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SOME FIXED POINT THEOREMS ON THE SUM AND PRODUCT OF OPERATORS IN TENSOR PRODUCT SPACES

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Abstract: Let X and Y be Banach spaces and P and Q be two subsets of X and Y respectively. Let $T_1 : X \otimes_\gamma Y \to X$ and $T_2 : X \otimes_\gamma Y \to Y$ be two mappings and S be a self mapping on $P\otimes Q$. Using T_1 and T_2 we define a self mapping T on $X\otimes_\gamma Y$. Different conditions under which $T+TS+S$ has a fixed point in $P\otimes Q$ are established here. Analogous results are also established taking the pair (T_1, T_2) as $(k, k^/)$ contraction mappings. Again considering $X \otimes_\gamma Y$ as a reflexive Banach space. We derive the conditions for $\frac{1}{m}(T + ST + S)$, $m > 2$, $m \in N$, for having a fixed point in $P \otimes Q$. Some iteration schemes converging to a fixed point of $T + ST + S$ in $P \otimes Q$ are also presented here.

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1. Introduction

In 1955, Krasnoselskii [13], proved a fixed point theorem for a sum of two mappings T and S on a non-empty closed convex bounded subset of a Banach space X. This theorem was generalized and extended by different researchers, viz., Cain and Nashed [4], Nashed and Wong [14], Edmunds [9], Reinermann [16], Sehgal and Sing [18], Vijayaraju [19], etc., considering S and T as different types

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of mappings. Taking Browder's fixed point theorem, in 1982, Olga Hadzic ([10], [11]) derived some fixed point theorems for $T+S$ on a Banach space X. In 2003, Dhage [5], discussed two fixed point theorems for the sum and product of two operators. In this paper, we consider two Banach spaces X and Y and there projective tensor product $X \otimes_\gamma Y$. We consider a pair of mappings $T_1 : X \otimes_\gamma Y \to X$ and $T_2 : X \otimes_\gamma Y \to Y$, and from this pair we construct a self mapping T on $X \otimes_\gamma Y$. Let P and Q be two subsets of X and Y respectively and S be a self mapping on $P \otimes Q$. We derive some fixed point theorems for $T+TS+S$ in the subset $P \otimes Q$ of $X \otimes_\gamma Y$. Some iteration schemes converging to this fixed point will also be discussed here.

2. Preliminaries

Definition 2.1. Let X and Y be normed spaces. A mapping $T : X \to Y$ is called non-expansive if and only if $\|Tx - Ty\| \leqslant \|x - y\|$, $\forall x, y \in X$. It is said to be demi-closed if its graph is sequentially closed in the product of the weak topology on X with the norm topology on Y (refer. to ([15])).

A mapping $T : X \to Y$ is called contraction if and only if

$$\|Tx - Ty\| \leqslant r\|x - y\|,$$

where r is real number with $0 \leqslant r < 1$, $\forall x, y \in X$.

Definition 2.2. (see [3]) Let X be a Banach space and f be a continuous (not necessarily linear) mapping of X into itself. The mapping f is said to be completely continuous if the image under f of each bounded set of X is contained in a compact set.

Theorem 2.3. (see [3]) *Let f be a completely continuous self mapping on a Banach space X. If for some positive integer m, $f^m(X)$ is bounded, then f has a fixed point.*

Theorem 2.4. (Schauder's Fixed-Point Theorem, see [17]) *Let K be a nonempty, convex and compact subset of a normed space. Any continuous mapping $T : K \to K$ has at least one fixed point.*

Theorem 2.5. (Banach's Contraction Mapping Principle, see [1]) *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $T : X \to X$ be a mapping such that for each $x, y \in X$, $d(Tx, Ty) \leqslant cd(x, y)$. Then T has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} T_n x = a$.*

Projective Tensor Product 2.7. (see [2]) *Given normed spaces X and Y, the projective tensor norm γ on $X \otimes_\gamma Y$ is defined by*

$$\|u\| = \inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$$

where the infimum is taken over all (finite) representations of u.

Lemma 2.8. (see [2]) *$X \otimes_\gamma Y$ can be represented as a linear subset of $BL(X^*, Y^*; F)$ consisting of all elements of the form $u = \sum_i x_i \otimes y_i$ where $\sum_i \|x_i\|\|y_i\| < \infty$. Moreover, $\|u\| = \inf\{\sum_i \|x_i\|\|y_i\|\}$ over all such representations of u.*

Definition 2.9 Let P and Q be bounded subsets of the Banach spaces X and Y respectively. A pair of mappings $T_1 : P \otimes Q \to P$ and $T_2 : P \otimes Q \to Q$ is called $(k, k^/)$ contraction mappings if:

(i) $\|T_1u - T_1v\| \leqslant \dfrac{k}{M_2}\|u - v\|, \ \dfrac{k}{M_2} < 1,$

(ii) $\|T_2u - T_2v\| \leqslant \dfrac{k^/}{M_1}\|u - v\|, \ \dfrac{k^/}{M_1} < 1,$

(iii) $\|T_1u\| \leqslant M_1$, $\|T_2u\| \leqslant M_2 \ \forall\, u, v \in P \otimes Q,$

where $P \otimes Q$ is bounded by M_1M_2.

3. Main Results

First, we consider P and Q as bounded subsets of the Banach spaces X and Y respectively.

Theorem 3.1. *Let P and Q be as defined above and let $T_1 : X \otimes_\gamma Y \to X$ and $T_2 : X \otimes_\gamma Y \to Y$ be two continuous mappings such that $T_1(P \otimes Q) \subseteq P$ and $T_2(P \otimes Q) \subseteq Q$. We define $T : X \otimes_\gamma Y \to X \otimes_\gamma Y$ by $T(u) = T_1(u) \otimes T_2(u)$, $u \in X \otimes_\gamma Y$. Let S be a completely continuous additive self mapping on $P \otimes Q$ such that S and T commute on $P \otimes Q$. Suppose for every $\sum_i p_i \otimes q_i$ in $P \otimes Q$, there exists one and only one solution $(\sum_i p_i \otimes q_i)^0$ in $P \otimes Q$ of the equation*

$$\sum_i a_i \otimes b_i = T(\sum_i a_i \otimes b_i) + T(\sum_i p_i \otimes q_i) + \sum_i p_i \otimes q_i \tag{1}$$

where $\sum_i a_i \otimes b_i \in X \otimes_\gamma Y$

Then $(T + TS + S)$ has a fixed point in $P \otimes Q$.

Proof. We define $J : P \otimes Q \to P \otimes Q$ by $J(\sum_i p_i \otimes q_i) = (\sum_i p_i \otimes q_i)^0$. First we show that J is continuous. Let $\{\sum_i p_{i_n} \otimes q_{i_n}\}_n$ be a sequence in $P \otimes Q$ such that

$$\sum_i p_{i_n} \otimes q_{i_n} \to \sum_i p_i \otimes q_i \qquad \text{as} \qquad n \to \infty\ . \text{ Now,}$$

$$\begin{aligned} J(\sum_i p_{i_n} \otimes q_{i_n}) &= T(\sum_i p_{i_n} \otimes q_{i_n})^0 + T(\sum_i p_{i_n} \otimes q_{i_n}) \\ &\quad + \sum_i p_{i_n} \otimes q_{i_n} \qquad [by\ (1)] \\ \lim_{n\to\infty} J(\sum_i p_{i_n} \otimes q_{i_n}) &= T(\lim_{n\to\infty}(\sum_i p_{i_n} \otimes q_{i_n})^0) + T(\lim_{n\to\infty}\sum_i p_{i_n} \otimes q_{i_n}) \\ &\quad + \sum_i p_i \otimes q_i \\ &\qquad [T \text{ is continuous as } T_1 \text{ and } T_2 \text{ are continuous }] \\ &= T(\lim_{n\to\infty} J(\sum_i p_{i_n} \otimes q_{i_n})) + T(\sum_i p_i \otimes q_i) + \sum_i p_i \otimes q_i. \end{aligned}$$

So, $\lim_{n\to\infty} J(\sum_i p_{i_n} \otimes q_{i_n})$ is a solution of the equation (1) . Therefore,

$$\lim_{n\to\infty} J(\sum_i p_{i_n} \otimes q_{i_n}) = (\sum_i p_i \otimes q_i)^0 = J(\sum_i p_i \otimes q_i)$$

showing that J is continuous.

For $u \in P \otimes Q$, $Ju = u^0 = T(u^0) + T(u) + u$, using (1). Now,

$$S(Ju) = S(u^0) = S(T(u^0) + T(u) + u) = T(S(u^0)) + T(S(u)) + S(u),$$

showing that $S(u^0)$ is a solution of the equation (1) for $S(u)$ in $P \otimes Q$. Hence

$$S(u^0) = (S(u))^0 \ i.e.\ S(Ju) = J(Su).$$

Thus S and J commute. Now we define

$$K : P \otimes Q \to P \otimes Q$$

by $K(u) = S(u^0) = S(Ju)$, for $u \in P \otimes Q$.

Since J is continuous and S is completely continuous so the mapping K is completely continuous. Again, since P and Q are bounded subsets, so, $P \otimes Q$ is bounded subset of $X \otimes_\gamma Y$. Now, $K^n(P \otimes Q) = S^n J^n(P \otimes Q)$ is bounded for

$n \in \mathbb{N}$. So, applying Theorem 2.3, we get K has a fixed point, say α in $P \otimes Q$. Therefore,

$$\begin{aligned}\alpha = K(\alpha) = SJ(\alpha) = JS(\alpha) = (S(\alpha))^0 &= T((S(\alpha))^0) + T(S(\alpha)) + S(\alpha) \\ &= T(\alpha) + T(S(\alpha)) + S(\alpha)\end{aligned}$$

Thus, α is a fixed point for $T + TS + S$ in $P \otimes Q$. □

In Theorem 3.1, instead of taking the bounded subsets P and Q, if we take the original space $X \otimes_\gamma Y$, the theorem will be as follows:

Theorem 3.2. *Let $T_1 : X \otimes_\gamma Y \to X$ and $T_2 : X \otimes_\gamma Y \to Y$ be two continuous mappings and $T : X \otimes_\gamma Y \to X \otimes_\gamma Y$ be defined by $T(u) = T_1(u) \otimes T_2(u)$, $u \in X \otimes_\gamma Y$. Let S be a completely continuous additive self mapping on $X \otimes_\gamma Y$ such that $ST = TS$ and for some $m > 1$, $S^m(X \otimes_\gamma Y)$ is bounded. Suppose for every $\sum_i x_i \otimes y_i$ in $X \otimes_\gamma Y$, there exists exactly one solution $(\sum_i x_i \otimes y_i)^0$ in $X \otimes_\gamma Y$, of the equation*

$$\sum_i a_i \otimes b_i = T(\sum_i a_i \otimes b_i) + T(\sum_i x_i \otimes y_i) + \sum_i x_i \otimes y_i, \qquad \sum_i a_i \otimes b_i \in X \otimes_\gamma Y.$$

Then $(T + TS + S)$ has a fixed point in $X \otimes_\gamma Y$.

Next, we consider (T_1, T_2) as a pair of (k, k') contraction mappings on $P \otimes Q$.

Lemma 3.3. (Refer to [6]) *If the pair (T_1, T_2) is as defined above (i.e.,(k, k') contraction), then the mapping $T : P \otimes Q \to P \otimes Q$ defined by $T(u) = T_1(u) \otimes T_2(u)$, $u \in P \otimes Q$ has a unique fixed point if $(k + k') < 1$.*

Now, we give an example of the above lemma:

Example 3.4. Let $D_{l^1 \otimes_\gamma \mathbb{K}}$ be a subset of $l^1 \otimes_\gamma \mathbb{K}$ bounded by a constant c. We define $T_1 : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1}$ by $T_1(\sum_i a_i \otimes x_i) = \frac{1}{2c} \sum_i \{a_{i_n} x_i\}_n$, where $a_i = \{a_{i_n}\}_n$ and $T_2 : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{\mathbb{K}}$ by $T_2(\sum_i a_i \otimes x_i) = \frac{1}{3} \sum_i \|a_i\|.|x_i|$. ($D_{l^1}$ and $D_{\mathbb{K}}$ are bounded subsets of l^1 and $\mathbb{K}$ respectively).

Then (T_1, T_2) is a pair of $(k, k^/)$ contraction mappings with $(k+k^/) < 1$. So the mapping $T : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ defined by $T(\sum_i a_i \otimes x_i) = \dfrac{1}{6c} \sum\limits_i \{M a_{i_n} x_i\}_n$, where $M = \|a_i\|.|x_i|$ has a unique fixed point in $D_{l^1 \otimes_\gamma \mathbb{K}}$.

Corollary 3.5. *Let (T_1, T_2) be a pair of $(k, k^/)$ contraction mappings and T be as defined in the Lemma 3.3. Let S be a completely continuous additive self mapping on $P \otimes Q$ such that S commutes with T. If $(k + k^/) < 1$, then $(T + TS + S)$ has a fixed point in $P \otimes Q$.*

Example 3.6. Let $D_{l^1 \otimes_\gamma \mathbb{K}}$ be a subset of $l^1 \otimes_\gamma \mathbb{K}$ bounded by a constant c. We define T_1 and T_2 as in the Example 3.4, and then we get the pair (T_1, T_2) as $(k, k^/)$ contraction mappings with $(k + k^/) < 1$. Now, $T : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ is defined by $T(\sum_i a_i \otimes x_i) = \dfrac{1}{6c} \sum\limits_i \{M a_{i_n} x_i\}_n$, where $M = \|a_i\|.|x_i|$.

Let $S : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ be defined by $S(\sum_i a_i \otimes x_i) = \sum_i \{\dfrac{a_{i_n} x_i}{n}\}_n$, where $a_i = \{a_{i_n}\}_n$.

To show that S is compact:

Let $S_m : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ be defined by

$$S_m(\sum_i a_i \otimes x_i) = \sum_i \{a_{i_1} x_i, \frac{a_{i_2} x_i}{2}, \frac{a_{i_3} x_i}{3}, ..., \frac{a_{i_m} x_i}{m}, 0, 0, 0, ...\}$$

Then each S_m is linear, bounded and compact. Also,

$$\begin{aligned} \|(S_m - S)(\sum_i a_i \otimes x_i)\| &= \|\sum_i \{0, 0, ..., 0, \frac{a_{i_{m+1}} x_i}{m+1}, \frac{a_{i_{m+2}} x_i}{m+2}, ...\}\| \\ &\leqslant \sum_i \sum_{j=m+1}^{\infty} \frac{1}{j} |a_{ij}|.|x_i| < \frac{1}{m+1} \sum_i \sum_{j=m+1}^{\infty} |a_{ij}|.|x_i| \\ &\leqslant \frac{1}{m+1} \sum_i \sum_{j=1}^{\infty} |a_{ij}|.|x_i| = \frac{1}{m+1} \sum_i \|a_i\|.|x_i| \end{aligned}$$

So, taking the projective tensor norm,

$$\|(S_m - S)(\sum_i a_i \otimes x_i)\| < \frac{1}{m+1} \|\sum_i a_i \otimes x_i\|$$

Therefore, $S_m \to S$ and so, S is compact.

Now, S is additive and clearly, S and T commutes. So, by Corollary 3.5, $(T + TS + S)$ has a fixed point in $D_{l^1 \otimes_\gamma \mathbb{K}}$.

Now we take P and Q as non-empty compact subsets.

Theorem 3.7. *Let P and Q be non empty compact subsets of the Banach spaces X and Y respectively. Let (T_1, T_2) be a pair of $(k, k^/)$ contraction mappings with $(k + k^/) < 1$, and the mapping T be as defined earlier. Let S be a continuous self mapping on $P \otimes Q$ such that $Tu + TS\alpha + S\alpha \in P \otimes Q$ for all $\alpha, u \in P \otimes Q$. Then $(T + TS + S)$ has fixed point in $P \otimes Q$.*

Proof. We fix an arbitrary point α in $P \otimes Q$ and define $J_\alpha : P \otimes Q \to P \otimes Q$ by $J_\alpha(u) = Tu + TS\alpha + S\alpha,\ u \in P \otimes Q$. Then for $u_1, u_2 \in P \otimes Q$

$$\begin{aligned} \|J_\alpha(u_1) - J_\alpha(u_2)\| &= \|(Tu_1 + TS\alpha + S\alpha) - (Tu_2 + TS\alpha + S\alpha)\| \\ &= \|Tu_1 - Tu_2\| \leqslant (k + k^/)\|u_1 - u_2\| \end{aligned}$$

So, J_α is a contraction on $P \otimes Q$ as $(k + k^/) < 1$, and has a unique fixed point, say $\hat{\alpha}$ in $P \otimes Q$. So, $\hat{\alpha} = J_\alpha(\hat{\alpha})$. Now we define

$$K : P \otimes Q \to P \otimes Q$$

by $K(\alpha) = \hat{\alpha},\ \alpha \in P \otimes Q$. For $\alpha, \beta \in P \otimes Q$,

$$\begin{aligned} \|K(\alpha) - K(\beta)\| &= \|\hat{\alpha} - \hat{\beta}\| \leqslant (k + k^/)\|\hat{\alpha} - \hat{\beta}\| + (k + k^/)\|S\alpha - S\beta\| \\ &\quad + \|S\alpha - S\beta\| \\ &\Rightarrow \|\hat{\alpha} - \hat{\beta}\| \leqslant \frac{1 + (k + k^/)}{1 - (k + k^/)}\|S\alpha - S\beta\| \end{aligned}$$

Since S is continuous, K is also continuous. So by Schauder's Theorem 2.4, K has a fixed point, say α, on $P \otimes Q$. Thus,

$$\alpha = K(\alpha) = \hat{\alpha} = T(\alpha) + TS(\alpha) + S(\alpha)$$

showing the result. □

Theorem 3.8. *Let P and Q be two non empty closed subsets of the Banach spaces X and Y respectively. Let $X \otimes_\gamma Y$ be reflexive and (T_1, T_2) be a pair of $(k, k^/)$ contraction mappings on $P \otimes Q$ with $(k + k^/) = 1$. Let S be a linear continuous self mapping on $P \otimes Q$ such that (i) $\|S\| \leqslant 1$ and (ii) $I - \frac{1}{m}(T + ST + S),\ m > 2,\ m \in \mathbb{N}$ is demiclosed. Then $\frac{1}{m}(T + ST + S)$ has a fixed point in $P \otimes Q$.*

Proof. Let $\{\alpha_n\}_n$ be a sequence in $(0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0.$$

For $n \in \mathbb{N}$, we define $f_n : P \otimes Q \to P \otimes Q$ by

$$f_n(u) = \frac{1}{m}(1 - \alpha_n)(Tu + STu + Su) + \alpha_n u_0, \text{ where } u_0 \in P \otimes Q.$$

For $u, v \in P \otimes Q$,

$$\begin{aligned}\|f_n(u) - f_n(v)\| &\leqslant \frac{1}{m}(1 - \alpha_n)[\|Tu - Tv\| + \|STu - STv\| + \|Su - Sv\|] \\ &\leqslant \frac{1}{m}(1 - \alpha_n)[\|u - v\| + \|S\|\|u - v\| + \|S\|\|u - v\|] \\ &\leqslant \frac{3}{m}(1 - \alpha_n)\|u - v\|, \text{ (T being nonexpansive)}\end{aligned}$$

showing that f_n is a contraction for $m > 2$. Now, $P \otimes Q$ being complete by Banach's contraction mapping principle, f_n has a unique fixed point, say p_n in $P \otimes Q$. Now,

$$\begin{aligned}\|p_n\| &\leqslant \frac{1}{m}(1 - \alpha_n)[\|Tp_n\| + \|S\|\|Tp_n\| + \|S\|\|p_n\|] + \alpha_n\|u_0\| \\ &\leqslant \frac{1}{m}(1 - \alpha_n)[2(\|Tp_n - Tu_0\| + \|Tu_0\|) + \|p_n\|] + \alpha_n\|u_0\| \\ &\leqslant \frac{1}{m}(1 - \alpha_n)[2(\|p_n\| + \|u_0\| + \|Tu_0\|) + \|p_n\|] + \alpha_n\|u_0\| \\ &= \frac{1}{m}(1 - \alpha_n)[3\|p_n\| + 2(\|u_0\| + \|Tu_0\|)] + \alpha_n\|u_0\|\end{aligned}$$

Dividing both sides by $\|p_n\|$, we get

$$1 \leqslant \frac{1}{m}(1 - \alpha_n)\left[3 + 2(\frac{\|u_0\|}{\|p_n\|} + \frac{\|Tu_0\|}{\|p_n\|})\right] + \alpha_n\frac{\|u_0\|}{\|p_n\|} \tag{2}$$

If $\{p_n\}_n$ is an unbounded sequence in $P \otimes Q$, then taking $n \to \infty$, from (2), we get

$$1 \leqslant \frac{3}{m}(1 - \alpha_n) \Rightarrow \frac{m}{3} \leqslant 1 - \alpha_n < 1 \Rightarrow m < 3,$$

a contradiction. So, $\{p_n\}_n$ must be bounded. Now, T and S being continuous, are bounded, and so $\{Tp_n\}_n$ and $\{Sp_n\}_n$ are bounded. Now,

$$\|p_n - \frac{1}{m}(Tp_n + STp_n + Sp_n)\| \leqslant \frac{\alpha_n}{m}\left[\|Tp_n\| + \|STp_n\| + \|Sp_n\|\right] + \alpha_n\|u_0\|$$

$$\to 0 \; as \; n \to \infty$$

Since $X \otimes_\gamma Y$ is reflexive, taking a subsequence if necessary, we may assume that $\{p_n\}_n$ has a weak limit (refer to [15]), say p. As $I - \frac{1}{m}(T + ST + S)$ is demiclosed, so, we get

$$p - \frac{1}{m}(T + ST + S)(p) = 0$$

showing that p is a fixed point for $\frac{1}{m}(T + ST + S)$, $m > 2$. □

Corollary 3.9. *Let $X \otimes_\gamma Y$ be reflexive and (T_1, T_2) be a pair of $(k, k^/)$ contraction mappings on $P \otimes Q$ with $(k + k^/) \leqslant 1$.*

(i) Let P and Q be two non-empty bounded compact subsets of X and Y respectively. If S is a non-expansive self mapping on $P \otimes Q$, such that $I - (T + ST + S)$ is demiclosed then $(T + ST + S)$ has a fixed point in $P \otimes Q$.

Proof. Defining $\{\alpha_n\}_n$ as in Theorem 3.8, we define $f_n : P \otimes Q \to P \otimes Q$ by

$$f_n(u) = (1 - \alpha_n)(Tu + STu + Su) + \alpha_n u_0, \text{ where } u_0 \in P \otimes Q.$$

Then f_n is a continuous mapping on $P \otimes Q$ and so, using Theorem 2.4, f_n has a fixed point say p_n in $P \otimes Q$. Since $P \otimes Q$ is bounded, so $\{p_n\}_n$ is bounded, and thus $\{Tp_n\}_n$ and $\{Sp_n\}_n$ are bounded. So, proceeding as in theorem 3.8, we get $T + ST + S$ has a fixed point in $P \otimes Q$. □

(ii) Let P and Q be two non-empty closed and bounded subsets of X and Y respectively. If S is a contraction self mapping with $r \in \left(0, \frac{1}{3}\right]$ on $P \otimes Q$ such that $I - (T + ST + S)$ is demiclosed and $(k + k^/) \in \left(0, \frac{1}{2}\right]$ then $(T + ST + S)$ has unique fixed point on $P \otimes Q$.

Proof. Defining $\{\alpha_n\}_n$ and f_n as above we get, for $u, v \in P \otimes Q$,

$$\begin{aligned}
\|f_n(u) - f_n(v)\| &\leqslant (1 - \alpha_n)[\|Tu - Tv\| + \|STu - STv\| + \|Su - Sv\|] \\
&\leqslant (1 - \alpha_n)[(k + k^/)\|u - v\| + r(k + k^/)\|u - v\| + r\|u - v\|] \\
&\leqslant (1 - \alpha_n)[(k + k^/) + r(k + k^/) + r]\|u - v\| \\
&\leqslant (1 - \alpha_n)\|u - v\|; \qquad max[(k + k^/) + r(k + k^/) + r] \leqslant 1
\end{aligned}$$

Then f_n is a contraction mapping on $P \otimes Q$ and so, using Theorem 2.5, f_n has unique fixed point say p_n in $P \otimes Q$. Since $P \otimes Q$ is bounded, so $\{p_n\}_n$ is bounded, and thus $\{Tp_n\}_n$ and $\{Sp_n\}_n$ are bounded. So, proceeding as in Theorem 3.8, we get $T + ST + S$ has a fixed point on $P \otimes Q$. □

(iii) Let P and Q be as in the above Corollary. If S is a linear continuous self mapping on $P \otimes Q$ such that $I - (T + ST + S)$ is demiclosed with $\|S\| \leqslant \frac{1}{3}$ and $(k + k^{/}) \in \left(0, \frac{1}{2}\right]$ then $T + ST + S$ has a unique fixed point in $P \otimes Q$.

Proof. For $\{\alpha_n\}_n$ and f_n as above we get,

$$\|f_n(u) - f_n(v)\| \leqslant (1 - \alpha_n)\|u - v\|, \ u, v \in P \otimes Q$$

Rest of the proof follows immediately as above. □

In [12], [17] we have the following sufficient conditions for a real sequence for converging to zero.

Lemma 3.10. *Let $\{\alpha_n\}_n$ be a non-negative real sequence satisfying*

$$\alpha_{n+1} \leqslant (1 - \gamma_n)\alpha_n + \gamma_n\delta_n + \sigma_n, \ n = 0, 1, 2, ...$$

If $\{\gamma_n\}_{n=1}^{\infty} \subset (0, 1)$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ satisfy the conditions:

I. $\sum_{n=1}^{\infty} \gamma_n = \infty$.

II. either $\limsup_{n\to\infty} \delta_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$.

III. $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

In 2013, Songnian He and Wenlong Zhu defined an iteration scheme which was named as boundary point method (refer to [12]). Considering this, here we derive the following iteration scheme:

$$\begin{aligned} x_0 &\in P \otimes Q \\ y_n &= \alpha_n x_n + (1 - \alpha_n)(Tx_n + STx_n + Sx_n) \\ x_{n+1} &= \alpha_n\lambda_n y_n + (1 - \alpha_n)(Tu + STu + Su) \end{aligned}$$

where $u \in P \otimes Q$ is an arbitrary (but fixed) element in $P \otimes Q$, $\{\alpha_n\}_n$ is a sequence in $(0,1)$ and $\{\lambda_n\}_n$ is a monotonic increasing sequence in $[0,1]$.

Theorem 3.11. *Let $\{\alpha_n\}_n$ and $\{\lambda_n\}_n$ satisfy the following conditions:*

A1. $\dfrac{\alpha_n}{1-\alpha_n\lambda_n} \to 0$.

A2. $\sum_{n=1}^{\infty}(1-\alpha_n\lambda_n) = \infty$.

A3. $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}_n$ of the above iteration scheme converges weakly to the unique fixed point of $(T+ST+S)$ in $P \otimes Q$.

Proof. We have $\{x_n\}_n$, $\{y_n\}_n$, $\{Tx_n\}_n$, $\{STx_n\}_n$ and $\{Sx_n\}_n$ are sequences in the bounded subset $P \otimes Q$ of $X \otimes_\gamma Y$.

$$\|x_{n+1} - (Tu + STu + Su)\| \leqslant \alpha_n(\lambda_n\|y_n\| + \|Tu\| + \|STu\| + \|Su\|) \to 0, \quad \text{[using A1]}$$

Now we will show that $\|x_{n+1} - x_n\| \to 0$

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n\lambda_n y_n + (1-\alpha_n)(Tu + STu + Su) \\ &\qquad - [\alpha_{n-1}\lambda_{n-1}y_{n-1} + (1-\alpha_{n-1})(Tu + STu + Su)]\| \\ &\leqslant \alpha_n\lambda_n\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\lambda_{n-1}\|y_{n-1}\|) \\ &\quad + \alpha_n|\lambda_n - \lambda_{n-1}|\|y_{n-1}\| \\ &\qquad + |\alpha_n - \alpha_{n-1}|[\|Tu\| + \|STu\| + \|Su\|]\end{aligned}$$

$$\begin{aligned}\|y_n - y_{n-1}\| \leqslant \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|[\|Tx_{n-1}\| \\ + \|STx_{n-1}\| + \|Sx_{n-1}\| + \|x_{n-1}\|]\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - x_n\| \leqslant\ & (1-(1-\alpha_n\lambda_n))\|x_n - x_{n-1}\| + \alpha_n|\lambda_n - \lambda_{n-1}|\|y_{n-1}\| \\ & + |\alpha_n - \alpha_{n-1}|[\|Tu\| \\ & + \|STu\| + \|Su\| + \lambda_{n-1}\|y_{n-1}\| \\ & + \|Tx_{n-1}\| + \|STx_{n-1}\| + \|Sx_{n-1}\| + \|x_{n-1}\|]\end{aligned}$$

Since, $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ as $\{\lambda_n\}_n \subset [0,1]$ is a monotonic increasing sequence and by the conditions $A1$, $A2$ and $A3$ we will easily get according to the lemma 3.10 $\|x_{n+1} - x_n\| \to 0$. Hence,

$$\begin{aligned}\|x_n - (Tx_n + STx_n + Sx_n)\| &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - (Tu + STu + Su)\| \\ &\quad + \|(Tu + STu + Su) - (Tx_n + STx_n + Sx_n)\| \\ &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - (Tu + STu + Su)\| \\ &+ \|u - x_n\|\end{aligned}$$

Since $\{x_n\}_n$ is a Cauchy sequence so, it converges to say, p. Without loss of generality, we can take $u = p$ here so that $\|u - x_n\| \to 0$. Thus

$$\|x_n - (Tx_n + STx_n + Sx_n)\| \to 0$$

Since $I - (T + ST + S)$ is demiclosed so proceeding Theorem 3.8 we have the iteration scheme converges weakly to the unique fixed point in $P \otimes Q$. □

4. Concluding Remarks

In [15], J. Penot discussed about asymptotically contractive maps on a subset C of a Banach space X. The map $f : C \to X$ is called asymptotically contractive on C if there exists some $x_0 \in C$ such that

$$\lim_{x \in C} \sup_{\|x\| \to \infty} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1$$

Here the following problem arises:

Can we obtain analogous result as in Theorem 3.1 or 3.7 taking S as asymptotically contractive map on $P \otimes Q$?

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