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PROPERTIES OF THE MODIFIED CAPUTO'S DERIVATIVE OPERATOR FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract: In this paper, a class $A_{\eta,\lambda}(\alpha,\beta,\gamma)$ of analytic functions involving the integral operator $J_{\eta,\lambda}f(z) = z + \sum_{n=2}^{\infty} \frac{(\Gamma(n+1))^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} a_n z^n$,given by Salah and Darus in [5] is defined. The extreme points for this class are provided, the coefficient bounds and radii of univalency and starlikeness are also provided.

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Key Words: Caputo's differentiation operator, univalency, starlikeness, Hadamard product

1. Introduction and Preliminaries

Let A denote the class of functions of the form $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $U = \{z : |z| < 1\}$. Also let S denote the familiar subclass of A consisting of all functions which are univalent in U.

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Definition 1.1. The fractional integral of order λ is defined, for a function $f(z) \in A$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \quad (\lambda > 0), \tag{1.1}$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$,

Definition 1.2. The fractional derivative of order λ is defined, for a function $f(z) \in A$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt \quad (0 \leq \lambda < 1), \tag{1.2}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Definition 1.3. The Caputo's definition of fractional-order derivative is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{1.3}$$

where $n-1 < Re(\alpha) \leq n, n \in N$ and α is allowed to be real or complex number, a is the initial value of the function f.

Definition 1.4. The Modified Caputo's derivative operator is given by

$$J_{\eta,\lambda} f(z) = \frac{\Gamma(2+\eta-\lambda)}{\Gamma(\eta-\lambda)} z^{\lambda-\eta} \int_0^z \frac{\Omega^{\eta} f(t)}{(z-t)^{\lambda+1-\eta}} dt, \tag{1.4}$$

where η (real number), $(\eta - < \lambda \leq \eta < 2)$, and $\Omega^{\eta} f(t) = \Gamma(2-\eta) t^{\eta} D_t^{\eta} f(t)$.

Now, if $z + \sum_{n=2}^{\infty} a_n z^n$ is an analytic function in A, then

$$J_{\eta,\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{(\Gamma(n+1))^2 \Gamma(2+\eta-\lambda) \Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1) \Gamma(n-\eta+1)} a_n z^n. \tag{1.5}$$

In this paper, we consider the following subclass of A

$A_{\eta,\lambda}(\alpha, \beta, \gamma)$

$$= \left\{ f(z) \in A : R\left\{ \alpha \frac{J_{\eta,\lambda} f(z)}{z} + \beta \left(J_{\eta,\lambda} f(z) \right)' \right\} > \gamma \, (z \in U) \right\}, \quad (1.6)$$

for some real $\alpha, \beta > 0$, and γ with $0 \le \gamma \le \alpha + \beta \le 1$. Where for η(real number) and $(\eta - 1 < \lambda \le \eta < 2)$.

Note that $J_{0,0} f(z) = f(z)$. Thus the class $A_{\eta,\lambda}(\alpha, \beta, \gamma)$ is a generalization of the class:

$$A(\alpha, \beta, \gamma) = \left\{ f(z) \in A : R\left\{ \alpha \frac{f(z)}{z} + \beta \left(f(z) \right)' \right\} > \gamma \, (z \in U) \right\}. \quad (1.7)$$

The class $(1, 7)$ was introduced and studied by Zhi-Gang Wang,Chun-Yi Gao and Shao-Mou Yuan [1]. While Saitoh [2] and Owa [3, 4] have determined and discussed certain related properties in a special case where $\alpha = 1 - \beta$.

In the present work, first we determine the extreme points of (1.6), then we find the coefficient bounds and radius of univalency for functions belonging to this class.

2. Extreme Points of the Class $A_{\eta,\lambda}(\alpha, \beta, \gamma)$

Theorem 2.1. *A function $f(z) \in A_{\eta,\lambda}(\alpha, \beta, \gamma)$ if and only if $f(z)$ can be expressed as*

$$f(z) = z + \frac{2(\alpha + \beta - \gamma)}{\Gamma(2 - \eta)\Gamma(2 + \eta - \lambda)} \int_{|x|=1} \left(\sum_{n=2}^{\infty} \frac{\Gamma(n + \eta - \lambda + 1)\Gamma(n - \eta + 1) x^{n-1}}{(\Gamma(n+1))^2 (\alpha + n\beta)} z^n \right) d\mu(x), \quad (2.1)$$

where $\mu(x)$is the probability measure on $X = \{x : |x| = 1\}$. For fixed α, β and γ, $A(\alpha, \beta, \gamma)$and the probability measures $\{\mu\}$ defined on X are one-to-one by the expression (2.1).

Proof. By the definition of $A_{\eta,\lambda}(\alpha, \beta, \gamma)$, we know $f(z) \in A_{\eta,\lambda}(\alpha, \beta, \gamma)$ if and only if

$$\frac{\alpha \left(\frac{J_{\eta,\lambda} f(z)}{z} \right) + \beta \left(J_{\eta,\lambda} f(z) \right)' - \gamma}{\alpha + \beta - \gamma} \in P,$$

where P denotes the normalized class of analytic functions which have positive real part. Using Herglotz expressions of functions in P, we have

$$\frac{\alpha \left(\frac{J_{\eta,\lambda} f(z)}{z} \right) + \beta \left(J_{\eta,\lambda} f(z) \right)' - \gamma}{\alpha + \beta - \gamma} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x)$$

or

$$\frac{\alpha J_{\eta,\lambda}f\left(z\right)}{\beta z}+\left(J_{\eta,\lambda}f\left(z\right)\right)'=\frac{1}{\beta}\int_{|x|=1}\frac{\alpha+\beta+\left(\alpha+\beta-2\gamma\right)xz}{1-xz}d\mu\left(x\right).$$

Thus we have

$$z^{-\frac{\alpha}{\beta}}\int_0^z\left[\frac{\alpha J_{\eta,\lambda}f\left(\zeta\right)}{\beta\zeta}+\left(J_{\eta,\lambda}f\left(\zeta\right)\right)'\right]\zeta^{\frac{\alpha}{\beta}}d\zeta$$
$$=\frac{1}{\beta}\int_{|x|=1}\left[z^{\frac{-\alpha}{\beta}}\int_0^z\frac{\alpha+\beta+\left(\alpha+\beta-2\gamma\right)x\zeta}{1-x\zeta}\zeta^{\frac{\alpha}{\beta}}d\zeta\right]d\mu\left(x\right).$$

That is

$$J_{\eta,\lambda}f\left(z\right)$$
$$=\frac{1}{\alpha+\beta}\int_{|x|=1}\left[\left(2\gamma-\alpha-\beta\right)z+2\left(\alpha+\beta-\gamma\right)\sum_{n=0}^{\infty}\frac{\left(\alpha+\beta\right)x^nz^{n+1}}{\left(n+1\right)\beta+\alpha}\right]d\mu\left(x\right)$$
$$=z+\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{2\left(\alpha+\beta-\gamma\right)}{\alpha+n\beta}x^{n-1}z^n\right)d\mu\left(x\right).$$

The last expression is equivalent to

$$f\left(z\right)=z+\frac{2\left(\alpha+\beta-\gamma\right)}{\Gamma\left(2-\eta\right)\Gamma\left(2+\eta-\lambda\right)}$$
$$\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{\Gamma\left(n+\eta-\lambda+1\right)\Gamma\left(n-\eta+1\right)x^{n-1}}{\left(\Gamma\left(n+1\right)\right)^2\left(\alpha+n\beta\right)}z^n\right)d\mu\left(x\right).$$

This deductive process can be conversed, so we have proved the first part of the theorem.

We know that both probability measures $\{\mu\}$ and class P, class P and $A_{\eta,\lambda}(\alpha,\beta,\gamma)$ are one-to-one, so the second part of the theorem is true. This completes the proof.

Corollary 2.2. *The extreme points of the class $A_{\eta,\lambda}(\alpha,\beta,\gamma)$ are*

$$f_x\left(z\right)=z+\frac{2\left(\alpha+\beta-\gamma\right)}{\Gamma\left(2-\eta\right)\Gamma\left(2+\eta-\lambda\right)}$$
$$\sum_{n=2}^{\infty}\frac{\Gamma\left(n+\eta-\lambda+1\right)\Gamma\left(n-\eta+1\right)x^{n-1}}{\left(\Gamma\left(n+1\right)\right)^2\left(\alpha+n\beta\right)}z^n\qquad\left(|x|=1\right).\quad(2.2)$$

Proof. Using the notation $f_x(z)$, (2.1) can be written as

$$f_\mu(z) = \int_{|x|=1} f_x(z)\, d\mu(x)$$

By Theorem (2.1) , the map $\mu \to f_\mu$ is one-to-one, so the assertion follows.

Corollary 2.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A_{\eta,\lambda}(\alpha,\beta,\gamma)$, then for $n \geq 2$, we have*

$$|a_n| \leq \frac{2(\alpha+\beta-\gamma)\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}{\Gamma(2-\eta)\Gamma(2+\eta-\lambda)(\Gamma(n+1))^2(\alpha+n\beta)}.$$

Proof. The coefficient bounds are maximized at an extreme point so the result follows from (2.2).

Corollary 2.4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A_{\eta,\lambda}(\alpha,\beta,\gamma)$, then for $|z| = r < 1$, we have*

$$|f(z)| \leq r + \frac{2(\alpha+\beta-\gamma)}{\Gamma(2-\eta)\Gamma(2+\eta-\lambda)} \sum_{n=2}^{\infty} \frac{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)r^n}{(\Gamma(n+1))^2(\alpha+n\beta)}.$$

This result follows from (2.3).

3. Radius of Univalency and Starlikeness

Theorem 3.1. *Let $f(z) \in A_{\eta,\lambda}(\alpha,\beta,\gamma)$, then $f(z)$ is univalent(close-to-convex) in $|z| < R(\alpha,\beta,\gamma)$, where*

$$R(\alpha,\beta,\gamma) = \inf_n \left\{ \frac{(n\beta+\alpha)(\Gamma(n+1))^2\Gamma(2-\eta)\Gamma(2+\eta-\lambda)}{2n(\alpha+\beta-\gamma)\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} \right\}^{\frac{1}{n-1}}.$$

Proof. It suffices to show that

$$\left| f'(z) - 1 \right| < 1. \tag{3.1}$$

For the left hand side of (3.1) we have

$$\left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}.$$

The last expression is less than 1 if

$$|z|^{n-1} < \frac{(n\beta+\alpha)\,(\Gamma\,(n+1))^2\,\Gamma\,(2-\eta)\,\Gamma\,(2+\eta-\lambda)}{2n\,(\alpha+\beta-\gamma)\,\Gamma\,(n+\eta-\lambda+1)\,\Gamma\,(n-\eta+1)}.$$

To show that the bound $R(\alpha,\beta,\gamma)$ is the best possible,we consider the function $f\,(z) \in A$ defined by

$$f\,(z) = z - \frac{\Gamma\,(n+\eta-\lambda+1)\,\Gamma\,(n-\eta+1)\,2\,(\alpha+\beta-\gamma)}{(\Gamma\,(n+1))^2\,(\alpha+n\beta)\,\Gamma\,(2-\eta)\,\Gamma\,(2+\eta-\lambda)}z^n.$$

If $\sigma > R(\alpha,\beta,\gamma)$, then there exists $n \geq 2$ such that

$$\left\{\frac{(n\beta+\alpha)\,(\Gamma\,(n+1))^2\,\Gamma\,(2-\eta)\,\Gamma\,(2+\eta-\lambda)}{2n\,(\alpha+\beta-\gamma)\,\Gamma\,(n+\eta-\lambda+1)\,\Gamma\,(n-\eta+1)}\right\}^{\frac{1}{n-1}} < \sigma.$$

Since $f^{'}\,(0) = 1 > 0$ and

$$f^{'}\,(\sigma) = 1 - \frac{\Gamma\,(n+\eta-\lambda+1)\,\Gamma\,(n-\eta+1)\,2n\,(\alpha+\beta-\gamma)}{(\Gamma\,(n+1))^2\,(\alpha+n\beta)\,\Gamma\,(2-\eta)\,\Gamma\,(2+\eta-\lambda)}\sigma^{n-1} < 0.$$

Thus, there exists $\sigma_0 \in (0,\sigma)$ such that $f^{'}\,(\sigma_0) = 0$,which implies that $f\,(z)$is not univalent in $|z| < \sigma$. This completes the proof of Theorem 3.1.

Theorem 3.2. *If $f\,(z) \in A_{\eta,\lambda}(\alpha,\beta,\gamma)$ then $f\,(z)$ is starlike of order μ, $|z| < r_0$, $0 \leq \mu < 1$ where*

$$r_0 = \inf_n \left\{\frac{(1-\mu)\,\Gamma\,(2-\eta)\,\Gamma\,(2+\eta-\lambda)\,(\Gamma\,(n+1))^2\,(n\beta+\alpha)}{2\,(n-\mu)\,(\alpha+\beta-\gamma)\,\Gamma\,(n+\eta-\lambda+1)\,\Gamma\,(n-\eta+1)}\right\}^{\frac{1}{n-1}}.$$

Proof. It suffices to show that

$$\left|\frac{zf^{'}\,(z)}{f\,(z)} - 1\right| < 1-\mu. \tag{3.2}$$

Using the same techniques of the previous theorem to proof of theorem (3.2) can be easily derived.

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