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$\psi - \phi$ CONTRACTION ON SUZUKI TYPE UNIQUE COMMON COUPLED FIXED POINT THEOREM IN PARTIALLY ORDERED MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we obtain a Suzuki type unique common coupled fixed point theorem by using $\psi - \phi$ contraction in partially ordered multiplicative metric spaces. We also give an example to illustrate our main theorem.

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1. Introduction

The notion of multiplicative metric was introduced by Bashirov et al. [3] in 2008. After that Ozavsar and Cevikel[8] investigated its topological properties and proved some fixed point theorems. For more works on fixed, common fixed

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point theorems in Multiplicative metric spaces, we refer [5, 2, 6, 9].

The Coupled fixed point is introduced by Bhaskar and Lakshmikantham [4]. Later some of authors proved coupled fixed and coupled common fixed point theorems (See [1, 7, 10, 11, 12]).

The aim of this paper is to prove Suzuki type unique common coupled fixed point theorem for Jungck type maps by using $\psi - \phi$ contraction condition in partially ordered multiplicative metric spaces.

First we give the following theorem of Suzuki [13].

Theorem 1.1. *(See [13])***:** *Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to \left(\frac{1}{2}, 1\right]$ by*

$$\theta(r) = \begin{cases} 1 & if 0 \le r \le \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2} & if \frac{(\sqrt{5}-1)}{2} \le r \le 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & if 2^{-\frac{1}{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim\limits_{n\to\infty} T^n x = z$ for all $x \in X$.

Now we recall some basic definitions and examples in multiplicative metric spaces.

Definition 1.2. (See[3]) Multiplicative metric on a nonempty set X is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(d_1) $d(x, y) \ge 1$ for all $x, y \in X$

(d_2) $d(x, y) = 1$ if and only if $x = y$,

(d_3) $d(x, y) = d(y, x)$,

(d_4) $d(x, y) \le d(x, z).d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) is called a multiplicative metric space.

Example 1.3. (See[9]) Let R^n_+ be the collection of all n - tupples of positive real numbers. And let $d^* : R^n_+ \times R^n_+ \to R$ be defined as

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*$$

where $x = (x_1, x_2 \cdots x_n), y = (y_1, y_2 \cdots y_n) \in R_+^n$ and $|.|^* : R_+ \to R_+$ is defined as follows

$$|a|^* = \begin{cases} a & if \quad a \geq 1 \\ \frac{1}{a} & if \quad a < 1 \end{cases}$$

Then clearly $d^*(x, y)$ is a multiplicative metric.

Example 1.4. (See[9]) Let $d : R \times R \to [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$ where $x, y \in R$ and $a > 1$. Then $d(x, y)$ is multiplicative metric.

Definition 1.5. (See[8]) Let (X, d) be a multiplicative metric space, x_0 an arbitrary point in X, and $\epsilon > 1$. A multiplicative open ball $B(x_0, \epsilon)$ of radius ϵ centered at x_0 is the set $\{z \in X : d(z, x_0) < \epsilon\}$.

A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative convergent to some point $x \in X$ if, for any given $\epsilon > 1$, then there exists $n_0 \in N$ such that $x_n \in B(x, \epsilon)$ for all $n \geq n_0$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 1.6. (See[8]) A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative convergent to x in X if and only if $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 1.7. (See [8]) Let (X, d_X) and (Y, d_Y) be two multiplicative metric space and x_0 an arbitrary but fixed element of X. A mapping $f : X \to Y$ is said to be multiplicative continuous at x_0 if and only if $x_n \to x_0$ in (X, d_X) implies that $f(x_n) \to f(x_0)$ in (Y, d_Y). That is, given arbitrary $\epsilon > 1$, then there exists $\delta > 1$ which depends on x_0 and ϵ such that $d_Y(fx, fx_0) < \epsilon$ for all those x in X for which $d_X(x, x_0) < \delta$.

Definition 1.8. (See[8]) A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative Cauchy sequence if, for any $\epsilon > 1$, there exists $n_0 \in N$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq n_0$.

Definition 1.9. (See[8]) A multiplicative metric space (X, d) is said to be complete if every multiplicative Cauchy sequence $\{x_n\}$ in X is multiplicative convergent in X.

Definition 1.10. (See [2]) Let Ψ be the set of all control functions $\varphi : [1, \infty) \to [1, \infty)$ such that

(i) φ is continuous and non - decreasing.

(ii) $\varphi(t) = 1$ if and only if $t = 1$.

Definition 1.11. [4] An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.12. [1] An element $(x, y) \in X \times X$ is called (g_1) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.

(g_2) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Definition 1.13. [1] The mappings $F : X \times X \to X$ and $f : X \to X$ are called w - compatible if $f(F(x, y)) = F(fx, fy)$ and $f(F(y, x)) = F(fy, fx)$ whenever $fx = F(x, y)$ and $fy = F(y, x)$.

2. Main Result

Theorem 2.1. *Let $(X, d, \preceq)$ be partially ordered multiplicative metric space and let $T : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

(2.1.1) $\frac{1}{2}\min\{d(gx, T(x, y)), d(gu, T(u, v))\} \leq \max\{d(gx, gu), d(gy, gv)\}$ *implies that* $\psi(d(T(x, y), T(u, v))) \leq \frac{\psi(M(x,y,u,v))}{\phi(M(x,y,u,v))}$, *for all x, y, u, v in X ,with $gx \preceq gu$ and $gy \succeq gv$ where $\psi,\ \phi\ \in \Psi$ and*

$$M(x, y, u, v) = \max\left\{\begin{array}{c} d(gx, gu), d(gy, gv), d(gx, T(x, y)), \\ d(gy, T(y, x)), d(gu, T(u, v)), d(gv, T(v, u)), \\ \sqrt{d(gx, T(u, v)).d(gu, T(x, y))}, \\ \sqrt{d(gy, T(v, u)).d(gv, T(y, x))} \end{array}\right\},$$

(2.1.2) $T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspaces of X,

(2.1.3) T has a mixed g - monotone property

(2.1.4) (a) If a non- decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,

(b) If a non- increasing sequence $\{y_n\} \to y$ then $y \preceq y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$, then T and g have a coupled coincidence point in $X \times X$.

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$. Since $T(X \times X) \subseteq g(X)$, we choose $x_1, y_1 \in X$ such that

$$gx_0 \preceq T(x_0, y_0) = gx_1$$

and

$$gy_0 \succeq T(y_0, x_0) = gy_1$$

and choose $x_2, y_2 \in X$ such that $gx_2 = T(x_1, y_1)$ and $gy_2 = T(y_1, x_1)$.

Since T has mixed g - monotone property, we obtain $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_0 \succeq gy_1 \succeq gy_2$.

Continuing this process , we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n, y_n) \quad \text{and} \quad gy_{n+1} = T(y_n, x_n), \quad n = 0, 1, 2, \cdots$$

with

$$gx_0 \preceq gx_1 \preceq gx_2 \cdots$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \cdots$$

Case (a) : If $gx_m = gx_{m+1}$ and $gy_m = gy_{m+1}$ for some m.

Then (x_m, y_m) is coupled coincidence point in $X \times X$.

Case (b): Assume $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n.

Since $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, clearly we have

$$\begin{aligned} \tfrac{1}{2} d(gx_n, T(x_n, y_n)) &\leq d(gx_n, gx_{n+1}) \\ &\leq \max \left\{ \begin{array}{c} d(gx_n, gx_{n+1}), \\ d(gy_n, gy_{n+1}) \end{array} \right\}. \end{aligned}$$

Thus

$$\tfrac{1}{2} \min \left\{ \begin{array}{c} d(gx_n, T(x_n, y_n)), \\ d(gx_{n+1}, T(x_{n+1}, y_{n+1})) \end{array} \right\} \leq \max \left\{ \begin{array}{c} d(gx_n, gx_{n+1}), \\ d(gy_n, gy_{n+1}) \end{array} \right\}.$$

From (2.1.1) , we get

$$\psi \left(d(T(x_n, y_n), T(x_{n+1}, y_{n+1})) \right) \leq \frac{\psi(M(x_n, y_n, x_{n+1}, y_{n+1}))}{\phi(M(x_n, y_n, x_{n+1}, y_{n+1}))},$$

where

$$\begin{aligned} & M\left(x_n, y_n, x_{n+1}, y_{n+1}\right) \\ &= \max \left\{ \begin{array}{c} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gx_n, T(x_n, y_n)), d(gy_n, T(y_n, x_n)), \\ d(gx_{n+1}, T(x_{n+1}, y_{n+1})), d(gy_{n+1}, T(y_{n+1}, x_{n+1})), \\ \\ \sqrt{d(gx_n, T(x_{n+1}, y_{n+1})).d(gx_{n+1}, T(x_n, y_n))}, \\ \\ \sqrt{d(gy_n, T(y_{n+1}, x_{n+1})).d(gy_{n+1}, T(y_n, x_n))} \end{array} \right\} \end{aligned}$$

$$= \max \left\{ \begin{array}{c} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2}), \\ \sqrt{d(gx_n, gx_{n+2}).d(gx_{n+1}, gx_{n+1})}, \\ \sqrt{d(gy_n, gy_{n+2}).d(gy_{n+1}, gy_{n+1})} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2}), \\ \sqrt{d(gx_n, gx_{n+2})}, \sqrt{d(gy_n, gy_{n+2})} \end{array} \right\}.$$

But

$$\begin{aligned} \sqrt{d(gx_n, gx_{n+2})} &\leq \sqrt{d(gx_n, gx_{n+1}).d(gx_{n+1}, gx_{n+2})} \\ &\leq \max \left\{ d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}) \right\}. \end{aligned}$$

Similarly

$$\sqrt{d(gy_n, gy_{n+2})} \leq \max \left\{ d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2}) \right\}.$$

Therefore

$$M\left(x_n, y_n, x_{n+1}, y_{n+1}\right) = \max \left\{ \begin{array}{c} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2}) \end{array} \right\}.$$

Put $R_n = \max \left\{ d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}) \right\}$.

Hence

$$\psi\left(d(gx_{n+1}, gx_{n+2})\right) \leq \frac{\psi(\max\{R_n, R_{n+1}\})}{\phi(\max\{R_n, R_{n+1}\})}.$$

Similarly

$$\psi\left(d(gy_{n+1}, gy_{n+2})\right) \leq \frac{\psi(\max\{R_n, R_{n+1}\})}{\phi(\max\{R_n, R_{n+1}\})}.$$

Now

$$\begin{aligned} \psi\left(R_{n+1}\right) &= \psi\left(\max\{d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2})\}\right) \\ &= \max\{\psi\left(d(gx_{n+1}, gx_{n+2})\right), \psi\left(d(gy_{n+1}, gy_{n+2})\right)\} \\ &\leq \frac{\psi\left(\max\{R_n, R_{n+1}\}\right)}{\phi\left(\max\{R_n, R_{n+1}\}\right)}. \end{aligned}$$

If R_{n+1} is maximum, we get

$$\psi\left(R_{n+1}\right) \leq \frac{\psi\left(R_{n+1}\right)}{\phi\left(R_{n+1}\right)} < \psi\left(R_{n+1}\right)$$

which is a contradiction.

Hence R_n is a maximum

$$\begin{aligned} \psi\left(R_{n+1}\right) &\leq \frac{\psi\left(R_n\right)}{\phi\left(R_n\right)} \\ &< \psi\left(R_n\right). \end{aligned} \tag{1}$$

Since ψ is non - decreasing, we obtain

$$R_{n+1} \leq R_n.$$

Thus $\{R_n\}$ is non- increasing sequence of non- negative real number and must converges to a real number $r \geq 1$ (say).

Suppose $r > 1$.

Letting $n \to \infty$ in (1), we get

$$\psi\left(r\right) \leq \frac{\psi\left(r\right)}{\phi\left(r\right)} < \psi\left(r\right)$$

which is a contradiction.

Hence $r = 1$. Thus

$$\lim_{n\to\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = 1$$

which implies that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 1 = \lim_{n\to\infty} d(gy_n, gy_{n+1}). \tag{2}$$

Now we prove that $\{gx_n\}$ and $\{gy_n\}$ are multiplicative Cauchy sequences.

On contrary suppose that $\{gx_n\}$ or $\{gy_n\}$ are not multiplicative Cauchy sequences.

Then there exist an $\epsilon > 1$ and monotone increasing sequence of natural numbers $\{m_k\}$ amd $\{n_k\}$ such that $n_k > m_k > k$,

$$\max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\} \geq \epsilon \tag{3}$$

and

$$\max\{d(gx_{m_k}, gx_{n_{k-1}}), d(gy_{m_k}, gy_{n_{k-1}})\} < \epsilon. \tag{4}$$

From (3) and (4), we have

$$\begin{aligned}\epsilon &\leq \max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\}\\ &\leq \max\{d(gx_{m_k},gx_{n_{k-1}}).d(gx_{n_{k-1}},gx_{n_k}),d(gy_{m_k},gy_{n_{k-1}}).d(gy_{n_{k-1}},gy_{n_k})\}.\end{aligned}$$

Letting $k\to\infty$, we have

$$\epsilon \leq \lim_{k\to\infty}\max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\}\leq \epsilon.$$

Thus

$$\lim_{k\to\infty}\max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\}=\epsilon. \quad (5)$$

Form (3), we have

$$\begin{aligned}\epsilon &\leq \max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\}\\ &\leq \max\{d(gx_{m_k},gx_{n_{k+1}}).d(gx_{n_{k+1}},gx_{n_k}),d(gy_{m_k},gy_{n_{k+1}}).d(gy_{n_{k+1}},gy_{n_k})\}\\ &\leq \max\left\{\begin{array}{c}(d(gx_{m_k},gx_{n_k}).d(gx_{n_k},gx_{n_{k+1}}).d(gx_{n_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{n_k}).d(gy_{n_k},gy_{n_{k+1}}).d(gy_{n_{k+1}},gy_{n_k})\end{array}\right\}.\end{aligned}$$

Letting $k\to\infty$, we have

$$\lim_{k\to\infty}\max\{d(gx_{m_k},gx_{n_{k+1}}),d(gy_{m_k},gy_{n_{k+1}})\}=\epsilon. \quad (6)$$

From (3), we have

$$\begin{aligned}\epsilon &\leq \max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\}\\ &\leq \max\left\{\begin{array}{c}d(gx_{m_k},gx_{m_{k+1}}).d(gx_{m_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{m_{k+1}}).d(gy_{m_{k+1}},gy_{n_k})\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c}d(gx_{m_k},gx_{m_{k+1}}).d(gx_{m_{k+1}},gx_{n_{k+2}}).d(gx_{n_{k+2}},gx_{n_k}),\\ d(gy_{m_k},gy_{m_{k+1}}).d(gy_{m_{k+1}},gy_{n_{k+2}}).d(gy_{n_{k+2}},gy_{n_k})\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c}d(gx_{m_k},gx_{m_{k+1}}).d(gx_{m_{k+1}},gx_{n_{k+2}}).d(gx_{n_{k+2}},gx_{n_{k+1}}).d(gx_{n_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{m_{k+1}}).d(gy_{m_{k+1}},gy_{n_{k+2}}).d(gy_{n_{k+2}},gy_{n_{k+1}}).d(gy_{n_{k+1}},gy_{n_k})\end{array}\right\} \quad (7)\\ &\leq \max\left\{\begin{array}{c}(d(gx_{m_k},gx_{m_{k+1}}))^2.d(gx_{m_k},gx_{n_k}).(d(gx_{n_{k+2}},gx_{n_{k+1}}))^2.(d(gx_{n_{k+1}},gx_{n_k}))^2,\\ (d(gy_{m_k},gy_{m_{k+1}}))^2.d(gy_{m_k},gy_{n_k}).(d(gy_{n_{k+2}},gy_{n_{k+1}}))^2.(d(gy_{n_{k+1}},gy_{n_k}))^2\end{array}\right\}.\end{aligned}$$

Letting $k\to\infty$, using (2), (5) and (7), we have

$$\lim_{k\to\infty}\max\{d(gx_{m_k+1},gx_{n_{k+2}}),d(gy_{m_k+1},gy_{n_{k+2}})\}=\epsilon. \quad (8)$$

Also from (3), we have

$$\epsilon \leq \max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\}$$

$$
\begin{aligned}
&\leq \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+2}}).d(gx_{n_{k+2}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_{k+2}}).d(gy_{n_{k+2}}, gy_{n_k}) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+2}}).d(gx_{n_k+2}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_{k+2}}).d(gy_{n_k+2}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}) \end{array}\right\} \qquad (9) \\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_k}).(d(gx_{n_k+2}, gx_{n_{k+1}}))^2.(d(gx_{n_{k+1}}, gx_{n_k}))^2, \\ d(gy_{m_k}, gy_{n_k}).(d(gy_{n_k+2}, gy_{n_{k+1}}))^2.(d(gy_{n_{k+1}}, gy_{n_k}))^2 \end{array}\right\}.
\end{aligned}
$$

Letting $k \to \infty$, from (2),(5) and (9)

$$\lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_{k+2}}), d(gy_{m_k}, gy_{n_{k+2}})\} = \epsilon. \qquad (10)$$

Again from (3), we have

$$
\begin{aligned}
\epsilon &\leq \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\} \\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{m_{k+1}}).d(gx_{m_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{m_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{m_{k+1}}).d(gx_{m_k+1}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{m_{k+1}}).d(gy_{m_k+1}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}) \end{array}\right\} \qquad (11) \\
&\leq \max\left\{\begin{array}{c} (d(gx_{m_k}, gx_{m_{k+1}}))^2.d(gx_{m_k}, gx_{n_k}).(d(gx_{n_{k+1}}, gx_{n_k}))^2, \\ (d(gy_{m_k}, gy_{m_{k+1}}))^2.d(gy_{m_k}, gy_{n_k}).(d(gy_{n_{k+1}}, gy_{n_k}))^2 \end{array}\right\}.
\end{aligned}
$$

Letting $k \to \infty$, from (2),(5) and (11), we have that

$$\lim_{k\to\infty} \max\{d(gx_{m_k+1}, gx_{n_{k+1}}), d(gy_{m_k+1}, gy_{n_{k+1}})\} = \epsilon. \qquad (12)$$

Now from (3), we get

$$
\begin{aligned}
\epsilon &\leq \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\} \\
&\leq \max\{d(gx_{m_k}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), d(gy_{m_k}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k})\} \\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{m_{k+1}}).d(gx_{m_k+1}, gx_{n_{k+2}}).d(gx_{n_{k+2}}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{m_{k+1}}).d(gy_{m_k+1}, gy_{n_{k+2}}).d(gy_{n_{k+2}}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}) \end{array}\right\}.
\end{aligned}
$$

Letting $k \to \infty$, using (2), we obtain that

$$\epsilon \leq \lim_{k\to\infty} \max\{d(gx_{m_{k+1}}, gx_{n_{k+2}}), d(gy_{m_{k+1}}, gy_{n_{k+2}})\}.$$

By property of ψ, (5), (6) and (8), we have

$$\psi(\epsilon) \leq \lim_{k\to\infty} \psi\left(\max\{d(gx_{m_{k+1}}, gx_{n_{k+2}}), d(gy_{m_{k+1}}, gy_{n_{k+2}})\}\right)$$

$$= \lim_{k\to\infty} \max\{\psi\left(d(gx_{m_k+1}, gx_{n_k+2})\right), \psi\left(d(gy_{m_k+1}, gy_{n_k+2})\right)\}. \quad (13)$$

Now we will show that

$$\frac{1}{2}\min\left\{ \begin{array}{c} d(gx_{m_k}, T(x_{m_k}, y_{m_k})), \\ d(gx_{n_k+1}, T(x_{n_k+1}, y_{n_k+1})) \end{array} \right\} \leq \max\left\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gx_{n_k+1})\right\}.$$

On contrary suppose that

$$\max\left\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gx_{n_k+1})\right\} < \frac{1}{2}\min\left\{ \begin{array}{c} d(gx_{m_k}, T(x_{m_k}, y_{m_k})), \\ d(gx_{n_k+1}, T(x_{n_k+1}, y_{n_k+1})) \end{array} \right\}.$$

Letting $k \to \infty$, we have

$$\epsilon \leq \frac{1}{2}\min\{1, 1\} = \frac{1}{2}$$

which is a contradiction.

Hence

$$\frac{1}{2}\min\left\{ \begin{array}{c} d(gx_{m_k}, T(x_{m_k}, y_{m_k})), \\ d(gx_{n_k+1}, T(x_{n_k+1}, y_{n_k+1})) \end{array} \right\} \leq \max\left\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gx_{n_k+1})\right\}.$$

Similarly

$$\frac{1}{2}\min\left\{ \begin{array}{c} d(gy_{m_k}, T(y_{m_k}, x_{m_k})), \\ d(gy_{n_k+1}, T(y_{n_k+1}, x_{n_k+1})) \end{array} \right\} \leq \max\left\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gx_{n_k+1})\right\}.$$

Now from (2.1.1), we get

$$\psi\left(d(T(x_{m_k}, y_{m_k}), T(x_{n_k+1}, y_{n_k+1}))\right) \leq \frac{\psi\left(M\left(x_{m_k}, y_{m_k}, x_{n_k+1}, y_{n_k+1}\right)\right)}{\phi\left(M\left(x_{m_k}, y_{m_k}, x_{n_k+1}, y_{n_k+1}\right)\right)} \quad (14)$$

where

$$M\left(x_{m_k}, y_{m_k}, x_{n_{k+1}}, y_{n_k+1}\right)$$

$$= \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gx_{m_k}, T(x_{m_k}, y_{m_k})), d(gy_{m_k}, T(y_{m_k}, x_{m_k})), \\ d(gx_{n_{k+1}}, T(x_{n_{k+1}}, y_{n_{k+1}})), d(gy_{n_{k+1}}, T(y_{n_{k+1}}, x_{n_{k+1}})), \\ \sqrt{d(gx_{m_k}, T(x_{n_{k+1}}, y_{n_{k+1}})).d(gx_{n_{k+1}}, T(x_{m_k}, y_{m_k}))}, \\ \sqrt{d(gy_{m_k}, T(y_{n_{k+1}}, x_{n_{k+1}})).d(gy_{n_{k+1}}, T(y_{m_k}, x_{m_k}))} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ \sqrt{d(gx_{m_k}, gx_{n_{k+2}}).d(gx_{n_{k+1}}, gx_{m_{k+1}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}}).d(gy_{n_{k+1}}, gy_{m_{k+1}})} \end{array} \right\}.$$

But

$$\max \left\{ \begin{array}{c} \sqrt{d(gx_{m_k}, gx_{n_{k+2}}).d(gx_{n_{k+1}}, gx_{m_{k+1}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}}).d(gy_{n_{k+1}}, gy_{m_{k+1}})} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+2}}), \\ d(gx_{n_{k+1}}, gx_{m_{k+1}}) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} d(gy_{m_k}, gy_{n_{k+2}}), \\ d(gy_{n_{k+1}}, gy_{m_{k+1}}) \end{array} \right\} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+2}}), \\ d(gx_{n_{k+1}}, gx_{m_{k+1}}), \\ d(gy_{m_k}, gy_{n_{k+2}}), \\ d(gy_{n_{k+1}}, gy_{m_{k+1}}) \end{array} \right\}$$

$\rightarrow \epsilon$ as $k \rightarrow \infty$.

So that

$$\lim_{k \to \infty} M\left(x_{m_k}, y_{m_k}, x_{n_{k+1}}, y_{n_k+1}\right) = \epsilon.$$

Hence letting $k \to \infty$ in (14), we have that

$$\lim_{k \to \infty} \psi\left(d\left(gx_{m_k+1}, gx_{n_k+2}\right)\right) \leq \frac{\psi(\epsilon)}{\phi(\epsilon)}.$$

Similarly

$$\lim_{k \to \infty} \psi\left(d\left(gy_{m_k+1}, gy_{n_k+2}\right)\right) \leq \frac{\psi(\epsilon)}{\phi(\epsilon)}.$$

From (13), we have

$$\begin{aligned}\psi(\epsilon) &\leq \tfrac{\psi(\epsilon)}{\phi(\epsilon)} \\ &< \psi(\epsilon)\end{aligned}$$

which is a contradiction.

Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the multiplicative metric space (X, d).

Hence we have

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 1 = \lim_{n,m\to\infty} d(gy_n, gy_m).$$

Suppose $g(X)$ is a complete subspace of X.

Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequence in the multiplicative metric space $(g(X), d)$.

It follows that $\{gx_n\}$ and $\{gy_n\}$ convergent to some α and β in $g(X)$.

Thus

$$\lim_{n\to\infty} d(gx_n, \alpha) = 1$$

and

$$\lim_{n\to\infty} d(gy_n, \beta) = 1.$$

Since $\alpha, \beta \in g(X)$, there exist $x, y \in X$ such that $\alpha = gx$ and $\beta = gy$.

Since $\{gx_n\}$ is non - decreasing sequence and $\{gx_n\} \to \alpha = gx$. It follows that $gx_{n+1} \preceq gx = \alpha$.

Also $\{gy_n\}$ is non - increasing sequence and $\{gy_n\} \to \beta = gy$. It follows that $gy_{n+1} \succeq gy = \beta$.

Now we claim that for each $n \geq 1$, at least one of the following holds

$$\frac{1}{2}\min\left\{\ d(gx_n, gx_{n+1}), d(gx, T(x,y))\ \right\} \leq \max\left\{\ d(gx_n, gx), d(gy_n, gy)\ \right\}$$

or

$$\frac{1}{2}\min\left\{\ d(gx_{n+1}, gx_{n+2}), d(gx, T(x,y))\ \right\} \leq \max\left\{\ d(gx_{n+1}, gx), d(gy_{n+1}, gy)\ \right\}.$$

On contrary suppose that

$$\frac{1}{2}\min\left\{\ d(gx_n, gx_{n+1}), d(gx, T(x,y))\ \right\} > \max\left\{\ d(gx_n, gx), d(gy_n, gy)\ \right\}$$

and

$$\frac{1}{2}\min\left\{\ d(gx_{n+1}, gx_{n+2}), d(gx, T(x,y))\ \right\} > \max\left\{\ d(gx_{n+1}, gx), d(gy_{n+1}, gy)\ \right\}.$$

Now

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(gx_n, gx).d(gx, gx_{n+1}) \\ &< \tfrac{1}{2}d(gx_n, gx_{n+1}).\tfrac{1}{2}d(gx_{n+1}, gx_{n+2}) \\ &= \tfrac{1}{4}d(gx_n, gx_{n+1}).d(gx_{n+1}, gx_{n+2}). \end{aligned}$$

Letting $n \to \infty$ and using (2), we have

$$1 \leq \frac{1}{4}$$

which is a contradiction.

Hence claim holds.

Sub case (i) : Suppose

$$\tfrac{1}{2}\min\left\{\ d(gx_n, T(x_n, y_n)), d(gx, T(x,y))\ \right\} \leq \max\left\{\ d(gx_n, gx), d(gy_n, gy)\ \right\}.$$

Now we prove that $T(x,y) = \alpha$, $T(y,x) = \beta$.
From (2.1.1), we get that

$$\psi\left(d\left(T(x_n, y_n), T(x,y)\right)\right) \leq \frac{\psi(M(x_n,y_n,x,y))}{\phi(M(x_n,y_n,x,y))},$$

where

$$M\left(x_n, y_n, x, y\right) = \max\left\{\begin{array}{c} d(gx_n, gx), d(gy_n, gy), d(gx_n, T(x_n, y_n)), \\ d(gy_n, T(y_n, x_n)), d(gx, T(x,y)), d(gy, T(y,x)), \\ \\ \sqrt{d(gx_n, T(x,y)).d(gx, T(x_n, y_n))}, \\ \\ \sqrt{d(gy_n, T(y,x)).d(gy, T(y_n, x_n))} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} d(gx_n, \alpha), d(gy_n, \beta), d(gx_n, T(x_n, y_n)), \\ d(gy_n, T(y_n, x_n)), d(\alpha, T(x,y)), d(\beta, T(y,x)), \\ \\ \sqrt{d(gx_n, T(x,y)).d(\alpha, T(x_n, y_n))}, \\ \\ \sqrt{d(gy_n, T(y,x)).d(\beta, T(y_n, x_n))} \end{array}\right\}.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} M\left(x_n, y_n, x, y\right) = \max\left\{ \begin{array}{c} 1,1,1,1,d(\alpha,T(x,y)),d(\beta,T(y,x)), \\ \\ \sqrt{d(\alpha,T(x,y))},\sqrt{d(\beta,T(y,x))} \end{array} \right\}$$

$$= \max\left\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \right\}.$$

Thus

$$\psi\left(d\left(\alpha,T(x,y)\right)\right) \leq \frac{\psi\Big(\max\big\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \big\}\Big)}{\phi\Big(\max\big\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \big\}\Big)}.$$

Similarly we can show that

$$\psi\left(d\left(\beta,T(y,x)\right)\right) \leq \frac{\psi\Big(\max\big\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \big\}\Big)}{\phi\Big(\max\big\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \big\}\Big)}.$$

Now consider

$$\begin{aligned}\psi\left(\max\left\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \right\}\right) &\\ = \max\left\{\ \psi\left(d(\alpha,T(x,y))\right),\psi\left(d(\beta,T(y,x))\right)\ \right\} &\\ \leq \frac{\psi\left(\max\left\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \right\}\right)}{\phi\left(\max\left\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \right\}\right)}.\end{aligned}$$

Therefore

$$\phi\left(\max\left\{\ d(\alpha,T(x,y)),d(\beta,T(y,x))\ \right\}\right) \leq 1.$$

Thus $\alpha = T(x,y) = gx$ and $\beta = T(y,x) = gy$.

Therefore (x,y) is a coupled coincidence point of T and g.

There exist coupled coincidence point of T and g when

$$\frac{1}{2}\min\left\{\ d(gx_{n+1},gx_{n+2}),d(gx,T(x,y))\ \right\} \leq \max\left\{\ d(gx_{n+1},gx),d(gy_{n+1},gy)\ \right\}$$

holds.

□

Theorem 2.2. *In addition to the hypothesis of Theorem 2.1, suppose that for every $(x,y),(x',y') \in X \times X$, there exists $(u,v) \in X \times X$ such that $(T(u,v),T(v,u))$ is comparable to $(T(x,y),T(y,x))$ and $(T(x',y'),T(y',x'))$.*

Moreover if (T,g) is w - compatible then T and g have a unique common coupled fixed point in $X \times X$.

Proof. From Theorem 2.1, there exists a coupled coincidence point $(x, y) \in X \times X$ of T and g.

Now let (x', y') be another coupled coincidence point of T and g.

That is

$$T(x', y') = gx' \text{ and } T(y', x') = gy'.$$

By additional assumption, there is $(u, v) \in X \times X$ such that $(T(u, v), T(v, u))$ is comparable to $(T(x, y), T(y, x))$ and $(T(x', y'), T(y', x'))$.

Let $u_0 = u$,$v_0 = v$, $x_0 = x$,$y_0 = y$,$x'_0 = x'$ and $y'_0 = y'$.

Since $T(X \times X) \subseteq g(X)$, we can construct the sequences $\{gu_n\}$, $\{gv_n\}$, $\{gx_n\}$, $\{gy_n\}$, $\{gx'_n\}$ and $\{gy'_n\}$ such that

$$\begin{aligned} gu_{n+1} &= T(u_n, v_n), gv_{n+1} = T(v_n, u_n), \\ gx_{n+1} &= T(x_n, y_n), gy_{n+1} = T(y_n, x_n), \\ gx'_{n+1} &= T(x'_n, y'_n) \text{ and } gy'_{n+1} = T(y'_n, x'_n), n = 0, 1, 2, \cdots \end{aligned}$$

Since $(gx, gy) = (T(x, y), T(y, x)) = (gx_1, gy_1)$ and $(T(u, v), T(v, u)) = (gu_1, gv_1)$ are comparable, then $gx \preceq gu_1$, and $gy \succeq gv_1$.

One can show that $gx \preceq gu_n$, and $gy \succeq gv_n$ for all n.

As in the Theorem 2.1, we conclude that $\{gu_{n+1}\} \to gx$ $\{gv_{n+1}\} \to gy$.

Analogously, we can show that $\{gu_{n+1}\} \to gx'$ $\{gv_{n+1}\} \to gy'$ in $(g(X), d)$. Since $g(X)$ is complete and $\{gu_{n+1}\}$ converges to gx and gx', we get $gx = gx'$.

Similarly $gy = gy'$.

Thus if (x, y) and (x', y') are coupled coincidence points of T and g, then

$$T(x, y) = gx = gx' = T(x', y') = \alpha, \text{ say} \tag{15}$$

and

$$T(y, x) = gy = gy' = T(y', x') = \beta, \text{ say.} \tag{16}$$

Since (T, g) is w - compatible, then

$$g\alpha = g(gx) = g(T(x, y)) = T(gx, gy) = T(\alpha, \beta)$$

and

$$g\beta = g(gy) = g(T(y, x)) = T(gy, gx) = T(\beta, \alpha).$$

Hence the pair (α, β) is also coupled coincidence point of T and g.

Thus we have

$$g\alpha = gx \text{ and } g\beta = gy.$$

Therefore

$$\alpha = gx = g\alpha = T(\alpha, \beta) \text{ and } \beta = gy = g\beta = T(\beta, \alpha).$$

Thus (α, β) is a common coupled fixed point of T and g.

Suppose $\left(\alpha^1, \beta^1\right)$ is another common coupled fixed point of T and g.

Then $\alpha^1 = T\left(\alpha^1, \beta^1\right) = g\alpha^1$ and $\beta^1 = T\left(\beta^1, \alpha^1\right) = g\beta^1$.

Since (α, β) and $\left(\alpha^1, \beta^1\right)$ are coupled coincidence points of T and g, it follows from (15) and (16) that $g\alpha = g\alpha^1$ and $g\beta = g\beta^1$ which implies that $\alpha = \alpha^1$ and $\beta = \beta^1$. Thus (α, β) is the unique common coupled fixed point of T and g.

□

Example 2.3. Let $X = [0,1]$ and $d(x,y) = e^{|x-y|}$ for all $x, y \in X$ then $(X, d, \preceq)$ is a partially ordered complete multiplicative metric space. The ordering $\preceq$ is defined by $x \preceq y \iff x \leq y$. Let $T : X \times X \to X$ and $g : X \to X$ be defined by $T(x,y) = \frac{x+y}{2}$ and $g(x) = x$. Then T and g satisfies all the conditions of Theorem 2.1 and Theorem 2.2. Clearly (0, 0) is the unique common coupled fixed point of T and g.

Corollary 2.4. *Let $(X, d, \preceq)$ be partially ordered complete multiplicative metric space and let $T : X \times X \to X$ be mapping satisfying*

(2.4.1) $\frac{1}{2}\min\{d(x, T(x,y)), d(u, T(u,v))\} \leq \max\{d(x,u), d(y,v)\}$ *implies that*

$$d\left(T(x,y), T(u,v)\right) \leq \max \left\{ \begin{array}{c} d(x,u), d(y,v), d(x, T(x,y)), \\ d(y, T(y,x)), d(u, T(u,v)), d(v, T(v,u)), \\ \sqrt{d(x, T(u,v)).d(u, T(x,y))}, \\ \sqrt{d(y, T(v,u)).d(v, T(y,x))} \end{array} \right\}^{\lambda},$$

for all $x, y, u, v \in X$,with $x \preceq u$, $y \succeq u$ and $\lambda \in \left(0, \frac{1}{2}\right)$

(2.4.2) (a) If a non- decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,
(b) If a non- increasing sequence $\{y_n\} \to y$ then $y \preceq y_n$ for all n
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$, then T have a unique coupled fixed point in $X \times X$.

Proof. The proof of this corollary follows from Theorem 2.1 by taking $\psi(t) = t, \phi(t) = t^{1-\lambda}$ and $g(x) = x$. □

Example 2.5. Let $X = [0,1]$ and $d(x,y) = 2^{|x-y|}$ for all $x, y \in X$ then $(X, d, \preceq)$ is a partially ordered complete multiplicative metric space. The ordering $\preceq$ is defined by $x \preceq y \iff x \leq y$. Let $T : X \times X \to X$ be defined by $T(x,y) = \frac{x+y}{4}$ and $\lambda = \frac{1}{4} \in \left(0, \frac{1}{2}\right)$. Consider

$$\begin{aligned} d(T(x,y),T(u,v)) &= 2^{\left|\frac{x+y}{4}-\frac{u+v}{4}\right|} \\ &= 2^{\left|\frac{x-u}{4}-\frac{v-y}{4}\right|} \\ &\leq \max\left\{2^{\left|\frac{x-u}{4}\right|,}, 2^{\left|\frac{y-v}{4}\right|,}\right\} \\ &= \max\left\{2^{\frac{1}{4}|x-u|,}, 2^{\frac{1}{4}|y-v|,}\right\} \\ &= \max\left\{2^{|x-u|,}, 2^{|y-v|,}\right\}^{\frac{1}{4}} = \max\left\{d(x,u), d(y,v)\right\}^{\lambda} \\ &\leq \max\left\{\begin{array}{c} d(x,u), d(y,v), d(x,T(x,y)), \\ d(y,T(y,x)), d(u,T(u,v)), d(v,T(v,u)), \\ \sqrt{d(x,T(u,v)).d(u,T(x,y))}, \\ \sqrt{d(y,T(v,u)).d(v,T(y,x))} \end{array}\right\}^{\lambda}. \end{aligned}$$

Clearly (0, 0) is the unique coupled fixed point of T.

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