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NOVEL IDENTITIES OF SYMMETRY FOR CARLITZ'S TWISTED q-BERNOULLI POLYNOMIALS UNDER S_4

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Abstract: In this paper, the authors discover some new symmetric identities of Carlitz's twisted q-Bernoulli polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree four shown by S_4.

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1. Introduction

As well known that the ordinary Bernoulli polynomials, $B_n(x)$, are defined by the following Taylor series expansion about $t = 0$:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt}, \qquad (|t| < 2\pi). \tag{1.1}$$

Upon setting $x = 0$ in the Eq. (1.1), we have $B_n(0) := B_n$ that is popularly known as n-th Bernoulli number (see, e.g., [1], [3], [4], [8], [9], [10], [11]).

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Imagine that p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of rational numbers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$. The notation "q" can be considered as an indeterminate, a complex number $q \in \mathbb{C}$ with $|q| < 1$, or a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. The q-analogue of x is defined by $[x]_q = (1 - q^x)/(1-q)$. It is clear that $\lim_{q\to 1} [x]_q = x$ for any x with $|x|_p \leq 1$ in the p-adic case (for details, see [1, 2, 4-12]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, which is denoted by $f \in UD(\mathbb{Z}_p)$. From here, Kim defined the q-Volkenborn integral or p-adic q-integral on $\mathbb{Z}_p$ of a function $f \in UD(\mathbb{Z}_p)$ in [9] as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x. \tag{1.2}$$

If we take $f_1(x) = f(x+1)$ in Eq. (1.2), then we see that

$$qI_q(f_1) = I_q(f) + (q-1) f(0) + \frac{q-1}{\log q} f'(0). \tag{1.3}$$

Also we observe that

$$q^n I_q(f_n) = I_q(f) + (q-1) \sum_{r=0}^{n-1} q^r f(r) + \frac{q-1}{\log q} \sum_{r=0}^{n-1} f'(r)$$

where $f_n(x) = f(x+n)$. For these related issues, see, e.g., [2], [4], [5], [6], [7], [8], [9], [10], [11], [12].

Let $T_p = \bigcup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \left\{\zeta : \zeta^{p^N} = 1\right\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we indicate by $\phi_\zeta : \mathbb{Z}_p \to C_p$ the locally constant function $x \to \zeta^x$. For $q \in C_p$ with $|1-q|_p < 1$ and $\zeta \in T_p$, the Carlitz's twisted q-Bernoulli polynomials with Witt's formula are defined by the following q-Volkenborn integral on $\mathbb{Z}_p$, with respect to μ_q, in [11]:

$$\int_{\mathbb{Z}_p} \zeta^y [x+y]_q^n \, d\mu_q(y) = \beta_{n,q,\zeta}(x) \quad (n \geq 0). \tag{1.4}$$

Substituting $x = 0$ into the Eq. (1.4) gives $\beta_{n,q,\zeta}(0) := \beta_{n,q,\zeta}$ that are called n-th Carlitz's twisted q-Bernoulli numbers. The following expression holds

$$\beta_{n,q,\zeta}(x) = \left([x]_q + q^x \beta_{q,\zeta}\right)^n$$

with the usual convention of replacing $\beta_{q,\zeta}^n$ by $\beta_{n,q,\zeta}$. Taking $\zeta = 1$ and $q \to 1$ in the Eq. (1.4) yields to

$$\beta_{n,q,\zeta}(x) \to B_n(x) := \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y),$$

where μ_1 is p-adic Haar distribution.

Recently, symmetric identities of some well-known polynomials arising from p-adic q-integrals on $\mathbb{Z}_p$ have been investigated extensively by many mathematicians. For example, Araci *et al.* [2] investigated some new symmetric identities of q-Frobenius polynomials under S_5, which are associated with the fermionic p-adic q-integral over the p-adic numbers field. Kim *et al.* [4] derived some novel identities of symmetry for the Carlitz q-Bernoulli polynomials invariant under S_4 which are derived from p-adic q-integral on $\mathbb{Z}_p$. Duran *et al.* [5] gave some new symmetric identities of q-Genocchi polynomials arising from the q-Volkenborn integral on $\mathbb{Z}_p$ under S_4. Duran *et al.* [6] obtained some new symmetric identities of weighted q-Genocchi polynomials using q-Volkenborn integral on $\mathbb{Z}_p$ under S_4. Duran *et al.* [7] considered some new symmetric identities of Carlitz's twisted (h,q)-Euler polynomials derived from p-adic invariant integral on $\mathbb{Z}_p$ under S_n. Moreover, Kim *et al.* [9] discovered new identities of symmetry for higher-order Carlitz's q-Bernoulli polynomials arising from p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree five. Furthermore, Kim [10] presented some novel identities of symmetry for Carlitz's-type q-Bernoulli polynomials resulting from p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree five.

In the next section, we consider the Carlitz's twisted q-Bernoulli polynomials and give some novel symmetric identities for these polynomials originating from the p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree four shown by S_4.

2. Novel identities of symmetry for $\beta_{n,q,\zeta}(x)$ under S_4

Let $\zeta \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ and $w_i \in \mathbb{N}$ with $i \in \{1,2,3,4\}$. By the Eqs. (1.2) and (1.4), we discover

$$\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 w_3 y} e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]_q t} d\mu_{q^{w_1 w_2 w_3}}(y)$$

$$= \lim_{N\to\infty} \frac{1}{[p^N]_{q^{w_1w_2w_3}}} \sum_{y=0}^{p^N-1} \zeta^{w_1w_2w_3y} q^{w_1w_2w_3y}$$
$$\times e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t}$$

$$= \lim_{N\to\infty} \frac{1}{[w_4p^N]_{q^{w_1w_2w_3}}} \sum_{l=0}^{w_4-1}\sum_{y=0}^{p^N-1} \zeta^{w_1w_2w_3(l+w_4y)} q^{w_1w_2w_3(l+w_4y)}$$
$$\times e^{[w_1w_2w_3(l+w_4y)+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t},$$

which yields

$$\begin{aligned} I =& \frac{1}{[w_1w_2w_3]_q} \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1} \zeta^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \\ &\times q^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \\ &\times \int_{\mathbb{Z}_p} \zeta^{w_1w_2w_3y} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} d\mu_{q^{w_1w_2w_3}}(y) \end{aligned} \tag{2.1}$$

$$\begin{aligned} =& \lim_{N\to\infty} \frac{1}{[w_1w_2w_3w_4p^N]_q} \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{l=0}^{w_4-1}\sum_{y=0}^{p^N-1} \\ &\times \zeta^{w_1w_2w_3(l+w_4y)+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \\ &\times q^{w_1w_2w_3(l+w_4y)+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \\ &\times e^{[w_1w_2w_3(l+w_4y)+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t}. \end{aligned}$$

Note that Eq. (2.1) is invariant for any permutation $\sigma \in S_4$. Therefore, we acquire the following theorem.

Theorem 1. *Let $\zeta \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ and $w_i \in \mathbb{N}$ with $i \in \{1,2,3,4\}$. Then the following*

$$\begin{aligned} I =& \frac{1}{\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q} \sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1} \\ &\times \zeta^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \\ &\times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \\ &\times \int_{\mathbb{Z}_p} \exp\Big(\big[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}y + w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}x + w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i \\ &+ w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k\big]_q t\Big) \end{aligned}$$

$$\times\zeta^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}y}d\mu_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}(y)$$

holds true for any $\sigma \in S_4$.

Using the definition of q-number, $[x]_q$, we easily derive that

$$\begin{aligned}&[w_1w_2w_3y + w_1w_2w_3w_4x + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k]_q \\ &= [w_1w_2w_3]_q\left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}.\end{aligned} \tag{2.2}$$

From Eq. (2.2), we compute

$$\begin{aligned}&\int_{\mathbb{Z}_p}\zeta^{w_1w_2w_3y}e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_qt}d\mu_{q^{w_1w_2w_3}}(y)\\ &\quad= \sum_{n=0}^{\infty}[w_1w_2w_3]_q^n\left(\int_{\mathbb{Z}_p}\zeta^{w_1w_2w_3y}\left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j\right.\right.\\ &\qquad\left.\left.+\frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}}d\mu_{q^{w_1w_2w_3}}(y)\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}[w_1w_2w_3]_q^n\,\beta_{n,q^{w_1w_2w_3},\zeta^{w_1w_2w_3}}\left(w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Thus, from Theorem 1 and Eq. (2.3), we conclude the following theorem.

Theorem 2. *Let* $\zeta \in T_p$, $q \in \mathbb{C}_p$ *with* $|q-1|_p < 1$ *and* $w_i \in \mathbb{N}$ *with* $i \in \{1,2,3,4\}$. *For* $n \geq 0$, *the following*

$$\begin{aligned}I &= \left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q^{n-1}\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}\\ &\quad\times\zeta^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k}\\ &\quad\times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k}\\ &\quad\times\beta_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}},\zeta^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x + \frac{w_{\sigma(4)}}{w_{\sigma(1)}}i + \frac{w_{\sigma(4)}}{w_{\sigma(2)}}j + \frac{w_{\sigma(4)}}{w_{\sigma(3)}}k\right)\end{aligned}$$

holds true for any $\sigma \in S_4$.

Using binomial theorem and the definition of q-number, we observe

$$\left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}} \tag{2.4}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_4]_q}{[w_1 w_2 w_3]_q} \right)^{n-m} [w_2 w_3 i + w_1 w_3 j + w_1 w_2 k]_{q^{w_4}}^{n-m}$$
$$\times q^{m(w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k)} [y + w_4 x]_{q^{w_1 w_2 w_3}}^{m},$$

which gives

$$[w_1 w_2 w_3]_q^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \zeta^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k} q^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k} \tag{2.5}$$
$$\times \int_{\mathbb{Z}_p} \zeta^{w_1 w_2 w_3 y} \left[y + w_4 x + \frac{w_4}{w_1} i + \frac{w_4}{w_2} j + \frac{w_4}{w_3} k \right]_{q^{w_1 w_2 w_3}}^{n} d\mu_{q^{w_1 w_2 w_3}}(y)$$
$$= \sum_{m=0}^{n} \binom{n}{m} [w_1 w_2 w_3]_q^{m-1} [w_4]_q^{n-m} \beta_{m, q^{w_1 w_2 w_3}, \zeta^{w_1 w_2 w_3}} (w_4 x) \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1}$$
$$\sum_{k=0}^{w_3-1} \zeta^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k} q^{(m+1)(w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k)}$$
$$\times [w_2 w_3 i + w_1 w_3 j + w_1 w_2 k]_{q^{w_4}}^{n-m}$$
$$= \sum_{m=0}^{n} \binom{n}{m} [w_1 w_2 w_3]_q^{m-1} [w_4]_q^{n-m} \beta_{m, q^{w_1 w_2 w_3}, \zeta^{w_1 w_2 w_3}} (w_4 x)$$
$$\times \widehat{C}_{n, q^{w_4}, \zeta^{w_4}} (w_1, w_2, w_3 \mid m),$$

where

$$\widehat{C}_{n,q,\zeta}(w_1, w_2, w_3 \mid m) \tag{2.6}$$
$$= \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \zeta^{w_2 w_3 i + w_1 w_3 j + w_1 w_2 k}$$
$$\times q^{(m+1)(w_2 w_3 i + w_1 w_3 j + w_1 w_2 k)} [w_2 w_3 i + w_1 w_3 j + w_1 w_2 k]_q^{n-m}.$$

Therefore, by (2.6), we arrive at the following theorem.

Theorem 3. *Let $\zeta \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, $w_i \in \mathbb{N}$ with $i \in \{1, 2, 3, 4\}$, and let $n \geq 0$. Then the following expression*

$$\sum_{m=0}^{n} \binom{n}{m} \left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} \right]_q^{m-1} \left[w_{\sigma(4)} \right]_q^{n-m}$$
$$\times \beta_{m, q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}, \zeta^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}} \left(w_{\sigma(4)} x \right)$$

$$\times \widehat{C}_{n,q^{w_{\sigma(4)}},\zeta^{w_{\sigma(4)}}}(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)} \mid m)$$

holds true for some $\sigma \in S_4$.

3. Conclusion

In this study, we have investigated not only new but also interesting symmetric identities for Carlitz's twisted q-Bernoulli polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$ under the symmetric group of degree four. We note that in the case $\zeta = 1$, the results derived in this paper reduce to the results in [4].

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