative metric space, see [2].

**Proposition 1.6.** Let $f$ and $g$ be compatible mappings of a multiplicative metric space $(X, d)$ into itself. If $ft = gt$ for some $t \in X$, then $fgt = fgt = ft = gt$.

**Proposition 1.7.** Let $f$ and $g$ be compatible mappings of a multiplicative metric space $(X, d)$ itself. Suppose that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$. Then

(i) $\lim_{n \to \infty} gfx_n = ft$ if $f$ is continuous at $t$.

(ii) $\lim_{n \to \infty} gfx_n = gt$ if $g$ is continuous at $t$.

(iii) $fgt = gft$ and $ft = gt$ if $f$ and $g$ are continuous at $t$.

**2. Main Results**

Now we give the following theorem for compatible mappings.

**Theorem 2.1.** Let $A, B, S$ and $T$ be mappings of a complete multiplicative metric space $(X, d)$ satisfying the following conditions

(C1) $SX \subset BX, \quad TX \subset AX$;

(C2) $d(Sx, Ty) \leq M^\lambda(x, y)$
for each $x, y \in X$ and $\lambda \in (0, 1/2)$, where

$$M(x, y) = \max \left\{ d(Ax, Sx), d(By, Ty), d(By, Ax), (d(Ax, Ty) \cdot d(By, Sx))^1/2, \frac{d(Ax, Sx) \cdot d(By, Ty)}{d(Ax, By)} \cdot \frac{d(Ax, Ty) \cdot d(By, Sx)}{d(Ax, By)}, \frac{d(Ax, Ty) \cdot d(By, Sx)}{d(Sx, Ty)} \right\}.$$  

(C3) one of $A, B, S$ and $T$ is continuous.

Assume that the pairs $A, S$ and $B, T$ are compatible. Then $A, B, S$ and $T$ have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $SX \subseteq BX$ and $TX \subseteq AX$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$ and for this point $x_1$, there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way, we can construct a sequence $\{y_n\}$ such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}; \quad y_{2n} = Sx_{2n} = Bx_{2n+1}.$$  

From (C2) by putting $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \leq M^\lambda(x_{2n}, x_{2n+1}),$$

where

$$M(x_{2n}, x_{2n+1}) = \max \left\{ d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, Ax_{2n}), (d(Ax_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n}))^{1/2}, \frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(Ax_{2n}, Bx_{2n+1})}, \frac{d(Ax_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n})}{d(Ax_{2n}, Bx_{2n+1})}, \frac{d(Ax_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n})}{d(Sx_{2n}, Tx_{2n+1})} \right\}$$

$$= \max \left\{ d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1}), (d(y_{2n}, y_{2n-1}) \cdot d(y_{2n+1}, y_{2n-1}))^{1/2}, \frac{d(y_{2n}, y_{2n-1}) \cdot d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n-1})} \right\},$$
Thus to some point $z$ follows from Proposition 1.7 that
\[
\frac{d(y_{2n}, y_{2n}) \cdot d(y_{2n+1}, y_{2n-1})}{d(y_{2n}, y_{2n-1})}, \frac{d(y_{2n+1}, y_{2n-1})}{d(y_{2n+1}, y_{2n})}
\]
\[
= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n})\}.
\]

Now if $d(y_{2n+1}, y_{2n}) > d(y_{2n-1}, y_{2n})$, then $M(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n})$. Then we have,
\[
d(y_{2n}, y_{2n+1}) \leq d^\lambda(y_{2n}, y_{2n+1}),
\]
which is a contradiction as $\lambda \in (0, 1/2)$. So, $d(y_{2n}, y_{2n-1}) \geq d(y_{2n+1}, y_{2n})$, which implies that $M(x_{2n}, x_{2n+1}) = d(y_{2n-1}, y_{2n})$ and hence
\[
d(y_{2n}, y_{2n+1}) \leq d^\lambda(y_{2n-1}, y_{2n}).
\]

Similarly, we have
\[
d(y_{2n-1}, y_{2n}) \leq d^\lambda(y_{2n-2}, y_{2n-1}).
\]

Hence, in general we get
\[
d(y_n, y_{n+1}) \leq d^\lambda(y_{n-1}, y_n)
\leq d^{2^2}(y_{n-2}, y_{n-1})
\leq \cdots \leq d^{\lambda^n}(y_0, y_1).
\]

Let $m, n \in \mathbb{N}$ with $m > n$. Then
\[
d(y_m, y_n) \leq d(y_m, y_{m-1}) \cdot d(y_{m-1}, y_{m-2}) \cdots d(y_{n+1}, y_n)
\leq d^{\lambda^{m-1}}(y_0, y_1) \cdot d^{\lambda^{m-2}}(y_0, y_1) \cdots d^\lambda(y_0, y_1)
\leq \frac{\lambda^n}{1-\lambda}(y_0, y_1).
\]

Taking $n \to \infty$, we have $d(y_m, y_n) \leq 1$ and hence $\lim_{n \to \infty} d(y_m, y_n) = 1$. Thus $\{y_n\}$ is a multiplicative Cauchy sequence in $X$ and hence it converges to some point $z \in X$. Consequently, the subsequence $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ and $\{Sx_{2n}\}$ of the sequence $\{y_n\}$ also converges to $z$.

Now suppose that $A$ is continuous. Since $A$ and $S$ are compatible on $X$, it follows from Proposition 1.7 that
\[
AAx_{2n} \to Az, \quad SAx_{2n} \to Az \quad \text{as } n \to \infty.
\]

Now putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in $(C_2)$, we have
\[
d(SAx_{2n}, Tx_{2n+1}) \leq M^\lambda(Ax_{2n}, x_{2n+1}),
\]
where
\[ M(Ax_{2n}, x_{2n+1}) = \max \left\{ d(AAx_{2n}, SaAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, AAx_{2n}), \right. \]
\[ \left. (d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SaAx_{2n}))^{1/2}, \right\} \]

\[ \min \left\{ \frac{d(AAx_{2n}, SaAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(AAx_{2n}, Bx_{2n+1})}, \right. \]
\[ \left. \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SaAx_{2n})}{d(AAx_{2n}, Bx_{2n+1})}, \right\} \]

\[ \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SaAx_{2n})}{d(SaAx_{2n}, Tx_{2n+1})} \} \}

Letting \( n \to \infty \), we have
\[ \lim_{n \to \infty} M(Ax_{2n}, x_{2n+1}) = \max \left\{ d(Az, Az), d(z, z), d(z, Az), (d(Az, z) \cdot d(z, Az))^{1/2}, \right. \]
\[ \left. \min \left\{ \frac{d(Az, Az) \cdot d(z, z)}{d(Az, z)}, d(z, Az), d(Az, z) \right\} \right\} \]
\[ = d(z, Az). \]

Hence
\[ d(z, Az) \leq d^\lambda(z, Az), \]

which a contradiction, we get \( z = Az \).

Next putting \( x = z \) and \( y = x_{2n+1} \) in (\( C_2 \)), we have
\[ d(Sz, Tx_{2n+1}) \leq M^\lambda(z, x_{2n+1}), \]

where
\[ M(z, x_{2n+1}) = \max \left\{ d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, Az), \right. \]
\[ \left. (d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz))^{1/2}, \right\} \]
\[ \min \left\{ \frac{d(Az, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(Az, Bx_{2n+1})}, \right. \]
\[ \left. \frac{d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz)}{d(Az, Bx_{2n+1})}, \right\} \]
Letting \( n \to \infty \), we have

\[
\lim_{n \to \infty} M(z, x_{2n+1}) = \max\{d(z, Sz), d(z, z), d(z, z) \cdot d(z, Sz)\}^{1/2},
\]

\[
\min\{d(z, Sz), d(z, Sz), d(z, z)\}
\]

\[= d(Sz, z). \]

Then we have

\[d(Sz, z) \leq d^{\lambda}(Sz, z),\]

which a contradiction, we get \( Sz = z \). Since \( SX \subset BX \), there exists a point \( u \in X \) such that \( z = Sz = Bu \).

Again putting \( x = z \) and \( y = u \) in \((C_2)\), we have

\[d(z, Tu) = d(Sz, Tu) \leq M^\lambda(z, u),\]

where

\[
M(z, u) = \max \left\{d(Az, Sz), d(Bu, Tu), d(Bu, Az), (d(Az, Tu) \cdot d(Bu, Sz))^{1/2}, \min \left\{ \frac{d(Az, Sz) \cdot d(Bu, Tu)}{d(Az, Bu)}, \frac{d(Az, Tu)}{d(Az, Bu)}, \frac{d(Bu, Sz)}{d(Sz, Tu)} \right\} \right\}
\]

\[= \max\{1, d(z, Tu), 1, d^{1/2}(z, Tu), 1\}
\]

\[= d(z, Tu). \]

Then

\[d(z, Tu) \leq d^{\lambda}(z, Tu),\]

which a contradiction. This implies that \( z = Tu \). Since \( B \) and \( T \) are compatible on \( X \) and \( Bu = Tu = z \), by Proposition 1.6, \( BTu = TBu \) and hence \( Bz = BTu = TBu = Tz \).

Also, we have

\[d(z, Bz) = d(Sz, Tz) \leq M^\lambda(z, z),\]
where

\[ M(z, z) = \max \left\{ d(Az, Sz), d(Bz, Tz), d(Bz, Az), \right. \]
\[ \left. (d(Az, Tz) \cdot d(Bz, Sz))^{1/2}, \min \left\{ \frac{d(Az, Sz) \cdot d(Bz, Tz)}{d(Az, Bz)}, \frac{d(Az, Tz) \cdot d(Bz, Sz)}{d(Sz, Tz)} \right\} \right\} \]
\[ = \max \left\{ 1, 1, d(Bz, z), d(Bz, z), \min \left\{ \frac{1}{d(z, Bz)}, d(Bz, z), d(Bz, z) \right\} \right\} \]
\[ = d(Bz, z). \]

This implies that \( z = Bz \). Hence, \( z = Bz = Tz = Az = Sz \). Therefore, \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Similarly, we can also complete the proof when \( B \) is continuous.

Next suppose that \( S \) is continuous. Since \( A \) and \( S \) are compatible on \( X \), it follows that

\[ S^2 x_{2n} \to Sz, \quad AS x_{2n} \to Sz \quad \text{as} \quad n \to \infty. \]

Now putting \( x = Sx_{2n} \) and \( y = x_{2n+1} \) in \( (C_2) \)

\[ d(SS x_{2n}, Tx_{2n+1}) \leq M^\lambda(Sx_{2n}, x_{2n+1}), \]

where

\[ M(Sx_{2n}, x_{2n+1}) \]
\[ = \max \left\{ d(AS x_{2n}, SS x_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, AS x_{2n}), \right. \]
\[ \left. (d(AS x_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SS x_{2n}))^{1/2}, \min \left\{ \frac{d(AS x_{2n}, SS x_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(AS x_{2n}, Bx_{2n+1})}, \frac{d(AS x_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SS x_{2n})}{d(AS x_{2n}, Bx_{2n+1})}, \frac{d(AS x_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SS x_{2n})}{d(SS x_{2n}, Tx_{2n+1})} \right\} \right\}. \]
Letting $n \to \infty$, we get
\[
\lim_{n \to \infty} M(Sx_{2n}, x_{2n+1}) = \max\left\{ d(Sz, Sz), d(z, z), d(z, Sz), d(z, Sz), \right. \\
\left. \min\left\{ \frac{1}{d(z, Sz)}, d(z, Sz), d(z, Sz) \right\} \right\} \\
= d(z, Sz).
\]
This implies that $Sz = z$. Since $SX \subset BX$, there exists a point $v \in X$ such that $z = Sz = Bv$.

Now putting $x = Sx_{2n}$ and $y = v$ in $(C_2)$
\[
d(SSx_{2n}, T v) \leq M^\lambda(Sx_{2n}, v),
\]
where
\[
M(Sx_{2n}, v) \\
= \max\left\{ d(ASx_{2n}, SSx_{2n}), d(Bv, T v), d(Bv, ASx_{2n}), \right. \\
\left. d(ASx_{2n}, T v) \cdot d(Bv, SSx_{2n}) \right\}, \\
\frac{d(ASx_{2n}, T v) \cdot d(Bv, SSx_{2n})}{d(ASx_{2n}, Bv)}, \min\left\{ \frac{d(ASx_{2n}, SSx_{2n}) \cdot d(Bv, T v)}{d(ASx_{2n}, Bv)}, \right. \\
\left. \frac{d(ASx_{2n}, T v) \cdot d(Bv, SSx_{2n})}{d(SSx_{2n}, T v)} \right\} \right\}.
\]
Letting $n \to \infty$, we get
\[
\lim_{n \to \infty} M(Sx_{2n}, v) \\
= \max\{1, d(z, T v), 1, d^{1/2}(z, T v), \min\{d(z, T v), d(z, T v), 1\}\} \\
= d(z, T v).
\]
This implies that $u = T v$. Since $B$ and $T$ are compatible on $X$ and $Bv = T v = z$, $BT v = TB v$ and hence $Bz = BT v = TB v = T z$.

Now putting $x = x_{2n}$ and $y = z$ in $(C_2)$
\[
d(Sx_{2n}, T z) \leq M^\lambda(x_{2n}, z),
\]
where
\[ M(x_{2n}, z) \]
\[ = \max \left\{ d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Bu, Ax_{2n}), \right. \]
\[ \left. \frac{(d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n}))^{1/2}, \min \left\{ \frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Tz)}{d(Ax_{2n}, Bz)}, \right. \right. \]
\[ \left. \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Sx_{2n}, Tz)} \right\} \right\}. \]

Letting \( n \to \infty \), we get
\[ \lim_{n \to \infty} M(x_{2n}, z) \]
\[ = \max \left\{ 1, 1, d(Tz, z), d(Tz, z), \min \left\{ \frac{1}{d(z, Tz)}, d(Tz, z), d(Tz, z) \right\} \right\} \]
\[ = d(Tz, z). \]

This implies that \( Tz = z \). Since \( TX \subset AX \), there exists a point \( w \in X \) such that \( z = Tz = Aw \).

Now putting \( x = w \) and \( y = z \) in \((C_2)\)
\[ d(Sw, Tz) \leq M^\lambda(w, z), \]
where
\[ M(w, z) \]
\[ = \max \left\{ d(Aw, Sw), d(Bz, Tz), d(Bz, Aw), \right. \]
\[ \left. \frac{(d(Aw, Tz) \cdot d(Bz, Sw))^{1/2}, \min \left\{ \frac{d(Aw, Sw) \cdot d(Bz, Tz)}{d(Aw, Bz)}, \right. \right. \]
\[ \left. \frac{d(Aw, Tz) \cdot d(Bz, Sw)}{d(Sw, Tz)} \right\} \right\} \]
\[ = \max \left\{ d(z, Sw), 1, 1, d^{1/2}(z, Sw), \min \left\{ d(z, Sw), d(z, Sw), 1 \right\} \right\} \]
\[ = d(z, Sw). \]

This implies that \( Sw = z \). Since \( A \) and \( S \) are compatible on \( X \) and \( Sw = Aw = z \), \( ASw = SAw \) and hence \( Az = ASw = SAw = Sz \). That is, \( z = Az = Sz = Bz = Tz \). Therefore, \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Similarly, we can complete the proof when \( T \) is continuous.
Finally, suppose that $z$ and $w$ are two common fixed points of $A, B, S$ and $T$.

Now putting $x = z$ and $y = w$ in $(C_2)$

$$d(z, w) = d(Sz, Tw) \leq M^\lambda(z, w),$$

where

$$M(z, w) = \max \left\{ \frac{d(Az, Sz)}{d(Az, Bw)} \cdot \frac{d(Bw, Sz)}{d(Az, Tw)} \cdot \frac{(d(Az, Tw) \cdot d(Bw, Sz))^{1/2}}{d(Az, Bw)}, \frac{d(Az, Tw) \cdot d(Bw, Sz)}{d(Sz, Tw)} \right\}$$

$$= \max \left\{ 1, 1, d(z, w), d(z, w), \min \left\{ \frac{1}{d(z, w)}, d(z, w), d(z, w) \right\} \right\} = d(z, w),$$

which implies that $z = w$. Therefore, $A, B, S$ and $T$ have a unique common fixed point in $X$. This completes the proof. \qed

References


