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SOLITARY SOLUTION OF A CLASS OF NONLINEAR TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: This paper presents a general solitary solution of a class of nonlinear time-fractional partial differential equations by Adomian decomposition method(ADM). This class of nonlinear time-fractional partial differential equations include a lot of standard nonlinear partial differential equations in mathematical physics. The solitary solution obtained by ADM is a general solitary solution and admit you investigate the solution for different initial conditions and different α(α is the order of derivative respect to time $0 < \alpha \leq 1$). Also the solution subject to the especial initial conditions and $\alpha = 1$ reduce to the solution of standard partial differential equation. Additionally, it use the fractional derivative of Caputo sense.

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1. Introduction

Nonlinear wave phenomena appear in a wide variety of scientific applications such as fluid mechanics, plasma physics, biology, hydrodynamics and another

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branch of sciences. They are described by wave dispersive equation. Therefore, the solitary solution of such equations is very important.

The soliton is defined as a nonlinear wave that has the following properties [8]:

• A localized wave propagates without change of its properties(shape, velocity, etc).

• Localized waves are stable against mutual collision and retain their identities. This means that soliton has the property of a particle.

In mathematical physics several nonlinear wave equation are proposed as wave dispersive equation, such as nonlinear Schrodinger equation(NLSe), Burger equation, Fisher equation and etc. The solitary solution of each equation is obtained by special method such Tanh method, variational iteration method and etc [1, 3, 4, 2, 5, 7].

This research work presents a general solitary solution for a class of nonlinear wave equation. This class of nonlinear wave equation include some of special nonlinear wave equations in mathematical physics such as Fisher equation, Burgers equation, nonlinear Schrodinger equation and etc. The obtained solitary solution by ADM for $\alpha = 1$ reduce to the solution of special nonlinear equation by another methods.

2. Solitary Solution

We consider a nonlinear time-fractional partial differential equation as

$$a_1\ D_t^{\alpha} u = a_2 u_{xx} + a_3 u_x + a_4 u + f(u, u_x, \bar{u}) \tag{1}$$

in which $f(u, u_x, \bar{u}) = b_1 u^2 + b_2 u u_x + b_3 |u|^2 u$ and $a_i, b_i \quad i = 1, 2, ...$; are complex numbers, $u = u(x, t)$, $\bar{u}$ is complex conjugate of u, $D_t^{\alpha} u = \frac{\partial^{\alpha}}{\partial t^{\alpha}} u, \quad 0 < \alpha \le 1$, $u_x = \frac{\partial}{\partial x} u$ and $u_{xx} = \frac{\partial^2}{\partial x^2} u$. If we choose $\xi = k(x - vt)$, after the change of this variable [6], the eq. (1) rewrite as

$$a_1 e^{i\pi\alpha} (kv)^{\alpha} \frac{d^{\alpha} U}{d\xi^{\alpha}} = a_2 k^2 \frac{d^2 U}{d\xi^2} + a_3 k \frac{dU}{d\xi} + a_4 U(\xi) + f(U, U', \bar{U}) \tag{2}$$

in which $f(U, U', \bar{U}) = b_1 U^2 + b_2 k U U' + b_3 |U|^2 U$. The equation (2) is an ordinary fractional differential equation and

$$a_2 k^2 \frac{d^2 U}{d\xi^2} = a_1 e^{i\pi\alpha} (kv)^{\alpha} \frac{d^{\alpha} U}{d\xi^{\alpha}} - a_3 k \frac{dU}{d\xi} - a_4 U(\xi) - f(U, U', \bar{U}). \tag{3}$$

By using the Adomian decomposition technique we have

$$U(\xi)=\sum_{n=0}^{\infty}U_n(\xi) \tag{4}$$

substituting the decomposition series (4) into (3) yield

$$\begin{aligned}\sum_{n=0}^{\infty}U_n &= U(0)+U'(0)\xi+\frac{a_1}{a_2k^2}e^{i\pi\alpha}(kv)^{\alpha}L^{-1}\sum_{n=0}^{\infty}\frac{d^{\alpha}U_n}{d\xi^{\alpha}}\\ &-\frac{a_3}{a_2k}L^{-1}\sum_{n=0}^{\infty}\frac{dU_n}{d\xi}-\frac{a_4}{a_2k^2}L^{-1}\sum_{n=0}^{\infty}U_n-\frac{1}{a_2k^2}L^{-1}\sum_{n=0}^{\infty}A_n,\end{aligned} \tag{5}$$

in which $L^{-1}=\int_0^{\xi}\int_0^{\xi}(.)d\xi d\xi$ and A_n is called Adomian's polynomials of equation (2) that we make some of them

$$\begin{aligned}A_0 &= b_1U_0^2+b_2kU_0U_0'+b_3|U_0|^2U_0\\ A_1 &= b_1(2U_0U_1)+b_2k(U_0U_1'+U_1U_0')+b_3(2|U_0|^2U_1+U_0^2\bar{U}_1)\\ A_2 &= b_1(U_1^2+2U_0U_2)+b_2k(U_1U_1'+U_2U_0'+U_0U_2')\\ &+b_3(2|U_0|^2U_2+U_1^2\bar{U}_0+2|U_1|^2U_0+U_0^2\bar{U}_2)\\ A_3 &= b_1(2U_1U_2+2U_0U_3)+b_2(U_1U_2'+U_2U_1'+U_3U_0'+U_0U_3')\\ &+b_3(U_1^2\bar{U}_1+U_0^2\bar{U}_3+2|U_0|^2U_3+2\bar{U}_0U_1U_2+2U_0\bar{U}_1U_2+2U_0U_1\bar{U}_2)\end{aligned} \tag{6}$$

the rest of polynomials can derive in the same manner[9]. From the eq. (5), the iterates are defined by the following recursive way

$$\begin{aligned}U_0 &= U(0)+U'(0)\xi\\ U_1 &= \frac{a_1}{a_2k^2}e^{i\pi\alpha}(kv)^{\alpha}L^{-1}\frac{d^{\alpha}U_0}{d\xi^{\alpha}}-\frac{a_3}{a_2k}L^{-1}\frac{dU_0}{d\xi}-\frac{a_4}{a_2k^2}L^{-1}U_0\\ &-\frac{1}{a_2k^2}L^{-1}A_0\\ U_2 &= \frac{a_1}{a_2k^2}e^{i\pi\alpha}(kv)^{\alpha}L^{-1}\frac{d^{\alpha}U_1}{d\xi^{\alpha}}-\frac{a_3}{a_2k}L^{-1}\frac{dU_1}{d\xi}-\frac{a_4}{a_2k^2}L^{-1}U_1\\ &-\frac{1}{a_2k^2}L^{-1}A_1\\ U_3 &= \frac{a_1}{a_2k^2}e^{i\pi\alpha}(kv)^{\alpha}L^{-1}\frac{d^{\alpha}U_2}{d\xi^{\alpha}}-\frac{a_3}{a_2k}L^{-1}\frac{dU_2}{d\xi}-\frac{a_4}{a_2k^2}L^{-1}U_2\\ &-\frac{1}{a_2k^2}L^{-1}A_2\end{aligned} \tag{7}$$

After the computing U_n $\quad n = 1, 2, ...,$ we have a solitary solution of as

$$u(x,t) = \sum_{n=0}^{\infty} U_n(k(x - vt)) \tag{8}$$

The terms U_1, U_2, U_3, ... can be computed by Mathematica software.

In the next section we apply for several time-fractional partial differential equations in mathematical physics.

3. Examples

3.1. Time-Fractional Burgers Equation

If we choose $a_1 = 1,\ a_2 = \epsilon,\ a_3 = 0,\ a_4 = 0,\ b_1 = 0,\ b_2 = -1,\ b_3 = 0$ the eq. (1) reduce to time fractional Burgers equation as

$${}_0^C D_t^{\alpha} u(x,t) = \epsilon\ u_{xx}(x,t) - u(x,t)u_x(x,t) \tag{9}$$

and general solitary solution of time fractional Burgers equation

$$U(\xi) = U_0 + U_1 + U_2 + ... \tag{10}$$

in which

$$\begin{aligned} U_0 =& U'(0)\xi + U(0), \\ U_1 =& \frac{U(0)U'(0)}{2\epsilon k}\xi^2 + \frac{U'(0)^2}{6\epsilon k}\xi^3 + e^{i\pi\alpha}(kv)^{\alpha}\frac{U'(0)}{\epsilon k^2(3-\alpha)!}\xi^{3-\alpha}, \\ U_2 =& \frac{U(0)^2U'(0)}{6\epsilon^2k^2}\xi^3 + \frac{U(0)U'(0)^2}{6\epsilon^2k^2}\xi^4 + \frac{U'(0)^3}{30\epsilon^2k^2}\xi^5 + e^{i\pi\alpha}(kv)^{\alpha}\frac{2U(0)U'(0)}{\epsilon^2k^3(4-\alpha)!}\xi^{4-\alpha} \\ &+ e^{i\pi\alpha}(kv)^{\alpha}\frac{U'(0)^2}{\epsilon^2k^3(4-\alpha)!}\xi^{5-\alpha} + e^{2i\pi\alpha}(kv)^{2\alpha}\frac{U'(0)}{\epsilon^2k^4(5-2\alpha)!}\xi^{5-2\alpha}. \end{aligned} \tag{11}$$

If we choose $U(0) = 2k\epsilon,\ U'(0) = -2k\epsilon,\ v = 2k\epsilon$ and $\alpha = 1$, the solitary solution by ADM reduce to

$$U(\xi) = 2k\epsilon[1 - \tanh(\xi)], \qquad \xi = k(x - 2k\epsilon t) \tag{12}$$

or $U(0) = -2k\epsilon,\ U'(0) = -2k\epsilon,\ v = -2k\epsilon$ and $\alpha = 1$, the solitary solution by ADM reduce to

$$U(\xi) = 2k\epsilon[-1 - \tanh(\xi)], \qquad \xi = k(x + 2k\epsilon t) \tag{13}$$

that compare accurately by Tanh method [3, 4].

3.2. Time-Fractional Nonlinear Schrodinger Equation

If we choose $a_1 = i,\ a_2 = -\frac{1}{2},\ a_3 = 0,\ a_4 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = -1$ the eq. (1) reduce to special form of time fractional NLS equation as

$$i\ {}_0^C D_t^\alpha u(x,t) = -\frac{1}{2}u_{xx}(x,t) - |u(x,t)|^2 u(x,t) \tag{14}$$

the general solitary solution of NLSe is

$$U(\xi) = U_0 + U_1 + U_2 + ... \tag{15}$$

in which

$$\begin{aligned}
U_0 =& U'(0)\xi + U(0),\\
U_1 =& -\frac{U^2(0)\bar{U}(0)}{k^2}\xi^2 - \frac{2U(0)U^\star(0)U'(0)}{3k^2}\xi^3 - \frac{U^2(0)\bar{U}'(0)}{3k^2}\xi^3 - \frac{U^\star(0)U'^2(0)}{6k^2}\xi^4\\
&- \frac{U(0)U'(0)U'^\star(0)}{3k^2}\xi^4 - \frac{U'^2(0)\bar{U}'(0)}{10k^2}\xi^5 - \frac{2iU'(0)e^{i\pi\alpha}(kv)^\alpha}{k^2(3-\alpha)!}\xi^{3-\alpha},\\
U_2 =& \frac{U^3(0)\bar{U}^2(0)}{2k^4}\xi^4 + \frac{U^2(0)\bar{U}^2(0)U'(0)}{30c^4}\xi^5 + \frac{U^3(0)\bar{U}(0)\bar{U}'(0)}{3k^4}\xi^5\\
&+ \frac{2U(0)\bar{U}^2(0)U'^2(0)}{9k^4}\xi^6 + \frac{19U^2(0)\bar{U}(0)U'(0)\bar{U}'(0)}{45k^4}\xi^6 + \frac{U^3(0)\bar{U}'^2(0)}{18k^4}\xi^6\\
&+ \frac{2\bar{U}^2(0)U'^3(0)}{63k^4}\xi^7 + \frac{58U(0)\bar{U}(0)U'^2(0)\bar{U}'(0)}{315k^4}\xi^7 + \frac{53U^2(0)U'(0)\bar{U}'^2(0)}{630k^4}\xi^7\\
&+ \frac{13\bar{U}(0)U'^3(0)U'(0)}{420k^4}\xi^8 + \frac{37U(0)U'^2(0)\bar{U}'^2(0)}{840k^4}\xi^8 + \frac{U'^3(0)\bar{U}'^2(0)}{120k^4}\xi^9\\
&+ \frac{4iU^2(0)\bar{U}(0)e^{i\pi\alpha}(kv)^\alpha}{k^4(4-\alpha)!}\xi^{4-\alpha} - \frac{4iU^2(0)\bar{U}'(0)e^{-i\pi\alpha}(kv)^\alpha}{k^4(5-\alpha)!}\xi^{5-\alpha}\\
&+ \frac{4iU(0)(4\bar{U}(0)U'(0) + U(0)\bar{U}'(0))e^{i\pi\alpha}(kv)^\alpha}{k^4(5-\alpha)!}\xi^{5-\alpha}\\
&+ \frac{8iU(0)U'(0)\bar{U}'(0)e^{-i\pi\alpha}(kv)^\alpha(\alpha-4)}{k^4(6-\alpha)!}\xi^{6-\alpha} -\\
&\frac{8iU'(0)e^{i\pi\alpha}(kv)^\alpha(U(0)\bar{U}'(0)\alpha + \bar{U}(0)U'(0)\alpha - 6U(0)\bar{U}'(0) - 5\bar{U}(0)U'(0))}{k^4(6-\alpha)!}\xi^{6-\alpha}\\
&- \frac{4iU'^2(0)\bar{U}'(0)e^{-i\pi\alpha}(kv)^\alpha(\alpha-5)(\alpha-4)}{k^4(7-\alpha)!}\xi^{7-\alpha}\\
&+ \frac{8iU'^2(0)\bar{U}'(0)e^{i\pi\alpha}(kv)^\alpha(\alpha^2-9\alpha+23)}{k^4(7-\alpha)!}\xi^{7-\alpha} - \frac{4U'(0)e^{2i\pi\alpha}(kv)^{2\alpha}}{c^4(5-2\alpha)!}\xi^{5-2\alpha}.
\end{aligned} \tag{16}$$

If we choose $U(0) = k,\ U'(0) = -ik,\ v = \frac{1}{2}k$, and $\alpha = 1$, the solitary solution by ADM reduce to

$$U(\xi) = ke^{-i\xi}, \qquad \xi = k(x - \frac{k}{2}t). \tag{17}$$

Also $U(0) = k,\ U'(0) = ik,\ v = -\frac{1}{2}k$ and $\alpha = 1$, the solitary solution by ADM reduce to

$$U(\xi) = ke^{i\xi}, \qquad \xi = k(x + \frac{k}{2}t) \tag{18}$$

that it is coincide to solution of NLSe subject to initial conditions [7].

4. Conclusion

In this research work present a solution for a class of nonlinear time-fractional differential equation. This solitary solution admit you investigate the solution for different initial conditions and different α.

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