International Journal of Pure and Applied Mathematics
Volume 109 No. 3 2016, 573-582
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v109i3.7

SOME FIXED POINT RESULTS IN FUZZY CONE METRIC SPACES

N. Priyobarta[1 §], Yumnam Rohen[2], B.B. Upadhyay[3]
[1,2,3]Department of Mathematics
NIT Manipur
Imphal, 795001, INDIA

Abstract: In this manuscript, we prove some common fixed point theorems for occasionally weakly compatible mappings in fuzzy cone metric space.

AMS Subject Classification: 46S40, 54H25
Key Words: t-norm, fuzzy cone metric space, occasionally weakly compatible mappings

1. Introduction

The concept of fuzzy set was introduced by L.A. Zadeh [1] in 1965. After that many authors developed the theory of fuzzy sets and its applications. Kramosil and Michaleck [2] introduced fuzzy metric space in 1975. In 1999, R. Vasuki [3] proved fixed point theorems for R-weakly commutating mappings. In 2007, Huang and Zhang [4] introduced the concept of cone metric space and proved some fixed point theorems for contractive mappings. In 2015, Tarkan Oner et al. [5] itroduced the concept of fuzzy cone metric space that generalized the corresponding notions of fuzzy metric space by George and Veeramani [6] and proved the fuzzy cone Banach contraction theorem. In recent past, several authors proved various fixed point theorems employing more generalized conditions.

Received: May 2, 2016
Revised: July 14, 2016
Published: September 30, 2016

§Correspondence author

The aim of this paper is to prove some common fixed point theorems for occasionally weakly compatible mappings in fuzzy cone metric space.

2. Preliminary Notes

In this section, we start with some definitions.

Definition 2.1. [1] Let X be any set. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 2.2. [5] Let E be a real Banach space, θ the zero of E and P a subset of E. Then P is called a cone if and only if

1. P is closed, nonempty and $P \neq \{\theta\}$,
2. if $a, b \in R$, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$,
3. if both $x \in P$ and $-x \in P$, then $x = \theta$.

For a given cone P, a partial ordering $\preceq$ on E with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$. Throughout this paper, we assume that all cones have nonempty interior.

A cone P is called normal if there exists a constant $K > 0$ such that for all $t, s \in E$, $\theta \preceq t \preceq s$ implies $\|t\| \leq K\|s\|$ and the least positive number K satisfying this property is called normal constant of P [5]. Rezapour and Hamlbarani [7] showed that there is no cone with normal constant $K < 1$ and there exist cones of normal constant 1, and cones of normal constant $M > K$ for each $K > 1$.

Definition 2.3. [8] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions;

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Example 2.4. $a * b = \min\{a, b\}$.

Definition 2.5. [6] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions;
FM1) $M(x, y, t) > 0$,
FM2) $M(x, y, t) = 1$ if and only if $x = y$,
FM3) $M(x, y, t) = M(y, x, t)$,
FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
FM5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous, $x, y, z \in X$ and $t, s > 0$.

Definition 2.6. [6] Let $(X, M, *)$ be a fuzzy metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

1. $\{x_n\}$ is said to converge to x if for any $t > 0$ and any $r \in (0, 1)$ there exists a natural number n_0 such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

2. $\{x_n\}$ is said to be a Cauchy sequence if for any $r \in (0, 1)$ and any $t > 0$ there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$.

3. $(X, M, *)$ is said to be a complete metric space if every Cauchy sequence is convergent.

Definition 2.7. [5] A 3-tuple $(X, M, *)$ is said to be a fuzzy cone metric space if P is a cone of E, X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times int(P)$ satisfying the following conditions;
For all $x, y, z \in X$ and $t, s \in int(P)$ (that is $t \gg \theta$, $s \gg \theta$)
FCM1) $M(x, y, t) > 0$,
FCM2) $M(x, y, t) = 1$ if and only if $x = y$,
FCM3) $M(x, y, t) = M(y, x, t)$,
FCM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
FCM5) $M(x, y, .) : int(P) \to [0, 1]$ is continuous.

If $E = \mathbb{R}$, $P = [0, \infty)$ and $a * b = ab$, then every fuzzy metric spaces became a fuzzy cone metric spaces.

Example 2.8. [5] Let $E = \mathbb{R}^2$. Then $P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \subset E$ is a normal cone with normal constant $K = 1$. Let $X = \mathbb{R}$, $a * b = ab$ and $M : X^2 \times int(P) \to [0, 1]$ defined by $M(x, y, t) = \frac{1}{e^{\frac{|x-y|}{\|t\|}}}$ for all $x, y \in X$ and $t \gg \theta$.

Definition 2.9. [5] Let $(X, M, *)$ be a fuzzy cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

1. $\{x_n\}$ is said to converge to x if for any $t \gg \theta$ and any $r \in (0,1)$ there exists a natural number n_0 such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

2. $\{x_n\}$ is said to be a Cauchy sequence if for any $0 < \varepsilon < 1$ and any $t \gg \theta$ there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

3. $(X, M, *)$ is called complete if every Cauchy sequence is convergent.

Definition 2.10. [5] Let $(X, M, *)$ be a fuzzy cone metric space. A subset A of X is said to be FC-bounded if there exists $t \gg \theta$ and $r \in (0,1)$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Definition 2.11. [9] Let X be a nonempty set. An element $x \in X$ is called a common fixed point of mappings $F : X \times X \to X$ and $T : X \to X$ if $x = T(x) = F(x, x)$.

Definition 2.12. [9] Let X be a nonempty set. The mappings $F : X \times X \to X$ and $T : X \to X$ are called commutative if $T(F(x, y)) = F(T(x), T(y))$ for all $x, y \in X$.

Definition 2.13. Let X be a set, F, T selfmaps of X. A point x in X is called a coincidence point of F and T if and only if $F(x) = T(x)$. We shall call $w = F(x) = T(x)$ a point of coincidence of F and T.

Definition 2.14. A pair of maps U and V is called weakly compatible pair if they commute at coincidence points.

Definition 2.15. [10] Two self maps F and T of a set X are occasionally weakly compatible if and only if there is a point x in X which is a coincidence point of F and T at which F and T commute.

A. Al-Thagafi and Naseer Shahzad [11] shown that occasionally weakly is weakly compatible but converse is not true.

Lemma 2.16. [12] Let X be a set, F, T occasionally weakly compatible self maps of X. If F and T have a unique point of coincidence, $w = F(x) = T(x)$, then w is the unique common fixed point of F and T.

3. Main Results

We have the following theorems.

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$\begin{aligned} M(Ax, By, k(t)) \geq & \min\{M(U(x), V(y), t), M(U(x), A(x), t), \\ & M(B(y), V(y), t), M(A(x), V(y), t), \\ & M(B(y), U(x), t)\} \end{aligned} \tag{1}$$

for all $x, y \in X$ and for all $t \gg \theta$, then there exists a unique point $w \in X$ such that $A(w) = U(w) = w$ and a unique point $z \in X$ such that $B(z) = V(z) = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, U and V.

Proof. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible, so there are points $x, y \in X$ such that $A(x) = U(x)$ and $B(y) = V(y)$. We claim that $A(x) = B(y)$. If not, by inequality (1)

$$\begin{aligned} M(A(x), B(y), k(t)) \geq & \min\{M(U(x), V(y), t), M(U(x), A(x), t), \\ & M(B(y), V(y), t), M(A(x), V(y), t), \\ & M(B(y), U(x), t)\} \\ = & \min\{M(A(x), B(y), t), M(A(x), A(x), t), \\ & M(B(y), B(y), t), M(A(x), B(y), t), \\ & M(B(y), A(x), t)\} \\ = & M(A(x), B(y), t). \end{aligned}$$

Therefore $A(x) = B(y)$, i.e. $A(x) = U(x) = B(y) = V(y)$. Suppose that there is another point z such that $A(z) = U(z)$ then by (1) we have $A(z) = U(z) = B(y) = V(y)$, so $A(x) = A(z)$ and $w = A(x) = U(x)$ is the unique point of coincidence of A and U. By Lemma 2.16, w is the only common fixed point of A and U. Similarly there is a unique point $z \in X$ such that $z = B(z) = V(z)$.

Assume that $w \neq z$. We have

$$\begin{aligned} M(w, z, k(t)) = & M(A(w), B(z), k(t)) \\ \geq & \min\{M(U(w), V(z), t), M(U(w), A(z), t), \\ & M(B(z), V(z), t), M(A(w), V(z), t), M(B(z), U(w), t)\} \\ = & \min\{M(w, z, t), M(w, z, t), M(z, z, t), \\ & M(w, z, t), M(z, w, t)\} \end{aligned}$$

$$= \quad M(w,z,t).$$

Therefore we have $z = w$ by Lemma 2.16 and z is a common fixed point of A, B, U and V. The uniqueness of the fixed point holds from (1). □

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible. If there exists $k \in (0,1)$ such that*

$$\begin{aligned} M(A(x), B(y), k(t)) \geq\ & \phi[\min\{M(U(x), V(y), t), M(U(x), A(x), t), \\ & M(B(y), V(y), t), M(A(x), V(y), t), \\ & M(B(y), U(x), t)\}] \end{aligned} \tag{2}$$

for all $x, y \in X$ and $\phi : [0,1] \to [0,1]$ such that $\phi(t) > t$ for all $\theta \ll t < 1$, then there exists a unique common fixed point of A, B, U and V.

Proof. The proof follows from Theorem 3.1. □

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible. If there exists $k \in (0,1)$ such that*

$$\begin{aligned} M(A(x), B(y), k(t)) \geq\ & \phi(M(U(x), V(y), t), M(U(x), A(x), t), \\ & M(B(y), V(y), t), M(A(x), V(y), t), \\ & M(B(y), U(x), t)) \end{aligned} \tag{3}$$

for all $x, y \in X$ and $\phi : [0,1]^5 \to [0,1]$ such that $\phi(t,1,1,t,t) > t$ for all $\theta \ll t < 1$, then there exists a unique common fixed point of A, B, U and V.

Proof. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible, there are points $x, y \in X$ such that $A(x) = U(x)$ and $B(y) = V(y)$. We claim that $A(x) = B(y)$. By inequality (3) we have

$$\begin{aligned} M(A(x), B(y), k(t)) \geq\ & \phi(M(U(x), V(y), t), M(U(x), A(x), t), \\ & M(B(y), V(y), t), M(A(x), V(y), t), \\ & M(B(y), U(x), t)) \\ =\ & \phi(M(A(x), B(y), t), M(A(x), A(x), t), \\ & M(B(y), B(y), t), M(A(x), B(y), t), \\ & M(B(y), A(x), t)) \\ =\ & \phi(M(A(x), B(y), t), 1, 1, M(A(x), B(y), t), \end{aligned}$$

$$\begin{aligned} & M(B(y), A(x), t)) \\ > \; & M(A(x), B(y), t) \end{aligned}$$

a contradiction, therefore $A(x) = B(y)$, i.e. $A(x) = U(x) = B(y) = V(y)$. Suppose that there is a another point z such that $A(z) = U(z)$ then by (3) we have $A(z) = U(z) = B(y) = V(y)$, so $A(x) = A(z)$ and $w = A(x) = V(x)$ is the unique point of coincidence of A and U . By Lemma 2.16, w is a unique common fixed point of A and U. Similarly there is a unique point $z \in X$ such that $z = B(z) = V(z)$. Thus z is a common fixed point of A, B, U and V . The uniqueness of the fixed point holds from (3). □

Theorem 3.4. *Let $(X, M, *)$ be a complete fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible. If there exists a point $k \in (0, 1)$ for all $x, y \in X$ and $t \gg \theta$ satisfying*

$$\begin{aligned} M(A(x), B(y), k(t)) \;\geq\; & M(U(x), V(y), t) * M(A(x), U(x), t) \\ & * M(B(y), V(y), t) * M(A(x), V(y), t) \end{aligned} \tag{4}$$

then there exists a unique common fixed point of A, B, U and V .

Proof. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible, there are points $x, y \in X$ such that $A(x) = U(x)$ and $B(y) = V(y)$. We claim that $A(x) = B(y)$. By inequality (4), we have

$$\begin{aligned} M(A(x), B(y), k(t)) \;\geq\; & M(U(x), B(y), t) * M(A(x), U(x), t) \\ & * M(By, B(y), t) * M(A(x), B(y), t) \\ = \; & M(A(x), B(y), t) * M(A(x), A(x), t) \\ & * M(B(y), B(y), t) * M(A(x), B(y), t) \\ \geq \; & M(A(x), B(y), t) * 1 * 1 * M(A(x), B(y), t) \\ \geq \; & M(A(x), B(y), t) \end{aligned}$$

Thus we have $A(x) = B(y)$, i.e. $A(x) = U(x) = B(y) = V(y)$. Suppose that there is a another point z such that $A(z) = U(z)$ then by (4) we have $A(z) = U(z) = B(y) = V(y)$, so $A(x) = A(z)$ and $w = A(x) = U(x)$ is the unique point of coincidence of A and U. Similarly there is a unique point $z \in X$ such that $z = B(z) = V(z)$. Thus w is a common fixed point of A, B, U and V . □

Corollary 3.5. *Let $(X, M, *)$ be a complete fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible. If there exists a point $k \in (0, 1)$ for all $x, y \in X$ and $t \gg \theta$ satisfying*

$$\begin{aligned} M(A(x), B(y), k(t)) \;\geq\; & M(U(x), V(y), t) * M(A(x), U(x), t) \\ & * M(B(y), V(y), t) * M(B(y), U(x), 2t) \\ & * M(A(x), V(y), t) \qquad (5) \end{aligned}$$

then there exists a unique common fixed point of A, B, U and V.

Proof. We have

$$\begin{aligned} M(A(x), B(y), k(t)) \;\geq\; & M(U(x), V(y), t) * M(A(x), U(x), t) \\ & * M(B(y), V(y), t) * M(B(y), U(x), 2t) \\ & * M(A(x), V(y), t) \\ \geq\; & M(U(x), V(y), t) * M(A(x), U(x), t) \\ & * M(B(y), V(y), t) * M(U(x), V(y), t) \\ & * M(Ty, B(y), t) * M(A(x), V(y), t) \\ \geq\; & M(U(x), V(y), t) * M(A(x), U(x), t) \\ & * M(B(y), V(y), t) * M(A(x), V(y), t) \end{aligned}$$

and therefore from Theorem 3.4, A, B, U and V have a common fixed point. □

Theorem 3.6. *Let $(X, M, *)$ be a complete fuzzy cone metric space. Then continuous self mappings U and V of X have a common fixed point in X if and only if there exists a self mapping A of X such that the following conditions are satisfied*

1. *$AX \subset VX \cap UX$,*
2. *the pairs $\{A, U\}$ and $\{B, V\}$ are weakly compatible,*
3. *there exists a point $k \in (0, 1)$ such that for every $x, y \in X$ and $t \gg \theta$*

$$\begin{aligned} M(A(x), A(y), k(t)) \;\geq\; & M(U(x), V(y), t) * M(A(x), U(x), t) * \\ & M(A(y), V(y), t) * M(A(x), V(y), t) \qquad (6) \end{aligned}$$

Then A, U and V have a unique common fixed point.

Proof. Since compatible implies occasionally weakly compatible, the result follows from Theorem 3.4. $\square$

Theorem 3.7. *Let $(X, M, *)$ be a complete fuzzy cone metric space and let A and B be self-mappings of X. Let the A and B are occasionally weakly compatible. If there exists a point $k \in (0,1)$ for all $x, y \in X$ and $t \gg \theta$*

$$\begin{aligned} M(U(x), U(y), k(t)) \quad &\geq \quad \alpha M(A(x), A(y), t) + \beta \min\{M(A(x), A(y), t), \\ &\quad\; M(U(x), A(x), t), M(U(y), A(y), t)\} \end{aligned} \tag{7}$$

for all $x, y \in X$, where $\alpha, \beta > 0$, $\alpha + \beta > 1$. Then A and U have a unique common fixed point.

Proof. Let the pairs $\{A, U\}$ be occasionally weakly compatible, so there is a point $x \in X$ such that $A(x) = U(x)$. Suppose that there exist another point $y \in X$ for which $A(y) = U(y)$. We claim that $U(x) = U(y)$. By inequality (7) we have

$$\begin{aligned} M(U(x), U(y), k(t)) \quad &= \quad \alpha M(A(x), A(y), t) + \beta \min\{M(A(x), A(y), t), \\ &\quad\; M(U(x), A(x), t), M(U(y), A(y), t)\} \\ &= \quad \alpha M(U(x), U(y), t) + \beta \min\{M(U(x), U(y), t), \\ &\quad\; M(U(x), U(x), t), M(U(y), U(y), t)\} \\ &= \quad (\alpha + \beta) M(U(x), U(y), t) \end{aligned}$$

a contradiction, since $(\alpha + \beta) > 1$. Therefore $U(x) = U(y)$. Therefore $A(x) = A(y)$ and $A(x)$ is unique. From Lemma 2.16, A and U have a unique fixed point. $\square$

References

[1] L. A. Zadeh, *Fuzzy sets*, Information and Computation, **8** (1965), 338-353.

[2] I. Kramosil and J. Michlek, *Fuzzy metric and statistical metric spaces*, Kybernetika, **11** (1975), 336-344.

[3] R. Vasuki, *Common fixed points for R-weakly commuting maps in fuzzy metric spaces*, Indian J. Pure Appl. Math. **30** (1999), 419-423.

[4] L. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468-1476.

[5] T. Oner, M.B. Kandemir and B. Tanay, *Fuzzy cone metric spaces*, J. Nonlinear Sci. and Appl. **8** (2015), 610-616.

[6] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst., **64** (1994), 395-399.

[7] S. Rezapour and R. Hamlbarani, *Some notes on paper Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **345** (2008), 719-724.

[8] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Maths., **10** (1960), 313-334.

[9] J. X. Fang, *Common fixed point theorems of compatible and weakly compatible maps in Menger spaces*, Nonlinear Analysis, Theory, Method and Applications, **71** (2009), 1833-1843.

[10] G. Jungck and B. E. Rhoades, *Fixed Point for Set Valued functions without Continuity*, Indian J. Pure Appl. Math., **29** (1998), 771-779.

[11] A. Al-Thagafi and Naseer Shahzad, *Generalized I-Nonexpansive Selfmaps and Invariant Approximations*, Acta Mathematica Sinica, English Series, **24** (2008), 867-876.

[12] G. Jungck and B. E. Rhoades, *Fixed Point Theorems for Occasionally Weakly compatible Mappings*, Fixed Point Theory, **7** (2006), 287-296.