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SOME RESULTS FOR CONTINUOUS FUNCTIONS ON WEAK SPACES

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Abstract: In this article, we consider the triple (X, w, τ_X) where w is a weak structure with an associated topology τ. The purpose of this article is to study the relationships among several continuities(continuity, $W(W^*)$-continuity, wO-continuity, wK-continuity) defined in w-spaces with associated topologies.

In particular, we investigate the new continuity, say Ow-continuity, defined in w-spaces with associated topologies, and study the characterizations of Ow-continuity and the relationships among Ow-continuity and several continuities.

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Key Words: weak spaces, W^*-continuous, W-continuity, WO-continuity, WK-continuity, wO-continuity, wK-continuity, Ow-continuity

1. Introduction

In [7], Siwiec introduced the notions of weak neighborhoods and weak base in a topological space. The author introduced the weak neighborhood systems

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defined by using the notion of weak neighborhoods in [4]. The weak neighborhood system induces a weak neighborhood space (briefly WNS) which is independent of neighborhood spaces [2] and general topological spaces [1, 3]. We introduced the notion of weak structure [5] which is defined by the properties of new interior operator and closure operator in a WNS, and investigated the notions of W-continuity and W^*-continuity [5]. In [6], we introduced the notions of WO-continuity and WK-continuity on associated w-spaces. In particular, we investigate some properties and relationships between WO-continuity, WK-continuity, W-continuity, W^*-continuity and continuity on associated w-spaces.

In this paper, for a w-space (X, w), a topology τ_X on X is called an associated topology on X if $w \subseteq \tau$. we will call the triple (X, w, τ_X) a *w-space* with an associated topology τ. The purpose of this paper is to study the relationships among several continuities(continuity, $W(W^*)$-continuity, wO-continuity, wK-continuity) defined in w-spaces with associated topologies. In particular, we investigate the new continuity, say Ow-continuity, defined in w-spaces with associated topologies, and study the characterizations of Ow-continuity and the relationships among Ow-continuity and several continuities.

2. Preliminaries

Definition 2.1 ([5]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:
(1) $\emptyset \in w_X$ and $X \in w_X$.
(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

The collection of all w-open sets (resp. w-closed sets) in a w-space X will be denoted by $WO(X)$ (resp. $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$.

Definition 2.2 ([5]). Let (X, w_X) be a w-space. For a subset A of X, the *w-closure*of A and the *w-interior* [5] of A are defined as the following:
(1) $wCl(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.
(2) $wInt(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3 ([5]). *Let (X, w_X) be a w-space and $A \subseteq X$.*
(1) If A is w-open, then $wInt(A) = A$.
(2) If A is w-closed, then $wCl(A) = A$.

3. Main Results

Definition 3.1. Let X be a nonempty set and let (X, w) be a w-space. A topology τ_X on X is called *an associated topology* on X if $w \subseteq \tau$. Then the triple (X, w, τ_X) is called a *w-space with an associated topology* τ. The triple (X, w, τ_X) simply will be denoted by (X, W_{τ_X}).

Remark 3.2. Let (X, w_X) be a w-space. Then clearly the family $\tau = \{U \subset X : U = wInt(U)\}$ is a topology and $w_X \subseteq \tau$ [5], and so it ia an associated topology on X for a w-space (X, w_X).

Definition 3.3. Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function on w-spaces with two associated topologies τ_X and τ_Y.

Then f is said to be

(1) *wO-continuous* if for $x \in X$ and for each τ_Y-open set V containing $f(x)$, there is a w-open set U containing x such that $f(U) \subseteq V$,

(2) *wK-continuous* if for every τ_Y-open set V in Y, $f^{-1}(V)$ is a w-open set in X.

(3) *W-continuous* [5] if for $x \in X$ and $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subset V$.

(4) *W^*-continuous* [5] if for every $A \in W(f(x))$, $f^{-1}(A)$ is in $W(x)$.

Remark 3.4. For a function $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$, from Definition 3.3 and the next examples, we get the following implications:

W^*-continuity → W-continuity
↗ ↗
wK-continuity → wO-continuity
↘
continuity

Example 3.5. For $X = \{a, b, c\}$, let $w = \{\emptyset, \{a\}, \{b\}, X\}$ be a w-structure.

(1) Let an associated topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Consider the identity function $f : (X, W_{\tau_X}) \to (X, W_{\tau_X})$. Then f is continuous, W^*-continuous and W-continuous but neither wO-continuous nor wK-continuous.

(2) Consider $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and the identity function $f : (X, W_{\mu_X}) \to (X, W_{\tau_X})$. Then f is W-continuous but not continuous.

(3) Consider a function $f : (X, W_{\tau_X}) \to (X, W_{\tau_X})$ defined as $f(b) = a; f(a) = f(c) = b$. Then f is continuous but not W-continuous.

(4) For an associated topology $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, consider the identity function $f : (X, W_{\mu_X}) \to (X, W_{\mu_X})$. Then f is wO-continuous but not

wK-continuous.

Remark 3.6. Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w_τ of the power set $P(X)$ is called an associated *weak structure* [6] on X if $\tau \subseteq w_\tau$.

Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space with τ and a topological space (Y, μ). Then f is said to be

(1) *WO-continuous* [6] if for $x \in X$ and $V \in O(f(x))$, there is $U \in W(x)$ such that $f(U) \subseteq V$,

(2) *WK-continuous* [6] if for every open set V in Y, $f^{-1}(V)$ is a w-open set in X.

$$\begin{array}{ccccc} & & W^*\text{-continuity} & \to & W\text{-continuity} \\ & & \downarrow & & \downarrow \\ \text{continuity} & \to & WK\text{-continuity} & \to & WO\text{-continuity} \end{array}$$

Theorem 3.7. *Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. Then the following statements are equivalent:*

(1) f is wO-continuous.

(2) $f(wCl(A)) \subseteq cl(f(A))$ for $A \subseteq X$.

(3) $wCl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for $V \subseteq Y$.

(4) $f^{-1}(int(V)) \subseteq wInt(f^{-1}(V))$ for $V \subseteq Y$

Proof. Obvious. □

Theorem 3.8. *Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function.*

(1) Then the following are equivalent:

(i) f is wK-continuous.

(ii) For every closed set F in Y, $f^{-1}(F)$ is w-closed in X.

(2) If f is wK-continuous, then wO-continuous.

Proof. Obvious. □

Now, we define a new notion of continuous function between an associated w-spaces with two associated topologies.

Definition 3.9. Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. The function f is said to be *Ow-continuous* if for each w-open set V containing $f(x)$, there is a τ_{w_X}-open set U containing x such that $f(U) \subseteq V$.

Obviously, it is that every W-continuous and every continuous function are Ow-continuous but the converses may not be true as the next examples:

Example 3.10. (1) As Example 3.5, for $X = \{a, b, c\}$ and a w-structure $w = \{\emptyset, \{a\}, \{b\}, X\}$, let an associated topology $\nu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Consider a function $f : (X, W_{\nu_X}) \to (X, W_{\nu_X})$ defined as $f(a) = a; f(b) = f(c) = b$. Then f is Ow-continuous but not W-continuous.

(2) In the above (1), consider $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and the function $f : (X, W_{\mu_X}) \to (X, W_{\nu_X})$. Then f is Ow-continuous but not continuous.

Remark 3.11. Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. Then we get the following implications:

$$
\begin{array}{ccccccc}
 & & W\text{-continuity} & \rightarrow & W\text{-continuity} & & \\
 & \nearrow & & \nearrow & & \searrow & \\
wK\text{-continuity} & \rightarrow & wO\text{-continuity} & & & & Ow\text{-continuity} \\
 & & & \searrow & & \nearrow & \\
 & & & & \text{continuity} & &
\end{array}
$$

Theorem 3.12. *Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. Then the following statements are equivalent:*

(1) f is Ow-continuous.

(2) $f(cl(A)) \subseteq wCl(f(A))$ for $A \subseteq X$.

(3) $cl(f^{-1}(B)) \subseteq f^{-1}(wCl(B))$ for $B \subseteq Y$.

(4) $f^{-1}(wInt(V)) \subseteq int(f^{-1}(V))$ for $V \subseteq Y$.

(5) If for every w-open set V in Y, $f^{-1}(V)$ is an open set in X.

Proof. (1) $\Rightarrow$ (2) For $x \in cl(A)$, assume that $f(x)$ is not in $wCl(f(A))$. Then there exists a w-open set V containing $f(x)$ such that $V \cap f(A) = \emptyset$. By Ow-continuity, there is an open set U containing x such that $f(U) \subseteq V$, and it implies that $f(U) \cap f(A) = \emptyset$ and $U \cap A = \emptyset$. So it is a contradiction and we have $f(cl(A)) \subseteq wCl(f(A))$.

(2) $\Rightarrow$ (3) Let $A = f^{-1}(B)$ for $B \subseteq Y$. Then by hypothesis, $f(cl(A)) \subseteq wCl(f(A)) = wCl(f(f^{-1}(B))) \subseteq wCl(B)$. Thus $cl(f^{-1}(B)) \subseteq f^{-1}(wCl(B))$.

(3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (5) Let V be a w-open set V in Y. Then $V = wInt(V)$ and by hypothesis, $f^{-1}(V) = f^{-1}(wInt(V)) \subseteq int(f^{-1}(V))$, and so $f^{-1}(V)$ is open in X.

(5) $\Rightarrow$ (1) For $x \in X$, let V be any w-open set containing $f(x)$. Then by (5), it follows that $f^{-1}(V)$ is an open set containing x and $x \in f^{-1}(V) \subseteq int(f^{-1}(V))$. From this fact, there exists an open set U such that $x \in U \subseteq f^{-1}(V)$. It implies that f is Ow-continuous. $\square$

We recall Remark 3.2: Let (X, w) be a w-space. Then $\tau_X = \{U \subset X : U = wInt(U)\}$ is an associated topology such that $w \subseteq \tau_X$.

Theorem 3.13. *Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. If the two associate topologies $\tau_X = \{U \subset X : U = wInt(U)\}$ and $\tau_Y = \{U \subset Y : U = wInt(U)\}$, then we get the following things:*
(1) wO-continuity = continuity:
(2) W-continuity = Ow-continuity:

Proof. (1) Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. From Remark 3.11, it is sufficient to show that the continuity implies wO-continuity. Suppose that f is continuous. For $x \in X$, let V be any τ_Y-open set containing $f(x)$. By continuity of f, there is an open set U containing x such that $f(U) \subseteq V$, and since $\tau_X = \{U \subset X : U = wInt(U)\}$, there exists a w-open set U_x containing x such that $U_x \subseteq U$. It implies f is wO-continuous.

(2) It is similar to the proof of (1). □

In summary, from the above theorem, the following diagram is obtained:

Remark 3.14. Let $f : (X, W_{\tau_X}) \to (Y, W_{\tau_Y})$ be a function. For the two associate topologies $\tau_X = \{U \subset X : U = wInt(U)\}$ and $\tau_Y = \{U \subset Y : U = wInt(U)\}$, the following diagram is obtained:

$$\begin{array}{ccccc} & \nearrow & W\text{-continuity} & \searrow & \\ wK\text{-continuity} & \to & wO\text{-continuity} & \to & W\text{-continuity} \\ & & \| & & \| \\ & & \text{continuity} & & Ow\text{-continuity} \end{array}$$

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