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ABOUT ONE INVERSE PROBLEM OF THE LINEAR-FRACTIONAL PROGRAMMING ON GENERALIZED NETWORK

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Abstract: For the one linear-fractional network programming problem with additional constraints of general kind and with inexact data we constructed a mathematical model for calculation of the new parameters of the restrictions for which the infeasible solution becomes feasible solution. For the selected feasible solution of this problem we minimally changed the parameters in the numerator of the linear-fractional objective function in order that the selected feasible solution has become an optimal. The measure proximity vectors estimated are using the norm l_1. That has allowed to remain within the framework of linear programming in solving of inverse problems.

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1. Mathematical Model

Let $G = (I, U)$ be a finite oriented connected graph (network) without multiple arcs and loops, where I is a set of nodes and $U \subset I \times I$ is a set of arcs. On

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the described graph G we consider the linear-fractional homogeneous flow programming optimization problem (1)–(4) with additional constraints of general kind:

$$f(x) = \frac{\sum\limits_{(i,j)\in U} c_{ij}x_{ij} + \beta}{\sum\limits_{(i,j)\in U} q_{ij}x_{ij} + \gamma} \longrightarrow \min, \tag{1}$$

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} \mu_{ji}x_{ji} = a_i, \ \ i \in I, \tag{2}$$

$$\sum_{(i,j)\in U} \lambda_{ij}^p x_{ij} = \alpha_p, \ \ p = \overline{1,l}, \tag{3}$$

$$x_{ij} \geq 0, \ \ (i,j) \in U, \tag{4}$$

where x_{ij} – an arc flow of arc $(i,j) \in U$, $x = (x_{ij},\ (i,j) \in U)$ – arc flows vector of the graph $G = (I,U)$, $I^+(U) = \{j \in I : (i,j) \in U\}, I_i^-(U) = \{j \in I : (j,i) \in U\}$, $\mu = (\mu_{ij},\ (i,j) \in U)$ – vector of coefficients transformation of arc flows, μ_{ij} - the transformation coefficient of the arc flow x_{ij} for arc (i,j): arc flow x_{ij} outgoing from the node i and incoming to the node j in a modified form $\mu_{ij}x_{ij}$, and, transformation arc flow x_{ij} is carried out immediately before the node j, $\mu_{ij} \in]0,1]$. Here c_{ij}, q_{ij}, μ_{ij}, a_i, λ_{ij}^p, α_p, β, γ – known parameters of problem (1)–(4).

Let us assume, that the denominator

$$q(x) = \sum_{(i,j)\in U} q_{ij}x_{ij} + \gamma$$

of the objective function $f(x)$ does not change a sign for the set of flows $x \in X$. Without restriction of a generality, we shall assume that $q(x) > 0, \forall x \in X$.

In practice, there are many situations when the values of the parameters are inexact data. For the first time the inverse optimization problems were considered in [1], [2] for changing the coefficients of the objective function for some special problems of linear programming. Let $x = (x_{ij},(i,j) \in U)$, $x_{ij} \geq 0, (i,j) \in U$ – basic infeasible solution extremal problem (1)–(4) for the given parameters (some constraints (2)–(3) are not met). We will change the parameters of restrictions (2)–(3) for the problem (1)–(4) so that all restrictions are performed (basic feasible solution).

2. Adjustment of Parameters Constraints

Consider the constraints of the problem (1)–(4). The following parameters

$$a_i,\ i \in I;\ \alpha_p,\ p = \overline{1,l};\ \mu_{ij},\ (i,j) \in U;\ \lambda_{ij}^p,\ p = \overline{1,l},\ (i,j) \in U \tag{5}$$

are inexact data. It is necessary minimally to change the parameters (5) so that a given infeasible basic solution x becomes feasible basic solution for new parameters, X is the set of the feasible basic solutions. To change the parameters in the network optimization problems in [3], [4] were considered some mathematical models in accordance with the l_1. The selected norm l_1 allows to remain within the framework of linear programming.

When adjusting the parameters (5) of restrictions (2)–(3), the following two cases can be presented.

In the case 1) if the transform coefficients $\mu_{ij}, (i,j) \in U$ of arc flows $x_{ij}, (i,j) \in U$ and the matrix of coefficients of additional constraints (3) do not change, then the adjusted parameters $\tilde{a}_i$, $i \in I$ and $\tilde{\alpha}_p$, $p = \overline{1,l}$ of problem (1)–(4) are computed as follows:

$$\tilde{a}_i = \sum_{j \in I_i^+(U)} x_{ij} - \sum_{j \in I_i^-(U)} \mu_{ji} x_{ji},\ \ i \in I,$$

$$\tilde{\alpha}_p = \sum_{(i,j) \in U} \lambda_{ij}^p x_{ij},\ \ p = \overline{1,l}.$$

In the case 2) when change the parameters a_i, $i \in I$; α_p, $p = \overline{1,l}$ of boundaries for restrictions (2)–(3) and also the transform coefficients $\mu_{ij}, (i,j) \in U$ of the arc flows $x_{ij}, (i,j) \in U$ and the matrix of coefficients of additional constraints (3), the new parameters have the following form:

$$\begin{gathered}
\widetilde{a}_i = a_i + k_i - \psi_i,\ k_i \geq 0,\ \psi_i \geq 0,\ \ i \in I; \\
\widetilde{\alpha}_p = \alpha_p + \varphi_p - \delta_p,\ \ \varphi_p \geq 0,\ \delta_p \geq 0,\ p = \overline{1,l}; \\
\widetilde{\mu}_{ij} = \mu_{ij} + \theta_{ij} - \gamma_{ij},\ \theta_{ij} \geq 0,\ \gamma_{ij} \geq 0,\ \ (i,j) \in U, \\
\widetilde{\lambda}_{ij}^p = \lambda_{ij}^p + r_{ij}^p - t_{ij}^p,\ p = \overline{1,l},\ \ (i,j) \in U.
\end{gathered} \tag{6}$$

The values of increase and decrease for the parameters (5) are defined as follows:

- k_i – increase and ψ_i – decrease of parameter a_i;
- φ_p – increase and δ_p – decrease of parameter α_p;

- θ_{ij} – increase and γ_{ij} – decrease of parameter μ_{ij};
- r^p_{ij} – increase and t^p_{ij} – decrease of parameter λ^p_{ij}.

It is necessary to find such new parameter (6) that the infeasible basic solution $x = (x_{ij},\ (i,j) \in U)$ of the problem (1)–(4) for the parameters (5) was a feasible basic solution for the new parameters (6) of the extremal problem of the following form:

$$f(x) = \frac{\sum\limits_{(i,j)\in U} c_{ij}x_{ij} + \beta}{\sum\limits_{(i,j)\in U} q_{ij}x_{ij} + \gamma} \longrightarrow \min, \tag{7}$$

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} \tilde{\mu}_{ji}x_{ji} = \tilde{a}_i,\ \ i \in I, \tag{8}$$

$$0 < \tilde{\mu}_{ij} \leq 1,\ \ (i,j) \in U,$$

$$\sum_{(i,j)\in U} \tilde{\lambda}^p_{ij}x_{ij} = \tilde{\alpha}_p,\ \ p = \overline{1,l}, \tag{9}$$

$$x_{ij} \geq 0,\ \ (i,j) \in U. \tag{10}$$

Let us note that the total parameter adjustments is a minimum in accordance with the selected norm l_p for the sum of vectors:

$$l_p = \|\tilde{a} - a\| + \|\tilde{\alpha} - \alpha\| + \|\tilde{\mu} - \mu\| + \|\tilde{\lambda} - \lambda\|,$$

$$a = (a_i, i \in I),\ \ \tilde{a} = (\tilde{a}_i, i \in I),$$

$$\alpha = (\alpha_p,\ p = \overline{1,l}),\ \ \tilde{\alpha} = (\tilde{\alpha}_p,\ p = \overline{1,l}),$$

$$\mu = (\mu_{ij},\ (i,j) \in U),\ \ \tilde{\mu} = (\tilde{\mu}_{ij},\ (i,j) \in U).$$

$$\lambda = \left(\lambda^p_{ij},\ p = \overline{1,l},\ (i,j) \in U\right),\ \ \tilde{\lambda} = (\tilde{\lambda}^p_{ij},\ p = \overline{1,l},\ (i,j) \in U).$$

Given that the increasing and decreasing for the value of each parameter is not may be simultaneously positive and using (6), the norm l_p for $p = 1$ is transformed to the following form:

$$l_1 = \sum_{i\in I} |k_i - \psi_i| + \sum_{p=1}^{l} |\varphi_p - \delta_p| + \sum_{(i,j)\in U} |\theta_{ij} - \gamma_{ij}| + \sum_{p=1}^{l}\sum_{(i,j)\in U} |r^p_{ij} - t^p_{ij}|$$

$$= \sum_{i\in I}(k_i + \psi_i) + \sum_{p=1}^{l}(\varphi_p + \delta_p) + \sum_{(i,j)\in U}(\theta_{ij} + \gamma_{ij}) + \sum_{p=1}^{l}\sum_{(i,j)\in U}(r_{ij}^p + t_{ij}^p).$$

The mathematical model of the extremal problem to determine the changes parameters (5) of the problem (1)–(4) has the form:

$$\begin{aligned} q(k,\psi,\varphi,\delta,\theta,\gamma,r,t) &= \sum_{i\in I}(k_i + \psi_i) + \sum_{p=1}^{l}(\varphi_p + \delta_p) \\ &+ \sum_{(i,j)\in U}(\theta_{ij} + \gamma_{ij}) + \sum_{p=1}^{l}\sum_{(i,j)\in U}(r_{ij}^p + t_{ij}^p) \longrightarrow \min, \end{aligned} \tag{11}$$

$$\begin{aligned} &\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)}(\mu_{ji} + \theta_{ji} - \gamma_{ji})x_{ji} = a_i + k_i - \psi_i, \\ &k_i \geq 0,\ \psi_i \geq 0,\ i \in I, \\ &0 < \mu_{ij} + \theta_{ij} - \gamma_{ij} \leq 1, \theta_{ij} \geq 0, \gamma_{ij} \geq 0, (i,j) \in U; \end{aligned} \tag{12}$$

$$\begin{aligned} &\sum_{(i,j)\in U}(\lambda_{ij}^p + r_{ij}^p - t_{ij}^p)x_{ij} = \alpha_p + \varphi_p - \delta_p,\ \ \varphi_p \geq 0, \\ &\delta_p \geq 0,\ p = \overline{1,l}; r_{ij}^p \geq 0,\ t_{ij}^p \geq 0,\ \ (i,j) \in U,\ p = \overline{1,l}; \end{aligned} \tag{13}$$

$$x_{ij} \geq 0,\ \ (i,j) \in U. \tag{14}$$

Thus, the infeasible basic solution $x = (x_{ij},\ (i,j) \in U)$ of the problem (1)–(4) for the parameters (5) is the feasible basic solution of the extremal problem (7)–(10) for the new parameters (6), where the numerical values of increasing and decreasing for the parameters (5) are obtained by solving the extremal problem (11)–(14). Now we can use the effective methods of decomposition [5] – [8] which designed to solve network problems of linear and fractional linear mathematical programming.

3. The Dual Problem

The dual problem for the linear-fractional network flow programming problem (1)–(4) has the following form:

$$g(y,r,z) = z \longrightarrow \max, \tag{15}$$

$$y_i - \mu_{ij} y_j + \sum\nolimits_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z \leq c_{ij}, \ \ (i,j) \in U,$$
$$- \sum_{j \in I} a_i y_i - \sum_{p=1}^{l} \alpha_p r_p + \gamma z = \beta, \tag{16}$$

where $z \in R^1$, $y = (y_i,\, i \in I)$, $r = (r_p,\, p = \overline{1,l})$.

The vector $\lambda = (y,\, r,\, z)$ is called the feasible solution of the dual problem (15) – (16), if for the components of the vector λ are performed the restrictions (16).

Theorem 1. *Let $x^0 = (x_{ij}^0,\, (i,j) \in U)$ be feasible solution of the problem* (1)–(4)*. If for some vector λ,*

$$\lambda = (y, r, z), \ \ y = (y_i,\, i \in I), \ \ r = (r_p,\, p = \overline{1,l}), \ \ z \in R^1$$

and for given feasible solution x^0 the following conditions are met:

$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z \leq c_{ij}, \ \ (i,j) \in U,$$
$$- \sum_{j \in I} a_i y_i - \sum_{p=1}^{l} \alpha_p r_p + \gamma z = \beta, \tag{17}$$
$$(y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z - c_{ij}) x_{ij}^0 = 0, \ \ (i,j) \in U,$$

then x^0 is the optimal solution of the problem (1)–(4)*.*

Proof. For the given feasible solution $x^0 = (x_{ij}^0,\, (i,j) \in U)$ of the problem (1)–(4) and for the some feasible solution (y, r, z), $y = (y_i,\, i \in I)$, $r = (r_p,\, p = \overline{1,l})$, $z \in R^l$ of the dual problem (15)–(16), the following relations are satisfied:

$$(y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z - c_{ij}) x_{ij}^0 = 0, \ (i,j) \in U.$$

Then we have:

$$\sum_{(ij)\in U} c_{ij}x_{ij}^0 + \beta = \sum_{(ij)\in U} (y_i - \mu_{ij}y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij}z)x_{ij}^0 + \beta$$

$$= \sum_{i\in I} a_i y_i + \sum_{p=1}^{l} \alpha_p r_p + \sum_{(ij)\in U} q_{ij}zx_{ij}^0 - \sum_{i\in I} a_i y_i - \sum_{p=1}^{l} \alpha_p r_p + yz$$

$$= z\sum_{(ij)\in U} q_{ij}x_{ij}^0 + yz = z(\sum_{(ij)\in U} q_{ij}x_{ij}^0 + \gamma).$$

Thus, the objective function of the primal problem (1)–(4) coincides with the objective function for the problem (15)–(16), which is dual problem to (1)–(4):

$$f(x^0) = \frac{\sum\limits_{(i,j)\in U} c_{ij}x_{ij}^0 + \beta}{\sum\limits_{(i,j)\in U} q_{ij}x_{ij}^0 + \gamma} = g(y, r, z) = z.$$

Hence, $x^0 = (x_{ij}^0,\ (i,j) \in U)$ is the optimal solution of the problem (1) – (4). □

4. The Inverse Problem

Let $x = (x_{ij},\ (i,j) \in U)$ be some feasible solution of the extremal problem (1)–(4). We apply the principles of inverse optimization for minimally change of the parameters in the numerator of linear-fractional objective function (1) in order that the selected feasible solution x becomes an optimal solution.

By Theorem 1, for some of vectors

$$\lambda = (y,\ r,\ z),\ \lambda \in \Lambda,\ y = (y_i,\ i \in I),\ r = (r_p,\ p = \overline{1,l}),\ z \in R^1,$$

the parameters $c = (c_{ij},\ (i,j) \in U)$ may be replaced by such values of the parameters $\widetilde{c} = (\widetilde{c}_{ij},\ (i,j) \in U)$, that are carried out the conditions: $(y_i - \mu_{ij}y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij}z - \tilde{c}_{ij})x_{ij}^0 = 0,\ (i,j) \in U.$

Depending on the arc flows x_{ij}^0, $(i,j) \in U$ a given feasible solution x^0 of the problem (1)–(4), we form the set of arcs B_1, B_2:

$$B_1 = \{(i,j) \in U : x_{ij}^0 = 0\},\ \ B_2 = \{(i,j) \in U : x_{ij}^0 > 0\}.$$

The inverse problem for minimally change of the parameters in the numerator of linear-fractional objective function (1) to the problem (1)–(4) has the form:

$$\| \widetilde{c} - c \| \longrightarrow \min, \tag{18}$$

$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z \le \widetilde{c}_{ij}, \ \ (i,j) \in B_1,$$
$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z = \widetilde{c}_{ij}, \ \ (i,j) \in B_2, \tag{19}$$
$$-\sum_{j \in I} a_i y_i - \sum_{p=1}^{l} \alpha_p r_p + \gamma z = \beta.$$

Let ζ_{ij} be an increase and η_{ij} – a decrease of the parameter c_{ij} of the numerator of linear-fractional objective function (1), $(i,j) \in U$.

We put $\widetilde{c}_{ij} = c_{ij} + \zeta_{ij} - \eta_{ij}$, $\zeta_{ij} \ge 0$, $\eta_{ij} \ge 0$ for all arcs $(i,j) \in U$. At the same time, ζ_{ij} and η_{ij} can not be simultaneously receive positive values: $\zeta_{ij}\eta_{ij} = 0, \ (i,j) \in U$.

The remaining parameters of the problem (1)–(4) do not changed. The inverse problem (18)–(19) we present as follows:

$$\| \widetilde{c} - c \| \longrightarrow \min, \tag{20}$$

$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z \le c_{ij} + \zeta_{ij} - \eta_{ij},$$
$$\zeta_{ij} \ge 0, \ \eta_{ij} \ge 0, \ \ (i,j) \in B_1,$$
$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z = c_{ij} + \zeta_{ij} - \eta_{ij}, \tag{21}$$
$$\zeta_{ij} \ge 0, \ \eta_{ij} \ge 0, \ \ (i,j) \in B_2,$$
$$-\sum_{j \in I} a_i y_i - \sum_{p=1}^{l} \alpha_p r_p + \gamma z = \beta.$$

In accordance with the norm l_1,

$$l_1 = \| \widetilde{c} - c \|_1 = \sum_{(ij) \in U} |\widetilde{c}_{ij} - c_{ij}| = \sum_{(ij) \in U} |\zeta_{ij} - \eta_{ij}| = \sum_{(ij) \in U} (\zeta_{ij} + \eta_{ij}),$$

the mathematical model of the inverse problem for changing parameters c_{ij}, $(i,j) \in U$ of the objective function (1) has the form:

$$u(\zeta,\eta) = \sum_{(ij)\in U} (\zeta_{ij} + \eta_{ij}) \longrightarrow \min, \tag{22}$$

$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z \le c_{ij} + \zeta_{ij} - \eta_{ij},$$

$$\zeta_{ij} \ge 0,\ \eta_{ij} \ge 0,\ \ (i,j) \in B_1,$$

$$y_i - \mu_{ij} y_j + \sum_{p=1}^{l} \lambda_{ij}^p r_p + q_{ij} z = c_{ij} + \zeta_{ij} - \eta_{ij}, \tag{23}$$

$$\zeta_{ij} \ge 0,\ \eta_{ij} \ge 0,\ \ (i,j) \in B_2,$$

$$-\sum_{j\in I} a_i y_i - \sum_{p=1}^{l} \alpha_p r_p + \gamma z = \beta,$$

where $\zeta = (\zeta_{ij},\ (i,j) \in U)$, $\eta = (\eta_{ij},\ (i,j) \in U)$.

As a result of solving the inverse problem (22)–(23) we obtained the values ζ_{ij}, η_{ij} increasing and decreasing respectively for each parameter c_{ij}, $\zeta_{ij} \ge 0$, $\eta_{ij} \ge 0$, $(i,j) \in U$. The new parameters $\widetilde{c}_{ij} = c_{ij} + \zeta_{ij} - \eta_{ij}$, $(i,j) \in U$ minimally different from the initial values of the parameters c_{ij}, $(i,j) \in U$. The feasible solution $x^0 \in X$ of the problem (1)–(4) is the optimal solution for the following extremal problem:

$$\frac{\displaystyle\sum_{(i,j)\in U} (c_{ij} + \zeta_{ij} - \eta_{ij}) x_{ij} + \beta}{\displaystyle\sum_{(i,j)\in U} q_{ij} x_{ij} + \gamma} \longrightarrow \min,$$

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} \mu_{ji} x_{ji} = a_i,\ \ i \in I,$$

$$\sum_{(i,j)\in U} \lambda_{ij}^p x_{ij} = \alpha_p,\ p = \overline{1,l},\ \ x_{ij} \ge 0,\ \ (i,j) \in U.$$

References

[1] D. Burton, P.L. Toint, On an instance of the inverse shortest paths problem, *J. Math. Programming*, **53** (1992), 45-61.

[2] R.K. Ahuja, J.B. Orlin, Inverse optimization, *Operation Research*, **49**, Issue 5 (2001), 771-783.

[3] L.A. Pilipchuk, A.S. Pilipchuk, Modeling parameters of the lower and upper bounds and parameters of the objective function for generalized network flow programming problems, In: *AIP Conf. Proc., American Institute of Physics*, **1690**, 060007 (2015), 10 pp., **doi:** 10.1063/1.4936745.

[4] L.A. Pilipchuk, Obratnaya zadacha korrektirovki parametrov ogranichenii dlya odnoi lineinoi neodnorodnoi zadachi setevoi optimizacii, *Vestnik BGU, Ser. 1*, **1** (2016), 136-143, In Russian.

[5] L.A. Pilipchuk, Primenenie konstruktivnih metodov dekompozicii dlya resheniya odnoi nelineinoi zadachi setevoi optimizacii, *Vesnik Grodzenskaga Dzyarjaynaga Yniversiteta imya Yanki Kupali, Ser. 2*, **6**, No. 2 (192) (2015), 54-61, In Russian.

[6] L.A. Pilipchuk, *Lineinie neodnorodnie zadachi potokovogo programmirovaniya*, Minsk: BGU (2009), In Russian.

[7] R. Gabasov, F.M. Kirillova, *Metodi Lineinogo Programmirovaniya*, Volume 3 Chastyah, Chast 3. Specialnie Zadachi, Minsk, Belorussian State Univ. (1980), In Russian.

[8] L.A. Pilipchuk, Linear-fractional programming: problems of optimization of inhomogeneous flows in the generalized networks, *Intern. J. Pure and Appl. Math.*, **82**, No 2 (2013), 261-273.