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DIRECT SOLUTION OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS USING A ONE STEP HYBRID BLOCK METHOD OF ORDER FIVE

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Abstract: In this article, a power series of order eight is adopted as a basis function to develop one step hybrid block method with three off step points for solving general fourth order ordinary differential equations. The strategy is employed for the developing this method are interpolating the power series at x_n and all off-step points and collocating its fourth derivative at all points in the selected interval. The method derived is proven to be consistent, zero stable and convergent with order five. Taylors series is used to supply the starting values for the implementation of the method while the performance of the method is tasted by solving linear and non-linear problems.

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1. Introduction

Ordinary differential equations are frequently used in mathematical modelling in order to analyse and understand variety of real world problems. Hence, finding the solution to these equations is very crucial. In this study, we are interested in finding the numerical solution of the general fourth order initial value problem of the form

$$\begin{aligned} y^{iv} =& f(x, y, y', y'', y'''), \quad x \in [a, b], \\ y(a) =& \eta_0, \quad y'(a) = \eta_1, \quad y''(a) = \eta_2, \quad y'''(a) = \eta_3. \end{aligned} \tag{1}$$

Equation (1) can be solved by reducing it into an equivalent system of four first-order IVPs equations and then several suitable numerical methods for solving this systems are employed. However, [11] had shown that replacing this approach by direct method will reduce computational work. Since then, direct methods were widely used for solving IVPs (see [3], [5], [6] and [8]).

Recently, scholars have proposed a hybrid block methods for solving equation (1) directly in order to avoid the shortcoming in reduction method and to take the advantages of hybrid and block methods which includes overcoming zero stability barrier and generating numerical solutions simultaneously[7]. Some work in hybrid block methods can be found in [1], [2], [9], [10], and [4].

2. Methodology

In deriving the new method, the following power series is used as an approximate solution to (2). Power series as below is used for the derivation the new method

$$y(x) = \sum_{i=0}^{d+c-1} a_i \left(\frac{x - x_n}{h} \right)^i, \quad n = 0, 1, 2, ..., N-1, \; x \in [x_n, x_{n+1}], \tag{2}$$

where $d = 4$ and $c = 5$ denote the number of interpolation and collocation points respectively, $h = x_n - x_{n-1}$ and $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.

Interpolating (2) at x_n, $x_{n+\frac{1}{4}}$, $x_{n+\frac{1}{3}}$, $x_{n+\frac{2}{3}}$ and collocating the fourth derivative of (2) at all points in the interval i.e at x_n, $x_{n+\frac{1}{4}}$, $x_{n+\frac{1}{3}}$, $x_{n+\frac{2}{3}}$ and x_{n+1} produce a system of equations which can be written in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} & \frac{1}{4096} & \frac{1}{16384} & \frac{1}{65536} \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \frac{1}{243} & \frac{1}{729} & \frac{1}{2187} & \frac{1}{6561} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} & \frac{32}{243} & \frac{64}{729} & \frac{128}{2187} & \frac{256}{6561} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{30}{h^4} & \frac{45}{(2h^4)} & \frac{105}{(8h^4)} & \frac{105}{(16h^4)} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{40}{h^4} & \frac{40}{h^4} & \frac{280}{(9h^4)} & \frac{560}{(27h^4)} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{80}{h^4} & \frac{160}{h^4} & \frac{2240}{(9h^4)} & \frac{8960}{(27h^4)} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix} \tag{3}$$

Using Gaussian elimination method, the values of $a_i's, i = 0(1)8$ in equation (2) are obtained. The values of $a_i's$ are then substituted into equation (2) to yield a continuous implicit scheme of the form

$$y(x) = \sum_{i=0,\frac{1}{4},\frac{1}{3},\frac{2}{3}} \alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \beta_i(x) f_{n+i} + \sum_{i=\frac{1}{4},\frac{1}{3},\frac{2}{3}} \beta_i(x) f_{n+i} \tag{4}$$

Differentiating equation (4) once, twice and thrice, we obtain

$$y^{'}(x) = \sum_{i=0,\frac{1}{4},\frac{1}{3},\frac{2}{3}} \frac{d}{dx}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{d}{dx}\beta_i(x) f_{n+i} + \sum_{i=\frac{1}{4},\frac{1}{3},\frac{2}{3}} \frac{d}{dx}\beta_i(x) f_{n+i}, \tag{5}$$

$$y^{''}(x) = \sum_{i=0,\frac{1}{4},\frac{1}{3},\frac{2}{3}} \frac{d^2}{dx^2}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{d^2}{dx^2}\beta_i(x) f_{n+i} + \sum_{i=\frac{1}{4},\frac{1}{3},\frac{2}{3}} \frac{d^2}{dx^2}\beta_i(x) f_{n+i}, \tag{6}$$

$$y^{'''}(x) = \sum_{i=0,\frac{1}{4},\frac{1}{3},\frac{2}{3}} \frac{d^3}{dx^3}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{d^3}{dx^3}\beta_i(x) f_{n+i} + \sum_{i=\frac{1}{4},\frac{1}{3},\frac{2}{3}} \frac{d^3}{dx^3}\beta_i(x) f_{n+i}, \tag{7}$$

where

$$\alpha_0 = \frac{((h-3x+3x_n)(h-4x+4x_n)(2h-3x+3x_n))}{(2h^3)}$$

$$\alpha_{\frac{1}{4}} = \frac{(64(x-x_n)(h-3x+3x_n)(2h-3x+3x_n))}{(5h^3)}$$

$$\alpha_{\frac{1}{3}} = \frac{-(9(x-x_n)(h-4x+4x_n)(2h-3x+3x_n))}{h^3}$$

$$\alpha_{\frac{2}{3}} = \frac{(9(x-x_n)(h-3x+3x_n)(h-4x+4x_n))}{(10h^3)}$$

$$\begin{aligned}\beta_0 = & \frac{((x-x_n)^4}{24} + \frac{(471353h^2(x-x_n)^2)}{209018880} - \frac{(19(x-x_n)^5)}{(240h)} \\ & + \frac{(31(x-x_n)^6)}{(360h^2)} - \frac{(27(x-x_n)^7)}{(560h^3)} + \frac{(3(x-x_n)^8)}{(280h^4)} \\ & - \frac{(14381h(x-x_n)^3)}{1105920} - \frac{(17329h^3(x-x_n))}{104509440)}\end{aligned}$$

$$\begin{aligned}\beta_{\frac{1}{4}} = & \frac{-((352(x-x_n)^6)}{(675h^2)} - \frac{(64(x-x_n)^5)}{(225h)} - \frac{(476041h^2(x-x_n)^2)}{24494400} \\ & + \frac{(16(x-x_n)^8)}{(175h^4)} + \frac{(7357h(x-x_n)^3)}{129600} + \frac{(24449h^3(x-x_n))}{12247200)} \\ & - \frac{(64(x-x_n)^7)}{(175h^3)}\end{aligned}$$

$$\begin{aligned}\beta_{\frac{1}{3}} = & \frac{-((1507h^2(x-x_n)^2)}{516096} + \frac{(9(x-x_n)^5)}{(40h)} - \frac{(39(x-x_n)^6)}{(80h^2)} \\ & + \frac{(207(x-x_n)^7)}{(560h^3)} - \frac{(27(x-x_n)^8)}{(280h^4)} - \frac{(7513h(x-x_n)^3)}{368640} \\ & + \frac{(73h^3(x-x_n))}{1290240)}\end{aligned}$$

$$\begin{aligned}\beta_{\frac{2}{3}} = & \frac{-((3(x-x_n)^6)}{(50h^2)} - \frac{(9(x-x_n)^5)}{(400h)} - \frac{(38957h^2(x-x_n)^2)}{38707200} \\ & + \frac{(27(x-x_n)^8)}{(1400h^4)} + \frac{(5561h(x-x_n)^3)}{1843200} + \frac{(2053h^3(x-x_n))}{19353600)}\end{aligned}$$

$$-\frac{(171(x-x_n)^7)}{(2800h^3)}$$

$$\beta_1 = \frac{((17(x-x_n)^6)}{(2160h^2)} - \frac{(x-x_n)^5}{(360h)} - \frac{(62887h^2(x-x_n)^2)}{627056640}$$
$$-\frac{(x-x_n)^7}{(112h^3)} + \frac{(x-x_n)^8}{(280h^4)}\frac{(215h(x-x_n)^3)}{663552}$$
$$+\frac{(3119h^3(x-x_n))}{313528320)}$$

Equation (4) is evaluated at the non-interpolating point x_{n+1} while its derivatives is evaluated at all points to produce the discreet schemes. The discreet schemes and its derivative are joined to form a block of the form

$$A^{[3]_4}Y_m \quad = \quad B^{[3]_4}R_1^{[3]_4} + h^4\left[D^{[3]_4}R_2^{[3]_4} + E^{[3]_4}R_3^{[3]_4}\right], \tag{8}$$

where $A^{[3]_4}, B^{[3]_4}, R_1^{[3]_4}, D^{[3]_4}, R_2^{[3]_4}, E^{[3]_4}$ and $R_3^{[3]_4}$ are shown in Appendix A.

Multiplying equation (8) by the inverse of $A^{[3]_4}$, we have

$$IY_m \quad = \quad \bar{B}_1^{[3]_4}R_1^{[3]_4} + +h^4\left[\bar{D}^{[3]_4}R_2^{[3]_4} + \bar{E}^{[3]_4}R_3^{[3]_4}\right], \tag{9}$$

so $\bar{B}_1^{[3]_4}$, $\bar{D}^{[3]_4}$ and $\bar{E}^{[3]_4}$ are also clear in Appendix A

3. Analysis of the Method

3.1. Order of the Methhod

The linear difference operator L associated with (9) is defined as

$$L[y(x);h] = \hat{I}\hat{Y}_m - \hat{\bar{B}}^{[3]_4}R_1^{[3]_4} - h^4\left[\hat{\bar{D}}^{[3]_4}R_2^{[3]_4} + \hat{\bar{E}}^{[3]_4}R_3^{[3]_4}\right] \tag{10}$$

where $\hat{I}$ is identity matrix and $\hat{\bar{B}}^{[3]_4}$, $\hat{\bar{D}}^{[3]_4}$, $\hat{\bar{E}}^{[3]_4}$ are 4×4 coefficients matrices of the main block $\hat{Y_m}$=$[y_{n+\frac{1}{4}}, y_{n+\frac{1}{3}}, y_{n+\frac{1}{4}}, y_{n+\frac{2}{3}}]^T$. Expanding $\hat{Y}_m$ and $R_3^{[3]_4}$ components in Taylors series respectively and collecting their terms in powers of h yields

$$L[y(x),h] = \bar{C}_0y(x) + \bar{C}_1hy^{'}(x) + \bar{C}_2hy^{''}(x) + \cdots \tag{11}$$

Definition 1. One step hybrid block method (9) and its linear operator (10) are said to have order p, if $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{p+4} = 0$ and $\bar{C}_{p+4} \neq 0$ with error constants vector $\bar{C}_{p+4}$.

Expanding (9) about $x = x_n$ using Taylor series

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(\frac{1}{4})^j h^j}{j!} y_n^j - y_n - \frac{h}{4} y_n^{'} - \frac{h^2}{32} y_n^{''} - \frac{h^3}{384} y_n^{'''} \\ - \frac{1525160066678784869h^4}{14708396095021850296320} y_n^{iv} \\ - \frac{18463342850998261}{107727510461585817600} \sum_{j=0}^{\infty} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{276458454908929}{2269814212194729984} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ - \frac{234407082948819889}{21790216437069407846400} \sum_{j=0}^{\infty} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{-904854089194540327}{706003012561048814223360} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \\ \\ \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^j}{j!} y_n^j - y_n - \frac{h}{3} y_n^{'} - \frac{h^2}{18} y_n^{''} - \frac{h^3}{162} y_n^{'''} \\ - \frac{1579262226933125475h^4}{551564853563319386112} y_n^{iv} \\ - \frac{491642959321279}{807956328461893632} \sum_{j=0}^{\infty} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{31525197391593517}{76606229661572136960} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ - \frac{432345564227566513}{12256996745851541913600} \sum_{j=0}^{\infty} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{110488310858156393}{26475112971039330533376} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \\ \\ \sum_{j=0}^{\infty} \frac{(\frac{2}{3})^j h^j}{j!} y_n^j - y_n - \frac{2h}{3} y_n^{'} - \frac{2h^2}{9} y_n^{''} - \frac{4h^3}{81} y_n^{'''} \\ - \frac{2041631831074625743h^4}{689456066954149232640} y_n^{iv} \\ - \frac{48038396025285287}{5049727052886835200} \sum_{j=0}^{\infty} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{36028797018963977}{7660622966157213696} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ - \frac{226466724119201927}{437749883780412211200} \sum_{j=0}^{\infty} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{1921535841011412749}{33093891213799163166720} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \\ \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - h y_n^{'} - \frac{h^2}{2} y_n^{''} - \frac{h^3}{6} y_n^{'''} \\ + \frac{904854089194540327h^4}{706003012561048814223360} y_n^{iv} \\ - \frac{50158840849838897}{1346593880769822720} \sum_{j=0}^{\infty} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{30399297484750853}{2837267765243412480} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ - \frac{5836665117072161713}{1361888527316837990400} \sum_{j=0}^{\infty} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\ + \frac{11673330234144328999}{44125188285065550888960} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_8 = 0,$$

and $\bar{C}_{5+4} \neq 0$.

Hence, the new method is of order $[5, 5, 5, 5]^T$ with error constant

$$[1.667949e^{-9}, 5.369218e^{-9}, 7.168291e^{-8} - 2.362547e^{-6}]^T.$$

3.2. Zero Stability

One step hybrid block method (9) is said to be zero stable if the first characteristic polynomial $\pi(z)$ having roots such that $|z_r| \leq 1$, and if $|z_r| = 1$ then the multiplicity of z_r must not greater than two. Applying this in our method $\hat{Y_m}$ where $\hat{I}$ represents 4×4 identity matrix and $\hat{B}$ denotes the coefficients matrix of y_n yields

$$\begin{aligned} \Pi(r) &= |z\,\hat{I} - \hat{B}| \\ &= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\ &= r^3(r-1) \end{aligned}$$

whose solutions are $r = 0, 0, 1, 1$. Hence, the new main block method in (9) is zero stable. This implies, the new hybrid block method is convergent since it is zero stable and consistent ($p \geq 1$), see Henrici, 1962.

3.3. Region of Absolute Stability

The locus method is employed to draw the region of absolute stability. Substituting the test equation $y^{''''} = -\lambda^4 y$ in (9) where $\bar{h} = \lambda^4 h^4$ and $\lambda = \frac{df}{dy}$ gives characteristic function $\Pi(r) = \rho(r) - \bar{h}\sigma(r)$. let $r = e^{i\theta} = \cos\theta - i\sin\theta$ and considering real part yields

$$\bar{h}(\theta, h) = \frac{2367600721920000(\cos(\theta) - 1))}{(3\cos(\theta) - 368)}, \tag{12}$$

which gives the region stability of the method as shown in Figure 1.

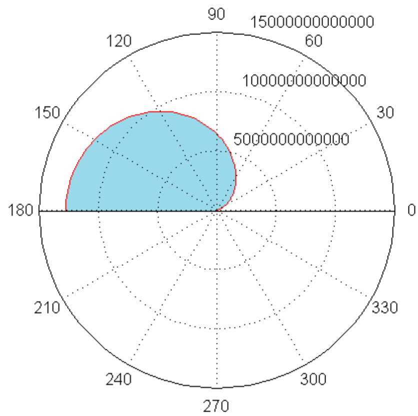

Figure 1: Region stability of new method

4. Numerical Experiments

In this section, the performance of new one step hybrid block has been tested on one non-linear IVP and two linear IVPs. For each example, computed solution (COP), exact solution (EXT) and absolute errors (ERR) were determined using *Matlab* code, see Table 1 – Table 3.

Problem 1.

$$y^{iv} - (y^{'})^2 + yy^{''} + 4x^2 - e^x(1 - 4x + x^2) = 0,$$

$$y(0) = 1, \quad y^{'}(0) = 1, \quad y^{''}(0) = 1, \quad h = \frac{1}{100}.$$

Exact solution: $y(x) = x^2 + e^x$.

Problem 2.

$$y^{iv} + y^{''} = 0, \quad y(0) = 0, \quad y^{'}(0) = \frac{-1.1}{72 - 50\pi},$$

$$y^{'''}(0) = \frac{1.2}{144 - 100\pi}, \quad h = \frac{1}{320}, \quad y^{''}(0) = \frac{1}{144 - 100\pi}.$$

Exact solution: $y(x) = \frac{1 - x - \cos x - 1.2 \sin x}{144 - 100\pi}$.

x		New method, $P = 5$	see [9], $P = 6$
0.003125	EXT	1.0031396535277390	1.003139653527739149
	CPS	1.0031396535277384	1.003139653526590265
	ERR	$6.661338e^{-16}$	$1.148884e^{-12}$
0.006250	EXT	1.0063086345037620	1.006308634503762010
	CPS	1.0063086345037602	1.006308634484910542
	ERR	$1.776357e^{-15}$	$1.8851468e^{-11}$
0.009375	EXT	1.0095069735890709	1.009506973589071086
	CPS	1.0095069735890683	1.009506973491318106
	ERR	$2.664535e^{-15}$	$9.7752980e^{-11}$
0.012500	EXT	1.0127347015406345	1.012734701540634377
	CPS	1.0127347015406303	1.012734701224875248
	ERR	$4.218847e^{-15}$	$3.15759129e^{-10}$
0.015625	EXT	1.0159918492116857	1.015991849211685747
	CPS	1.0159918492116808	1.015991848424806972
	Error	$4.884981e^{-15}$	$1.15463e^{-10}$

Table 1: Comparison of the new method with [9] for solving Problem 1, where $h = \frac{1}{320}$

Problem 3.

$$y^{iv} - x = 0, \quad y(0) = 0, \quad y^{'}(0) = 1, \quad y^{''}(0) = 0,$$

$$y^{'''}(0) = 0, \quad h = \frac{1}{320}.$$

Exact solution: $y(x) = \frac{x^5}{120} + x$.

5. Conclusion

We have derived a new hybrid block method for the solution of linear and non-linear forth order initial value problems. The obtained numerical results show the efficiency of the developed method in term of accuracy. Its performance is clear better than existing method through Table 1, Table 2 and Table 3. The new method developed is also found to be consistent, zero-stable, and then convergent.

x		New method, $P = 5$	[10], $P = 6$
0.103125	EXT	0.001300799589367158	0.00130079934027
	CPS	0.001300799589367138	0.00130079934027
	ERR	$1.929880e^{-17}$	$0.498732999343e^{-15}$
0.206250	EXT	0.002531773700195635	0.00253177321538
	CPS	0.002531773700195563	0.00253177321538
	ERR	$7.199102e^{-17}$	$0.676542155631e^{-15}$
0.306250	EXT	0.003652478978884993	0.00365247827946
	CPS	0.003652478978884832	0.00365247827947
	ERR	$1.608956e^{-16}$	$0.313507900196e^{-14}$
0.406250	EXT	0.004695953231804849	0.00469595233257
	CPS	0.004695953231804570	0.00469595233258
	ERR	$2.792905e^{-16}$	$0.943602834758e^{-14}$
0.506250	EXT	0.005657642360593461	0.00565764127720
	CPS	0.005657642360593029	0.00565764127722
	ERR	$4.310788e^{-16}$	$0.221168569570e^{-13}$

Table 2: Comparison of the new method with [10] for solving Problem 2, where $h = \frac{1}{320}$.

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x		New method, $P = 5$	[1], $P = 8$
	EXT	0.100000083333333340	0.100000083333334000
0.1	CPS	0.100000083333333340	0.10000008333351720
	ERR	$0.000000e^{0}$	$1.832e^{-13}$
	EXT	0.200002666666666690	0.200002666666666900
0.2	CPS	0.200002666666666580	0.20000266667150250
	ERR	$1.110223e^{-16}$	$4.835e^{-12}$
	EXT	0.300020250000000040	0.300020250000000004
0.3	CPS	0.300020249999999770	0.30002025000721480
	ERR	$2.775558e^{-16}$	$7.214e^{-12}$
	EXT	0.400085333333333350	0.400008533333333333
0.4	CPS	0.400085333333332740	0.40000853340160457
	ERR	$6.106227e^{-16}$	$6.832e^{-11}$
	EXT	0.500260416666666650	0.500260416666666665
0.5	CPS	0.500260416666665760	0.50026041674083458
	ERR	$8.881784e^{-16}$	$7.416e^{-11}$

Table 3: Comparison of the new method with [1] for solving Problem 3, where $h = \frac{1}{10}$

[9] B. Olabode, et al, Implicit hybrid block numerov-type method for the direct solution of fourth-order ordinary differential equations, *American Journal of Computational and Applied Mathematics*, **5**, No. 5 (2015), 129-139.

[10] S.J. Kayode, An efficient zero-stable numerical method for fourth-order differential equations, *International Journal of Mathematics and Mathematical Sciences* (2008)

[11] D. Awoyemi, *On Some Continuous Linear Multistep Methods for Initial Value Problems*, Unpublished Doctoral Dissertation, University of Ilorin, Nigeria (1992).

[12] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, New York, 1962.

6. Appendix A

$$A^{[3]_4} = \begin{pmatrix}
\frac{-128}{5} & 27 & \frac{-27}{5} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-128}{5h} & \frac{18}{h} & \frac{-9}{10h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{52}{5h} & \frac{-45}{4h} & \frac{9}{40h} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{64}{5h} & \frac{-12}{h} & \frac{-3}{10h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-128}{5h} & \frac{30}{h} & \frac{-69}{10h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-704}{5h} & \frac{144}{h} & \frac{-207}{10h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1152}{5h^2} & \frac{-198}{h^2} & \frac{63}{5h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{288}{5h^2} & \frac{-36}{h^2} & \frac{-18}{5h^2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{18}{h^2} & \frac{-9}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1152}{5h^2} & \frac{234}{h^2} & \frac{-153}{5h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2304}{5h^2} & \frac{450}{h^2} & \frac{-261}{5h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{-3456}{5h^3} & \frac{648}{h^3} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-3456}{5h^3} & \frac{648}{h^3} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{-3456}{5h^3} & \frac{648}{h^3} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{-3456}{5h^3} & \frac{648}{h^3} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{-3456}{5h^3} & \frac{648}{h^3} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$Y_m = \begin{pmatrix}
y_{n+\frac{1}{4}} \\
y_{n+\frac{1}{3}} \\
y_{n+\frac{2}{3}} \\
y_{n+1} \\
y'_{n+\frac{1}{4}} \\
y'_{n+\frac{1}{3}} \\
y'_{n+\frac{2}{3}} \\
y'_{n+1} \\
y''_{n+\frac{1}{3}} \\
y''_{n+\frac{2}{3}} \\
y''_{n+\frac{1}{4}} \\
y''_{n+1} \\
y'''_{n+\frac{1}{4}} \\
y'''_{n+\frac{1}{3}} \\
y'''_{n+\frac{2}{3}} \\
y'''_{n+1}
\end{pmatrix},$$

$$B^{[3]_4} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ \frac{-17}{(2h)} & -1 & 0 & 0 \\ \frac{-5}{(8h)} & 0 & 0 & 0 \\ \frac{1}{(2h)} & 0 & 0 & 0 \\ \frac{-5}{(2h)} & 0 & 0 & 0 \\ \frac{-35}{(2h)} & 0 & 0 & 0 \\ \frac{45}{h^2} & 0 & -1 & 0 \\ \frac{18}{h^2} & 0 & 0 & 0 \\ \frac{9}{h^2} & 0 & 0 & 0 \\ \frac{-27}{h^2} & 0 & 0 & 0 \\ \frac{-63}{h^2} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & -1 \\ \frac{-108}{h^3} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & 0 \end{pmatrix}, \; R_1^{[3]_4} = \begin{pmatrix} y_n \\ y_n' \\ y_n'' \\ y_n''' \end{pmatrix},$$

$$D^{[3]_4} = \begin{pmatrix} \frac{979h^4}{4976640} \\ \frac{-17329h^3}{104509440} \\ \frac{5935h^3}{668860416} \\ \frac{-5357h^3}{627056640} \\ \frac{7625h^3}{62705664} \\ \frac{82569326393009h^3}{78812993478983680} \\ \frac{2843665867144587h^2}{630503947831869440} \\ \frac{-17719067273067h^2}{78812993478983680} \\ \frac{-41h^2}{204120} \\ \frac{296061227673923h^2}{1576259869579967360} \\ \frac{1832684344077h^2}{1231453023109120} \\ \frac{-1537285236573981h}{1970324836974592 0} \\ \frac{47401167953007h}{90071992547409920} \\ \frac{40668356010069h}{630503947831869440} \\ \frac{105105845795691h}{7881299347898368} \\ \frac{-3629h}{184320} \end{pmatrix},$$

$$E^{[3]_4} = \begin{pmatrix}
\frac{(-1213h^4)}{583200} & \frac{(137h^4)}{20480} & \frac{(1999h^4)}{921600} & \frac{(-461h^4)}{14929920} \\
\frac{(-24449h^3)}{12247200} & \frac{(-2548074565957h^3)}{45035996273704960} & \frac{(-2053h^3)}{19353600} & \frac{(3119h^3)}{313528320} \\
\frac{(12353h^3)}{78382080} & \frac{(167h^3)}{917504} & \frac{(253h^3)}{17694720} & \frac{(-2201h^3)}{2006581248} \\
\frac{(-7549h^3)}{73483200} & \frac{(-857h^3)}{3317760} & \frac{(-2033h^3)}{116121600} & \frac{(2419h^3)}{1881169920} \\
\frac{(-6887h^3)}{7348320} & \frac{(3233h^3)}{774144} & \frac{(2047h^3)}{3870720} & \frac{-(5911h^3)}{188116992} \\
\frac{(-1108604311169h^3)}{96207267430400} & \frac{(6739h^3)}{172032} & \frac{(14635578915999891h^3)}{788129934789836800} & \frac{(7336424722119h^3)}{39406496739491840} \\
\frac{(5128479470881h^2)}{131941395333120} & \frac{(-1507h^2)}{258048} & \frac{(50765836424615901h^2)}{2522015791327477600} & \frac{(-2023447286282923h^2)}{1008806316530991104O} \\
\frac{(-468278332203551h^2)}{98516241848729600} & \frac{(-2895136283892861h^2)}{1261007895663738880} & \frac{(-3422820419919567h^2)}{1261007895663738800} & \frac{(112547357585937h^2)}{504403158265495552O} \\
\frac{(-352h^2)}{382725} & \frac{(-29h^2)}{3780} & \frac{(-19h^2)}{37800} & \frac{(11h^2)}{306180} \\
\frac{(-181382466546167h^2)}{9235897673318400} & \frac{(2398533712677243h^2)}{39406496739491840} & \frac{(4210783391832579h^2)}{394064967394918400} & \frac{(-520746202572989h^2)}{945755921747804160} \\
\frac{(-199588131815h^2)}{11544872091648} & \frac{(75119h^2)}{645120} & \frac{(418379808958713h^2)}{3518437208883200} & \frac{(3370668535027h^2)}{461794883665920} \\
\frac{(-7357h)}{21600} & \frac{(7513h)}{61440} & \frac{(-5561h)}{307200} & \frac{(215h)}{110592} \\
\frac{(79h)}{4320} & \frac{(-4853h)}{61440} & \frac{(-151h)}{61440} & \frac{(73h)}{552960} \\
\frac{(294205782948971h)}{4925812092436480} & \frac{(-443h)}{12288} & \frac{(-953h)}{307200} & \frac{(123018609613061h)}{630503947831869440} \\
\frac{(-529654659952219h)}{3078632557772800} & \frac{(7663550232614229h)}{19703248369745920} & \frac{(2633h)}{20480} & \frac{(-731428617866741h)}{157625986957967360} \\
\frac{(4931h)}{21600} & \frac{(-6311h)}{61440} & \frac{(93964355335705923h)}{197032483697459200} & \frac{(515840206952979h)}{4925812092436480}
\end{pmatrix},$$

and

$$R_3^{[3]_4} = \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix}, \; R_2^{[3]_4} = \begin{pmatrix} f_n \end{pmatrix},$$

$$\bar{B}^{[3]_4} = \begin{pmatrix} 1 & \frac{h}{4} & \frac{h^2}{32} & \frac{h^3}{384} \\ 1 & \frac{h}{3} & \frac{h^2}{18} & \frac{h^3}{162} \\ 1 & \frac{2h}{3} & \frac{2h^2}{9} & \frac{4h^3}{81} \\ 1 & h & \frac{h^2}{2} & \frac{h^3}{6} \\ 0 & 1 & \frac{h}{4} & \frac{h^2}{32} \\ 0 & 1 & \frac{h}{3} & \frac{h^2}{18} \\ 0 & 1 & \frac{2h}{3} & \frac{2h^2}{9} \\ 0 & 1 & h & \frac{h^2}{2} \\ 0 & 0 & 1 & \frac{h}{4} \\ 0 & 0 & 1 & \frac{h}{3} \\ 0 & 0 & 1 & \frac{2h}{3} \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \bar{D}^{[3]_4} = \begin{pmatrix} \frac{152516006667878486 9h^4}{1470839609502185029632 0} \\ \frac{157926226933125475h^4}{551564853563319386112} \\ \frac{204163183107462574 3h^4}{689456066954149232640} \\ \frac{145916627926804129 1h^4}{1313249651341236633 60} \\ \frac{273086032844213593 7h^3}{183854951187773128704 0} \\ \frac{412079365904400578 9h^3}{137891213390829846528 0} \\ \frac{100790559660551741 57h^3}{689456066954149232640} \\ \frac{328312412835309203 3h^3}{919274755938865643 52} \\ \frac{1862572697452545h^2}{126100789566373888} \\ \frac{279673536859707919h^2}{131324965134123663 36} \\ \frac{31135997423796033h^2}{630503947831869440} \\ \frac{6755399441055747h^2}{90071992547409920} \\ \frac{49524935746038441h}{630503947831869440} \\ \frac{7033399418053923h}{90071992547409920} \\ \frac{514299957446631h}{5629499534213120} \\ \frac{10344205394116613h}{177329235327713280} \end{pmatrix},$$

$$\bar{E}^{[3]_4} =$$

$$\begin{pmatrix}
\frac{18463342850998261h^4}{1077275104615858176 00} & \frac{-276458454908929h^4}{2269814212194729984} & \frac{2344070829488198 89h^4}{217902164370694078464 00} & \frac{-904854089194540327h^4}{706003012561048814223360} \\
\frac{491642959321279h^4}{8079563284618936 32} & \frac{-31525197391593517h^4}{766062296615721369 60} & \frac{432345564227566513h^4}{122569967458515419136 00} & \frac{-110488310858156393h^4}{264751129710393305333 76} \\
\frac{48038396025285287h^4}{504972705288683520 0} & \frac{-36028797018963977h^4}{7660622966157213696} & \frac{226466724119201927h^4}{437749883780412211200} & \frac{-192153584101141274 9h^4}{330938912137991631667 20} \\
\frac{50158840849838897h^4}{134659388076982272 0} & \frac{-30399297484750853h^4}{283726776524341248 0} & \frac{583666511707216171 3h^4}{136188852731683799040 0} & \frac{-116733302341443289 99h^4}{441251882850655508889 60} \\
\frac{41479900791570421h^3}{134659388076982272 00} & \frac{-860387482870931h^3}{405323966463344640} & \frac{498025990824393649h^3}{272377705463367598080 0} & \frac{-191088084033051370 3h^3}{882503765701311017779 20} \\
\frac{79375943432404981h^3}{100994541057736704 00} & \frac{-128915539333480493h^3}{255354098871907123 20} & \frac{864691128455134129h^3}{204283279097525698560 0} & \frac{-329663492723520643 9h^3}{661877824275983263334 40} \\
\frac{256705178760118261h^3}{504972705288683520 0} & \frac{-54043195528445957h^3}{283726776524341248 0} & \frac{112890230659420396 3h^3}{340472131829209497600} & \frac{-112409846699167613 83h^3}{330938912137991631667 20} \\
\frac{410390516044136261h^3}{336648470192455680 0} & \frac{-45598946227126277h^3}{283726776524341248 0} & \frac{175099953512164832 81h^3}{680944263658418995200} & \frac{-131324965134124974 47h^3}{220625941425327754444 80} \\
\frac{110461336172888059h^2}{265993852991569920 0} & \frac{-61344832346180833h^2}{2269814212194729984} & \frac{169571081282125711h^2}{756604737398243328 00} & \frac{-716705659449508 81h^2}{272377705463367598080} \\
\frac{248260929458798581h^2}{336648470192455680 0} & \frac{-23h^2}{540} & \frac{239591500176110276 9h^2}{680944263658418995200} & \frac{-129703669268270332 9h^2}{315179916321896792064 0} \\
\frac{1681343860884985h^2}{9974769487183872} & \frac{-250199979298361h^2}{16888498602639360} & \frac{470776281047794892 3h^2}{226981421219472998400} & \frac{-134507508870798893 3h^2}{817133116390102794240} \\
\frac{11821949021847541h^2}{415615395299328 00} & 0 & \frac{102141639548762799 13h^2}{756604737398243328 00} & \frac{151320947479648256 9h^2}{272377705463367598080} \\
\frac{323h}{900} & \frac{-2061h}{10240} & \frac{801h}{51200} & \frac{-167h}{92160} \\
\frac{266212777973455981h}{664984632478924800} & \frac{-19h}{120} & \frac{3h}{200} & \frac{-29773797536504953h}{170236065914604748 80} \\
\frac{14011198840708199h}{8312307905986560 0} & \frac{31525197391593 47h}{11821949021847552} & \frac{11h}{75} & \frac{-28022397681416407h}{425590164786511872 0} \\
\frac{128h}{225} & \frac{-9h}{40} & \frac{292593238290726857h}{591097451092377600} & \frac{13669128556511233h}{132996926495784960}
\end{pmatrix}.$$