

SOME RESULTS FOR CONTINUOUS FUNCTIONS ON WEAK SPACES

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Abstract: In this article, we consider the triple (X, w, τ_X) where w is a weak structure with an associated topology τ . The purpose of this article is to study the relationships among several continuities (continuity, $W(W^*)$ -continuity, wO -continuity, wK -continuity) defined in w -spaces with associated topologies.

In particular, we investigate the new continuity, say Ow -continuity, defined in w -spaces with associated topologies, and study the characterizations of Ow -continuity and the relationships among Ow -continuity and several continuities.

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1. Introduction

In [7], Siwiec introduced the notions of weak neighborhoods and weak base in a topological space. The author introduced the weak neighborhood systems

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defined by using the notion of weak neighborhoods in [4]. The weak neighborhood system induces a weak neighborhood space (briefly WNS) which is independent of neighborhood spaces [2] and general topological spaces [1, 3]. We introduced the notion of weak structure [5] which is defined by the properties of new interior operator and closure operator in a WNS, and investigated the notions of W -continuity and W^* -continuity [5]. In [6], we introduced the notions of WO -continuity and WK -continuity on associated w -spaces. In particular, we investigate some properties and relationships between WO -continuity, WK -continuity, W -continuity, W^* -continuity and continuity on associated w -spaces.

In this paper, for a w -space (X, w) , a topology τ_X on X is called an associated topology on X if $w \subseteq \tau$. we will call the triple (X, w, τ_X) a w -space with an associated topology τ . The purpose of this paper is to study the relationships among several continuities(continuity, $W(W^*)$ -continuity, wO -continuity, wK -continuity) defined in w -spaces with associated topologies. In particular, we investigate the new continuity, say Ow -continuity, defined in w -spaces with associated topologies, and study the characterizations of Ow -continuity and the relationships among Ow -continuity and several continuities.

2. Preliminaries

Definition 2.1 ([5]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

- (1) $\emptyset \in w_X$ and $X \in w_X$.
- (2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a w -space on X . Then $V \in w_X$ is called a w -open set and the complement of a w -open set is a w -closed set.

The collection of all w -open sets (resp. w -closed sets) in a w -space X will be denoted by $WO(X)$ (resp. $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$.

Definition 2.2 ([5]). Let (X, w_X) be a w -space. For a subset A of X , the w -closure of A and the w -interior [5] of A are defined as the following:

- (1) $wCl(A) = \cap \{F : A \subseteq F, X - F \in w_X\}$.
- (2) $wInt(A) = \cup \{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3 ([5]). Let (X, w_X) be a w -space and $A \subseteq X$.

- (1) If A is w -open, then $wInt(A) = A$.
- (2) If A is w -closed, then $wCl(A) = A$.

3. Main Results

Definition 3.1. Let X be a nonempty set and let (X, w) be a w -space. A topology τ_X on X is called an associated topology on X if $w \subseteq \tau$. Then the triple (X, w, τ_X) is called a w -space with an associated topology τ . The triple (X, w, τ_X) simply will be denoted by (X, W_{τ_X}) .

Remark 3.2. Let (X, w_X) be a w -space. Then clearly the family $\tau = \{U \subset X : U = wInt(U)\}$ is a topology and $w_X \subseteq \tau$ [5], and so it is an associated topology on X for a w -space (X, w_X) .

Definition 3.3. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function on w -spaces with two associated topologies τ_X and τ_Y .

Then f is said to be

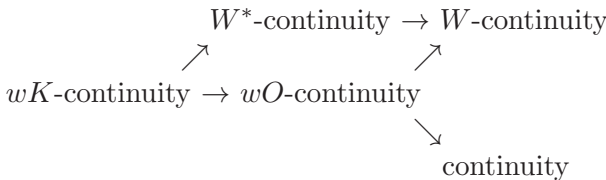
(1) wO -continuous if for $x \in X$ and for each τ_Y -open set V containing $f(x)$, there is a w -open set U containing x such that $f(U) \subseteq V$,

(2) wK -continuous if for every τ_Y -open set V in Y , $f^{-1}(V)$ is a w -open set in X .

(3) W -continuous [5] if for $x \in X$ and $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subset V$.

(4) W^* -continuous [5] if for every $A \in W(f(x))$, $f^{-1}(A)$ is in $W(x)$.

Remark 3.4. For a function $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$, from Definition 3.3 and the next examples, we get the following implications:



Example 3.5. For $X = \{a, b, c\}$, let $w = \{\emptyset, \{a\}, \{b\}, X\}$ be a w -structure.

(1) Let an associated topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Consider the identity function $f : (X, W_{\tau_X}) \rightarrow (X, W_{\tau_X})$. Then f is continuous, W^* -continuous and W -continuous but neither wO -continuous nor wK -continuous.

(2) Consider $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and the identity function $f : (X, W_{\mu_X}) \rightarrow (X, W_{\tau_X})$. Then f is W -continuous but not continuous.

(3) Consider a function $f : (X, W_{\tau_X}) \rightarrow (X, W_{\tau_X})$ defined as $f(b) = a; f(a) = f(c) = b$. Then f is continuous but not W -continuous.

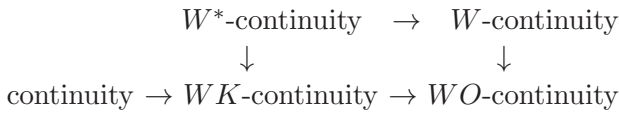
(4) For an associated topology $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, consider the identity function $f : (X, W_{\mu_X}) \rightarrow (X, W_{\mu_X})$. Then f is wO -continuous but not

wK -continuous.

Remark 3.6. Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w_τ of the power set $P(X)$ is called an associated *weak structure* [6] on X if $\tau \subseteq w_\tau$.

Let $f : (X, w_\tau) \rightarrow (Y, \mu)$ be a function on an associated w -space with τ and a topological space (Y, μ) . Then f is said to be

- (1) *WO-continuous* [6] if for $x \in X$ and $V \in O(f(x))$, there is $U \in W(x)$ such that $f(U) \subseteq V$,
- (2) *WK-continuous* [6] if for every open set V in Y , $f^{-1}(V)$ is a w -open set in X .



Theorem 3.7. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. Then the following statements are equivalent:

- (1) f is wO -continuous.
- (2) $f(wCl(A)) \subseteq cl(f(A))$ for $A \subseteq X$.
- (3) $wCl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for $V \subseteq Y$.
- (4) $f^{-1}(int(V)) \subseteq wInt(f^{-1}(V))$ for $V \subseteq Y$

Proof. Obvious. □

Theorem 3.8. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function.

- (1) Then the following are equivalent:
 - (i) f is wK -continuous.
 - (ii) For every closed set F in Y , $f^{-1}(F)$ is w -closed in X .
- (2) If f is wK -continuous, then wO -continuous.

Proof. Obvious. □

Now, we define a new notion of continuous function between an associated w -spaces with two associated topologies.

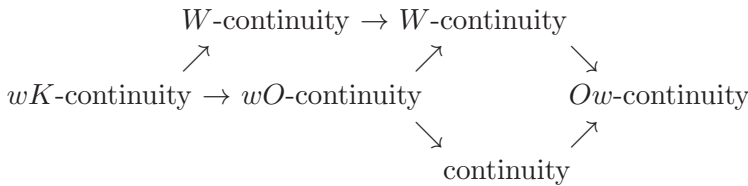
Definition 3.9. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. The function f is said to be *Ow-continuous* if for each w -open set V containing $f(x)$, there is a τ_{w_X} -open set U containing x such that $f(U) \subseteq V$.

Obviously, it is that every W -continuous and every continuous function are *Ow*-continuous but the converses may not be true as the next examples:

Example 3.10. (1) As Example 3.5, for $X = \{a, b, c\}$ and a w -structure $w = \{\emptyset, \{a\}, \{b\}, X\}$, let an associated topology $\nu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Consider a function $f : (X, W_{\nu_X}) \rightarrow (X, W_{\nu_X})$ defined as $f(a) = a; f(b) = f(c) = b$. Then f is Ow -continuous but not W -continuous.

(2) In the above (1), consider $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and the function $f : (X, W_{\mu_X}) \rightarrow (X, W_{\nu_X})$. Then f is Ow -continuous but not continuous.

Remark 3.11. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. Then we get the following implications:



Theorem 3.12. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. Then the following statements are equivalent:

- (1) f is Ow -continuous.
- (2) $f(cl(A)) \subseteq wCl(f(A))$ for $A \subseteq X$.
- (3) $cl(f^{-1}(B)) \subseteq f^{-1}(wCl(B))$ for $B \subseteq Y$.
- (4) $f^{-1}(wInt(V)) \subseteq int(f^{-1}(V))$ for $V \subseteq Y$.
- (5) If for every w -open set V in Y , $f^{-1}(V)$ is an open set in X .

Proof. (1) \Rightarrow (2) For $x \in cl(A)$, assume that $f(x)$ is not in $wCl(f(A))$. Then there exists a w -open set V containing $f(x)$ such that $V \cap f(A) = \emptyset$. By Ow -continuity, there is an open set U containing x such that $f(U) \subseteq V$, and it implies that $f(U) \cap f(A) = \emptyset$ and $U \cap A = \emptyset$. So it is a contradiction and we have $f(cl(A)) \subseteq wCl(f(A))$.

(2) \Rightarrow (3) Let $A = f^{-1}(B)$ for $B \subseteq Y$. Then by hypothesis, $f(cl(A)) \subseteq wCl(f(A)) = wCl(f(f^{-1}(B))) \subseteq wCl(B)$. Thus $cl(f^{-1}(B)) \subseteq f^{-1}(wCl(B))$.

(3) \Rightarrow (4) It is obvious.

(4) \Rightarrow (5) Let V be a w -open set V in Y . Then $V = wInt(V)$ and by hypothesis, $f^{-1}(V) = f^{-1}(wInt(V)) \subseteq int(f^{-1}(V))$, and so $f^{-1}(V)$ is open in X .

(5) \Rightarrow (1) For $x \in X$, let V be any w -open set containing $f(x)$. Then by (5), it follows that $f^{-1}(V)$ is an open set containing x and $x \in f^{-1}(V) \subseteq int(f^{-1}(V))$. From this fact, there exists an open set U such that $x \in U \subseteq f^{-1}(V)$. It implies that f is Ow -continuous. □

We recall Remark 3.2: Let (X, w) be a w -space. Then $\tau_X = \{U \subset X : U = wInt(U)\}$ is an associated topology such that $w \subseteq \tau_X$.

Theorem 3.13. *Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. If the two associate topologies $\tau_X = \{U \subset X : U = wInt(U)\}$ and $\tau_Y = \{U \subset Y : U = wInt(U)\}$, then we get the following things:*

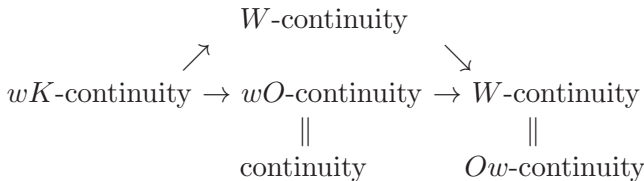
- (1) wO -continuity = continuity:
- (2) W -continuity = Ow -continuity:

Proof. (1) Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. From Remark 3.11, it is sufficient to show that the continuity implies wO -continuity. Suppose that f is continuous. For $x \in X$, let V be any τ_Y -open set containing $f(x)$. By continuity of f , there is an open set U containing x such that $f(U) \subseteq V$, and since $\tau_X = \{U \subset X : U = wInt(U)\}$, there exists a w -open set U_x containing x such that $U_x \subseteq U$. It implies f is wO -continuous.

(2) It is similar to the proof of (1). □

In summary, from the above theorem, the following diagram is obtained:

Remark 3.14. Let $f : (X, W_{\tau_X}) \rightarrow (Y, W_{\tau_Y})$ be a function. For the two associate topologies $\tau_X = \{U \subset X : U = wInt(U)\}$ and $\tau_Y = \{U \subset Y : U = wInt(U)\}$, the following diagram is obtained:



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