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THE FOURIER TRANSFORM OF $\left(G(x,m)_+^\lambda\right)$

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Abstract: In this article we give a sense the Fourier transform of $\left(G(x,m)_+^\lambda\right) = \left(\left(_{i=1}^{p} x_i^2\right)^m - \left(_{i=p+1}^{p+q} x_i^2\right)^m\right)$ In particular if $m = 1$ we obtain the Fourier transform $(P(x))_+^\lambda = \left(x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2)\right)_+^\lambda$, where $p + q = n$ is the dimension of the space.

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1. Introduction

Let $x = (x_1, ...x_p, x_{p+1}, ...x_{p+q})$ be a point of R^n, where $p + q = n$ is the dimension of the space and let G be the quadratic form defined by

$$G = G(x,m) = \left(_{i=1}^{p} x_i^2\right)^m - \left(_{i=p+1}^{p+q} x_i^2\right)^m \quad (1)$$

where $m = 1, 2, 3, ...$

If $m = 1$, we have

$$G = G(x,1) = P(x_1, ...x_p, x_{p+1}, ...x_{p+q}) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 = P(x) = P \quad (2)$$

The $P = 0$ hypersurface is a hypercone with a singular point(the vertex) at the origin.

We defined the generalized function G_+^λ where λ is a complex number, by the following form

$$\left(G_+^\lambda, \varphi\right) =_{P>0} (G(x,m))^\lambda \varphi(x)dx \quad (3)$$

where $dx = dx_1...dx_n, \varphi \in C_o^\infty$ is the space of infinitely differentiable function with compact support.For $Re(\lambda) \geq 0$,the integral(3) converges and are analytic function of λ. Analytic continuation to $Re(\lambda) < 0$ can be used to extend the definition $\left(G_+^\lambda, \varphi\right)$.

We know from([10]and [1]) that

where

$$< \delta^{(k-1)}(G),\varphi >= \int_0^\infty \left[\left(\frac{1}{2ms^{2m-1}}\frac{\partial}{\partial s}\right)^{k-1}\left\{S^{q-2m}\frac{\Psi(r,s)}{2m}\right\}\right]_{s=r} r^{p-1}dr, \tag{9}$$

$$L_m = \left(\sum_{j=1}^{p} x_j^2\right)^{1-m}\left(\sum_{j=1}^{p}\frac{\partial^2}{\partial x_j^2}\right) - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^{1-m}\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right), \tag{10}$$

$$A_{p,q,m,k} = \begin{array}{c}\frac{(-1)^k\pi^{\frac{n}{2}}\Gamma(\frac{q}{2m})\Gamma(\frac{n}{2m})\Gamma(1-\frac{n}{2m})}{m^2\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})\Gamma(1-\frac{p}{2m})}.\\ \frac{1}{(2m)^{2k}}\frac{\Gamma(\frac{n}{2m}+\frac{2(m-1)}{m})}{\Gamma(\frac{n}{2m}+k)\left(\frac{2}{m}(m-1)-1\right)\ldots\left(\frac{2}{m}(m-1)-k\right)},\end{array} \tag{11}$$

if $p = 2ms + m, m \neq 2, n$ odd ([1],page39,formula(31)),

$$A_{p,q,m,k} = 0 \tag{12}$$

if $p = 2ms, q$ odd (n odd)([1],page40,formula(37)),

$$A_{p.q,m,k} = \begin{array}{c}\frac{(-1)(-1)^{\frac{q}{2m}}\pi^{\frac{n}{2}}\Gamma(\frac{q}{2m})\Gamma(\frac{p}{2m})}{m^2\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})\Gamma(\frac{p+q}{2m})}.\\ \frac{1}{k!(2m)^{2k}}\frac{\Gamma(\frac{n}{2m}+\frac{2(m-1)}{m})}{\Gamma(\frac{n}{2m}+\frac{2(m-1)}{m}+k)},\end{array} \tag{13}$$

if $p = (2s)m, q = (2l)m; k \geq \frac{n}{2m}, s, l = 0, 1, ...$ ([2]) and

$$A_{p,q,m,k} = \begin{array}{c}\frac{(-1)(-1)^{\frac{q-m}{2m}}\pi^{\frac{n}{2}-1}\Gamma(\frac{q}{2m})\Gamma(\frac{p}{2m})}{m^2\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})\Gamma(\frac{p+q}{2m})}[\psi(\frac{p}{2m}) - \psi(\frac{n}{2m})]\\ \frac{1}{k!(2m)^{2k}}\frac{\Gamma(\frac{n}{2m}+\frac{2(m-1)}{m})}{\Gamma(\frac{n}{2m}+\frac{2(m-1)}{m}+k)}.\end{array} \tag{14}$$

if $p = (2s)m + m, q = (2l)m + m; k \geq \frac{n}{2m}, s, l = 0, 1, ...$ ([2])

Now we are going to study the Fourier transform of $G_+^\lambda(x,m)$.

2. The Fourier transform of $G^{\lambda}_{+}(x,m)$

Let $G^{\lambda}_{+} = G^{\lambda}_{+}(x,m)$ be defined by

$$G^{\lambda}_{+} = G^{\lambda}_{+}(x,m) = \begin{cases} G^{\lambda}(x.m) \text{ if } G = G(x.m) \geq 0 \\ \\ 0 \text{ if } G = G(x,m) < 0 \end{cases} \tag{15}$$

the Fourier transform of $G^{\lambda}_{+} = G^{\lambda}_{+}(x,m)$ is defined by the following formula

$$F\left\{G^{\lambda}_{+}(x,m)\right\} =_{R^n} e^{-\langle x,y\rangle} G^{\lambda}_{+}(x,m)dx \tag{16}$$

where

$$\begin{gathered} \langle x,y\rangle = x_1y_1 + ... + x_py_p + x_{p+1}y_{p+1} + ... + x_{p+q}y_{p+q}, \\ dx = dx_1...dx_pdx_{p+1}...dx_{p+q} \end{gathered} \tag{17}$$

and $p+q=n$ dimension of the space.

From(16), using(17) and(2), we have

$$\begin{gathered} F\left\{G^{\lambda}_{+}(x,m)\right\} =_{R^p} \left(_{R^q} e^{-i\langle x,y\rangle} G^{\lambda}_{+}(x,m)\right) dx_1...dx_pdx_{p+1}...dx_{p+q} = \\ = I_p\left(I_q((G(x_1,...x_p,x_{p+1},...x_{p+q},m)^{\lambda}\right) \end{gathered} \tag{18}$$

where

$$\begin{gathered} I_q((G(x_1,...x_p,x_{p+1},...x_{p+q},m))^{\lambda} = \\ =_{R^q} e^{-i(x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q})} \left(\left(^{p}_{i=1} x_i^2\right)^m - \left(^{p+q}_{i=p+1} x_i^2\right)^m\right)^{\lambda} dx_{p+1}...dx_{p+q} \end{gathered} \tag{19}$$

and

$$\begin{gathered} I_p\left(I_q((G(x_1,...x_p,x_{p+1},...x_{p+q},m)^{\lambda}\right) = \\ {}_{R^p} e^{-i(x_1y_1+...+x_py_p)} I_q((G(x_1,...x_p,x_{p+1},...x_{p+q}),m)^{\lambda} dx_p...dx_1. \end{gathered} \tag{20}$$

By calling

$$r^2 = x_1^2 + ... + x_p^2, \tag{21}$$

$$s^2 = x_{p+1}^2 + ... + x_{p+q}^2, \tag{22}$$

$$\bar{y_p} = (y_1., , , y_p), \tag{23}$$

$$\bar{y_q} = (y_{p+1}., , , y_{p+q}) \tag{24}$$

and without loss of generality we may assume that the component of y_q are given by $\bar{y_q} = (|\bar{y_q}|, 0, 0, ...0)$ so that the integration(19) becomes

$$I_q((G(x_1, ...x_p, x_{p+1}, ...x_{p+q}, m)^\lambda =$$

$$=_{R^q} e^{-i|\bar{y_q}|x_{p+1}} \left(\left(^p_{i=1} x_i^2\right)^m - \left(^{p+q}_{i=p+1} x_i^2\right)^m \right)_+^\lambda dx_{p+1}...dx_{p+q}. \tag{25}$$

We shall perform the integration(25) by going to polar coordinates. After integration over angles $\varphi_{p+2}, \varphi_{p+3}, ...\varphi_{p+q-1}$ and using the fact that

$$\Omega_{q-1} = \frac{2\,(\sqrt[2]{\pi})^{q-1}}{\Gamma(\frac{q-1}{2})} \tag{26}$$

area of unit sphere in R^{q-1} we arrive at

$$I_q((G(x_1, ...x_p, x_{p+1}, ...x_{p+q}, m)^\lambda =$$

$$= \frac{2(\sqrt[2]{\pi})^{q-1}}{\Gamma(\frac{q-1}{2})} {}_0^\infty {}_0^\pi \left(r^{2m} - s^{2m}\right)_+^\lambda e^{-i|\bar{y_q}|s\cos\varphi_{p+1}} sen^{q-2}(\varphi_{p+1}) s^{q-1} d\varphi_{p+1} ds. \tag{27}$$

Now using the integral representation of the Bessel function $J_\gamma(x)$:

$$J_\gamma(x) = \frac{1}{2^\gamma \sqrt[2]{\pi}\Gamma(\gamma + \frac{1}{2})} {}_0^\pi e^{\pm ix\cos\theta} x^\gamma sen^{2\gamma}\theta d\theta \tag{28}$$

([4].page 409 and [5], page953,formula7) where

$$J_\gamma(x) =_{j\geq 0} (-1)^j \frac{(-1)^j \left(\frac{x}{2}\right)^{\gamma+2j}}{j!\Gamma(\gamma + j + 1)} \tag{29}$$

we have

$$I_q((G(x_1, ...x_p, x_{p+1}, ...x_{p+q}), m)^\lambda = \frac{2(\sqrt[2]{\pi})^{q-1}}{\Gamma(\frac{q-1}{2})}.$$

$${}_0^\infty s^{q-1} \left(r^2 - s^2\right)_+^\lambda \frac{\sqrt[2]{\pi} 2^{\frac{q-2}{2}} \Gamma(\frac{q-1}{2}) J_{\frac{q-2}{2}}(s|\bar{y_q}|)}{(s|\bar{y_q}|)^{\frac{q-2}{2}}} ds = \tag{30}$$

$$= 2\,(\sqrt[2]{\pi})^{q-1} \sqrt[2]{\pi} 2^{\frac{q-2}{2}} \,(|\bar{y_q}|)_0^{1-\frac{q}{2}\,\infty} s^{\frac{q}{2}} \left(r^{2m} - s^{2m}\right)_+^\lambda J_{\frac{q-2}{2}}(s\,|\bar{y_q}|) ds.$$

Now taking into account the formula(15),from(30) we have

$$I_q((G(x_1,...x_p,x_{p+1},...x_{p+q},m)^{\lambda} =$$

$$= 2^{\frac{q}{2}}\pi^{\frac{q-1}{2}}\left(|\bar{y_q}|\right)_0^{1-\frac{q}{2}\ r} s^{\frac{q}{2}}\left(r^{2m}-s^{2m}\right)^{\lambda} J_{\frac{q-2}{2}}(s\,|\bar{y_q}|)ds. \quad (31)$$

On the other hand, usin the formula(29)), we have

$$_0^r s^{\frac{q}{2}}\left(r^{2m}-s^{2m}\right)^{\lambda} J_{\frac{q-2}{2}}(s\,|\bar{y_q}|)ds =$$

$$=_{j\geq 0}\frac{(-1)^j}{j!\Gamma(\frac{q-2}{2}+j+1)}{}_0^r s^{q+2j-1}\left(r^{2m}-s^{2m}\right)^{\lambda} ds. \quad (32)$$

Now making changes of variables and using the formula

$$_0^1 t^{a-1}\left(1-t\right)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (33)$$

for $Re(a)$ >0 and $Re(b) > 0$,([7],page8,formula(1)), from(31) and (33), we have

$$I_q((G(x_1,...x_p,x_{p+1},...x_{p+q},m)^{\lambda} =$$

$$\frac{\pi^{\frac{q}{2}}\Gamma(\lambda+1)}{m}{}_{j\geq 0}\frac{(-1)^j\left(\frac{|\bar{y_q}|}{2}\right)^{2j}\Gamma(\frac{q}{2m}+\frac{j}{m})}{j!\Gamma(\frac{q-2}{2}+j+1)\Gamma(\lambda+1+\frac{q}{2m}+\frac{j}{m})}r^{q+2j+2\lambda m}. \quad (34)$$

On the other hand,from(20)and(34) we have

$$I_p\left(I_q((G(x_1,...x_p,x_{p+1},...x_{p+q},m)^{\lambda}\right) = \frac{\pi^{\frac{q}{2}}\Gamma(\lambda+1)}{m}{}_{j\geq 0}\frac{(-1)^j\left(\frac{|\bar{y_q}|}{2}\right)^{2j}\Gamma(\frac{q}{2m}+\frac{j}{m})}{j!\Gamma(\frac{q-2}{2}+j+1)\Gamma(\lambda+1+\frac{q}{2m}+\frac{j}{m})}$$

$$_{R^p}e^{-i(x_1y_1+...+x_py_p)}r^{q+2j+2\lambda m}dx_p...dx_1. \quad (35)$$

Now we are going to study the integral

$$_{R^p}e^{-i(x_1y_1+...+x_py_p)}r^{q+2j+2\lambda m}dx_p...dx_1. \quad (36)$$

and without loss of generality we may assume that the component of $\bar{y_p}$ are given by $\bar{y_p} = (|\bar{y_p}|, 0, 0, ...0)$ so that the integration(36) becomes

$$_{R^p}e^{-i|\bar{y_p}|x_1}r^{q+2j+2\lambda m}dx_p...dx_1. \quad (37)$$

We shall perform the integration(37) by going to polar coordinate. After integration over angles $\theta_2, ...\theta_{p-1}$ and using the fact

$$\Omega_{p-1} = \frac{2\left(\sqrt[2]{\pi}\right)^{p-1}}{\Gamma(\frac{p-1}{2})} \tag{38}$$

we arrive at

$$\begin{gathered} {}_{R^p} e^{-i(x_1y_1+...+x_py_p)} r^{q+2j+2\lambda m} dx_p...dx_1 = \\ = \frac{2\left(\sqrt[2]{\pi}\right)^{p-1}}{\Gamma(\frac{p-1}{2})} {}_0^{\infty} \left({}_0^{\pi} e^{-i\left|\bar{y_p}\right| r\cos\theta_1} sen^{p-2}(\theta_1) r^{p-1} d\theta_1 \right) r^{q+2j+2\lambda m} dr. \end{gathered} \tag{39}$$

On the other hand, from(39) and using the formula(28) we have

$$\begin{gathered} {}_{R^p} e^{-i(x_1y_1+...+x_py_p)} r^{q+2j+2\lambda m} dx_p...dx_1 = \\ 2^{\frac{p}{2}} \pi^{\frac{p}{2}} \left(\left|\bar{y_p}\right|\right)_0^{1-\frac{p}{2}} {}^{\infty} r^{q+2j+2\lambda m+p-\frac{p}{2}} J_{\frac{p-2}{2}}(r\left|\bar{y_p}\right|) dr. \end{gathered} \tag{40}$$

Now considering the formula

$$\begin{gathered} {}_0^{\infty} x^{\mu} J_{\nu}(ax) dx = \\ = \frac{2^{\mu} a^{-\mu-1} \Gamma(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2})} \end{gathered} \tag{41}$$

si $-Re\nu - 1 < Re\mu < \frac{1}{2}$([5],page684,formula4), from(39), we have

$$\begin{gathered} {}_{R^p} e^{-i(x_1y_1+...+x_py_p)} r^{q+2j+2\lambda m} dx_p...dx_1 = \\ 2^{p+q} \pi^{\frac{p}{2}} 2^{2\lambda m} 2^{2j} \left(\left|\bar{y_p}\right|\right)^{-2\lambda m-q-2j-\frac{p}{2}-\frac{p}{2}} \frac{\Gamma(\lambda m+\frac{p}{2}+\frac{q}{2}+j)}{\Gamma(-\lambda m-\frac{q}{2}-j)}. \end{gathered} \tag{42}$$

Finally,from(18),using(35)and(42),we obtain the following formula

$$\begin{gathered} F\left\{G_+^{\lambda}(x,m)\right\} = F\left\{\left(\left({}_{i=1}^{p} x_i^2\right)^m - \left({}_{i=p+1}^{p+q} x_i^2\right)^m\right)_+^{\lambda}\right\} = \\ \frac{2^{n+2\lambda m}}{m} \pi^{\frac{n}{2}} \Gamma(\lambda+1). \\ {}_{j\geq 0} \frac{(-1)^j \Gamma(\frac{q}{2m}+\frac{j}{m}) \Gamma(\lambda m+\frac{n}{2}+j) \left(\left|\bar{y_q}\right|^2\right)^j \left(\left|\bar{y_p}\right|^2\right)^{-\lambda m-\frac{n}{2}-j}}{j! \Gamma(\frac{q-2}{2}+j+1) \Gamma(\frac{q}{2m}+\frac{j}{m}+\lambda+1) \Gamma(-\lambda m-\frac{q}{2}-j)}. \end{gathered} \tag{43}$$

We observe that by putting $m=1$ in(38),considerin(2) and the formulae

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{sen(\pi z)} \tag{44}$$

and

$$(1-z))^\alpha =_{j\geq 0} (-1)^j \binom{\alpha}{j} z^j \tag{45}$$

if $\mid z<1$,where.

$$\binom{\alpha}{j}=\frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)}=\frac{(-1)^j\Gamma(-\alpha+j)}{j!\Gamma(-\alpha)} \tag{46}$$

we obtain the following formula

$$F\left\{P_+^\lambda(x)\right\}= \\ =\frac{\pi^{\frac{n}{2}}\Gamma(\lambda+\frac{n}{2})\ 2^{\lambda+\frac{n}{2}}}{\Gamma(1+\lambda+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}\left(|\bar{y_p}|^2-|\bar{y_q}|^2\right)^{-\lambda-\frac{n}{2}} \tag{47}$$

for $|\bar{y_q}|<|\bar{y_p}|$.

By calling

$$Q(y)=|\bar{y_p}|^2-|\bar{y_q}|^2= \\ =y_1^2+...+y_p^2-y_{p+1}^2-...-y_{p+q}^2 \tag{48}$$

the formula(42) can be rewrritten in the following form

$$F\left\{P_+^\lambda(x)\right\}=\frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+\frac{n}{2})\Gamma(\lambda+1)}{\Gamma(1+\lambda+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}\left(Q(y)\right)^{-\lambda-\frac{n}{2}} \tag{49}$$

if $Q(y)>0$. The formula (44)appear in([3]).
On the other hand, by putting $\lambda=\frac{\alpha-n}{2m}$ in (43) we have

$$F\left\{G_+^{\frac{\alpha-n}{2m}}(x,m)\right\}=F\left\{\left(\left(_{i=1}^{p}x_i^2\right)^m-\left(_{i=p+1}^{p+q}x_i^2\right)^m\right)_+^{\frac{\alpha-n}{2m}}\right\}= \\ \frac{2^\alpha}{m}\pi^{\frac{n}{2}}\Gamma(1+\frac{\alpha-n}{2m}). \tag{50}$$

$$_{j\geq 0}\frac{(-1)^j\Gamma(\frac{q}{2m}+\frac{j}{m})\Gamma(\frac{\alpha}{2}+j)\left(|\bar{y_q}|^2\right)^j\left(|\bar{y_p}|^2\right)^{-\frac{\alpha}{2}-j}}{j!\Gamma(\frac{q-2}{2}+j+1)\Gamma(\frac{q}{2m}+\frac{j}{m}+\frac{\alpha-n}{2}+1)\Gamma(\frac{n-\alpha}{2}-\frac{q}{2}-j)}.$$

where n is the dimension of the space and $p+q=n$.

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