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ON GRADED 2-ABSORBING PRIMARY SUBMODULES

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Abstract: In this paper, we introduce and study the concept of graded 2-absorbing primary submodules of graded modules over graded commutative rings generalizing graded 2-absorbing submodules. Let R be a graded ring and M be a graded R-module. A proper graded submodule N of M is called a graded *2-absorbing primary submodule* of M if whenever $a, b \in h(R)$ and $m \in h(M)$ and $abm \in N$, then $am \in M\text{-Gr}(N)$ or $bm \in M\text{-Gr}(N)$ or $ab \in (N :_R M)$.

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1. Introduction and Preliminaries

Let G be a multiplicative group with identity e and R be a commutative ring with $1 \neq 0$. R is called a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and the set of all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $a \in R$, then the element a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is called the g-component of a in R_g. In this case, R_e is a subring of R and $1_R \in R_e$. Let R be a G-graded ring and M an R-module. We call M as a graded R-module if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$ where $R_g M_h$ denotes the additive subgroup of M

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consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \underset{g\in G}{\cup} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \oplus_{g\in G} M_g$ is a graded R-module, then the subgroup M_g of M is an R_e -module for all $g \in G$. Let N be a submodule of M. Then N is called a graded submodule of M if $N = \oplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$ [11]. The graded radical of I (in abbreviation $Gr(I)$) is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of $G(R)$, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in N$ [13]. Let R be a graded ring, M a graded R-module and N a graded submodule of M. We will denote by $(N :_R M)$ *the residual of* N *by* M, that is, the set of all $r \in h(R)$ such that $rM \subseteq N$. Graded multiplication modules are introduced by S.E Atani and R.E Atani in [7]: A graded R-module M is said to be a graded multiplication module if for each graded submodule N of M, $N = IM$ for some graded ideal I of R. We say that I is a presentation graded ideal of N. Note that since $I \subseteq (N :_R M)$, then $N = IM \subseteq (N :_R M)M \subseteq N$. So $N = (N :_R M)M$. Let $N = IM$ and $K = JM$ be graded submodules of a graded multiplication R-module M. Then the product of N and K is independent from their graded presentation ideals I and K. [12] It can be easily understood that if a graded module is multiplication, then it is a graded multiplication module. However the converse of this observation is not true. There is an example in [14] for a graded multiplication ring which is not multiplication.The group ring $R[Z]$, where R is a Dedekind domain is a graded Dedekind domain and so it is a graded multiplication domain. On the other hand, if R is not a field, then $R[Z]$ is not a Dedekind domain and so it is not a multiplication domain. Thus a graded multiplication module need not be multiplication module. Let N be a proper graded submodule of a nonzero graded R-module M. Then the M-graded radical of N, denoted by M-$Gr(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If M has no graded prime submodule containing N, then we say M-$Gr(N) = M$. It is shown in [12, Theorem 9] that if N is a proper graded submodule of a graded multiplication R-module M, then M-$Gr(N) = Gr((N :_R M))M$.

Let R be a graded commutative ring with $1 \neq 0$ and M be a graded R-module. Graded prime and graded primary ideals have been introduced and studied in [13]. A proper graded ideal P of R is said to be graded prime (resp. graded weakly prime) ideal if whenever $a, b \in h(R)$ with $ab \in P$ (resp. $0 \neq ab \in P$), then either $a \in P$ or $b \in P$ [13]. A proper graded ideal I of R is said to be a graded primary ideal of R if whenever $a, b \in h(R)$ with $ab \in I$, then

$a \in I$ or $a \in Gr(I)$. [13]. Graded prime and graded primary submodules of M have been studied in [3], [4], [5], [12]. A proper graded submodule N of M is said to be graded prime (resp. graded weakly prime) submodule if whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$ (resp. $0 \neq am \in N$), then either $a \in (N :_R M)$ or $m \in N$. A proper graded ideal I of R is said to be graded 2-absorbing (graded weakly 2-absorbing) ideal if whenever $a, b, c \in h(R)$ with $abc \in I$ (resp. $0 \neq abc \in I$), then either $ab \in I$ or $bc \in I$ or $ac \in I$. Recently, graded 2-absorbing and weakly graded 2-absorbing submodules of M have been introduced in [1]. A proper graded submodule N of M is said to be graded 2-absorbing (graded weakly 2-absorbing) submodule if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$ (resp. $0 \neq abm \in N$) then either $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. As it is defined in [15], a graded ideal I of R is said to be a graded 2-absorbing primary ideal if whenever $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. In this paper we intoduce the concept of graded 2-absorbing primary submodules as follows: Let R be a G-graded ring, N be a proper graded submodule of a graded R-module M and $g \in G$. We say a proper R_e-submodule N_g is to be a *g-2-absorbing primary submodule* of M_g if whenever $a, b \in R_e$ and $m \in M_g$ with $abm \in N_g$, then $ab \in (N_g :_{R_e} M_g)$ or $am \in M_g\text{-Gr}(N_g)$ or $bm \in M_g\text{-Gr}(N_g)$. We say N to be a graded *2-absorbing primary submodule* of M if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in M\text{-}Gr(N)$ or $bm \in M\text{-}Gr(N)$. We give some basic results of this class of graded submodules and discuss on the relations among graded 2-absorbing ideals, graded 2-absorbing primary ideals, graded 2-absorbing submodules and graded 2-absorbing primary submodules.

2. Properties of Graded 2-Absorbing Primary Submodules

Lemma 1. *Let R be a graded ring and M be a graded R-module. If N is a graded 2-absorbing primary submodule of M, then N_g is a g-2-absorbing primary R_e-submodule of M_g for all $g \in G$.*

Proof. Let $a, b \in R_e$, $m \in M_g$ with $abm \in N_g$. Since N is a graded 2-absorbing primary submodule of M and $N_g = (N \cap M_g) \subset N$ we get either $ab \in (N :_R M)$ or $am \in M\text{-}Gr(N)$ or $bm \in M\text{-}Gr(N)$. If $ab \in (N :_R M)$, then $ab \in (N_g :_{R_e} M_g)$ as $(N :_R M) \subset ((N \cap M_g) :_{R_e} M_g) = (N_g :_{R_e} M_g)$. Suppose that $am \in M\text{-}Gr(N)$. Since $am \in M_g$ and $am \in M\text{-}Gr(N)$, we have $am \in M\text{-}Gr(N) \cap M\text{-Gr}(M_g) = M\text{-Gr}(N \cap M_g) = M\text{-Gr}(N_g)$. If $bm \in M\text{-}Gr(N)$, then similarly we conclude that $bm \in M\text{-Gr}(N_g)$. Thus N_g is a g-2-absorbing primary R_e-submodule of M_g. □

We need the following Lemma as it is used in the most of the proofs in this paper.

Lemma 2. *[6], [7], [12] Let M be a graded module over a G-graded ring R. Then*

1. If N is a graded submodule of M, I is a graded ideal of R, $a \in h(R)$ and $m \in h(M)$, then Rm, IN and aN are graded submodules of M, and Ra, $(N :_R M)$ are graded ideals of R.

2. If $\{N_i\}_{i\in\Lambda}$ is a collection of graded submodules of M, then $\sum\limits_{i\in\Lambda} N_i$ and $\bigcap\limits_{i\in\Lambda} N_i$ are graded submodules of M.

3. M is a graded multiplication R-module if and only if for each m in $h(M)$ there exists a graded ideal I of R such that $Rm = IM$.

4. Every homomorphic image of a graded multiplication module is a graded multiplication module.

Theorem 3. *Let M be a finitely generated graded multiplication R-module, and N be a graded 2-absorbing primary submodule of M. Then the following statements hold.*

1. *If $abK \subseteq N$ where $a, b \in R$ and K is a R-submodule of M, then $ab \in (N :_R M)$ or $aK \subseteq M\text{-}Gr(N)$ or $bK \subseteq M\text{-}Gr(N)$.*

2. *If $abK \subseteq N_g$ where $a, b \in R_e$ and K is a R_e-submodule of M_g, then $ab \in (N_g :_{R_e} M_g)$ or $aK \subseteq M\text{-Gr}(N_g)$ or $bK \subseteq M\text{-Gr}(N_g)$.*

3. *$(N_g :_{R_e} M_g)$ is a g-2-absorbing primary ideal of R_e.*

Proof. (1) Assume that $abK \subseteq N$ but $ab \notin (N :_R M)$, $aK \nsubseteq M\text{-}Gr(N)$ and $bK \nsubseteq M\text{-}Gr(N)$. Then $ak_1 \notin M\text{-}Gr(N)$ and $bk_2 \notin M\text{-}Gr(N)$ for some $k_1, k_2 \in h(K)$. Since $abk_1 \in N$ and $ab \notin (N :_R M)$ and $ak_1 \notin M\text{-}Gr(N)$, we have $bk_1 \in M\text{-}Gr(N)$. Since $abk_2 \in N$ and $ab \notin (N :_R M)$ and $bk_2 \notin M\text{-}Gr(N)$, we have $ak_2 \in M\text{-}Gr(N)$. Now, since $ab(k_1 + k_2) \in N$ and $ab \notin (N :_R M)$, we have $a(k_1 + k_2) \in M\text{-}Gr(N)$ or $b(k_1 + k_2) \in M\text{-}Gr(N)$. Suppose that $a(k_1 + k_2) = ak_1 + ak_2 \in M\text{-}Gr(N)$. Since $ak_2 \in M\text{-}Gr(N)$, we have $ak_1 \in M\text{-}Gr(N)$, a contradiction. Suppose that $b(k_1 + k_2) = bk_1 + bk_2 \in M\text{-}Gr(N)$. Since $bk_1 \in M\text{-}Gr(N)$, we have $bk_2 \in M\text{-}Gr(N)$, a contradiction again. Thus $aK \subseteq M\text{-}Gr(N)$ or $bK \subseteq M\text{-}Gr(N)$.

(2) Suppose that N is a graded 2-absorbing primary submodule of M. Then N_g is a g-2-absorbing primary R_e-submodule of M_g by Lemma 1. So we are done from part (1).

(3) Let $a, b, c \in R_e$ such that $abc \in (N_g :_{R_e} M_g)$, $ac \notin Gr(N_g :_{R_e} M_g)$ and $bc \notin Gr(N_g :_{R_e} M_g)$. Put $K = cM_g$. Then K is a R_e-submodule of M_g. Since N_g is a g-2-absorbing primary submodule of M_g and $abK \subseteq N_g$, we have that either $ab \in (N_g :_{R_e} M_g)$ or $aK \subseteq M\text{-Gr}(N_g)$ or $bK \subseteq M\text{-Gr}(N_g)$ by (1). Since $ac \notin Gr(N_g :_{R_e} M_g)$ and $bc \notin Gr(N_g :_{R_e} M_g)$, we conclude that $aK \nsubseteq Gr((N_g :_{R_e} M_g))M = M\text{-Gr}(N_g)$ and $bK \nsubseteq Gr((N_g :_{R_e} M_g))M = M - \text{Gr}(N_g)$. Thus $ab \in (N_g :_{R_e} M_g)$, which shows that $(N_g :_{R_e} M_g)$ is a g-2-absorbing primary ideal of R_e. □

Recall that a graded R-module M is caled graded cyclic if $M = Rm$ where $m \in h(M)$. It is well-known that M is a graded cyclic module, then M is a graded multiplication module.

Theorem 4. (1) *Let M_g be a cyclic R_e-submodule and N a graded submodule of M_g. Then N_g is a g-2-absorbing primary R_e-submodule of M_g if and only if $(N_g :_{R_e} M_g)$ is a g-2-absorbing primary ideal of R_e.*

(2) If M is a cyclic R-module and N a graded submodule of M, then N is a graded 2-absorbing primary submodule of M if and only if $(N :_R M)$ is a graded 2-absorbing primary ideal of R.

Proof. (1) If N_g is a g-2-absorbing primary R_e-submodule of M_g, then $(N_g :_{R_e} M_g)$ is a g-2-absorbing primary ideal of R_e by Theorem 3. Conversely, let $a, b \in R_e$, $m \in M_g$ such that $abm \in N_g$. Since M_g is a graded cyclic R_e-module, $M_g = R_e x$ for some $x \in M$. Then there is $c \in R_e$ such that $m = cx$. Since $abc \in (N_g :_{R_e} M_g)$ and $(N_g :_{R_e} M_g)$ is a g-2-absorbing primary ideal of R_e, we get either $ab \in (N_g :_{R_e} M_g)$ or $ac \in Gr(N_g :_{R_e} M_g)$ or $bc \in Gr(N_g :_{R_e} M_g)$. If $ab \in (N_g :_{R_e} M_g)$, then we are done. So suppose that $ac \in Gr(N_g :_{R_e} M_g)$. Thus $am = acx \in Gr(N_g :_{R_e} M_g)M_g = M_g\text{-Gr}(N_g)$. Similary if $bc \in Gr(N_g :_{R_e} M_g)$, then we have $bm \in M_g\text{-Gr}(N_g)$, so we are done.

(2) One can easily verify similar to part (1). □

Theorem 5. *Let M be a graded R-module and N be a graded submodule of M. If M-$Gr(N)$ is a graded prime submodule of M, then N is a graded 2-absorbing primary submodule of M.*

Proof. Suppose that M-$Gr(N)$ is a graded prime submodule of M and $abm \in N$, $am \notin M$-$Gr(N)$ for some $a, b \in h(R)$ and $m \in h(M)$. Since M-$Gr(N)$ is a graded prime submodule and $abm \in M$-$Gr(N)$, then $b \in (M$-

$Gr(N) :_R M)$. So $bm \in M\text{-}Gr(N)$. Consequently N is a graded 2-absorbing primary submodule of M. □

We say that a graded R-module M is a graded divided module if for every graded prime submodule N of M, we have $N \subseteq Rm$ for all $m \in M \backslash N$.

Theorem 6. *Let R be a graded ring. If M is a divided graded R-module, then every proper submodule of M is a graded 2-absorbing primary submodule of M. In particular, every proper graded submodule of a graded chained module is a graded 2-absorbing primary submodule.*

Proof. Let N be a proper graded submodule of M. Since the graded prime submodules of a graded divided module are linearly ordered, we conclude that $M\text{-}Gr(N)$ is a graded prime submodule of M. Hence N is a graded 2-absorbing primary submodule of M by Theorem 5. □

Lemma 7. *Let R be a graded ring and I be a proper graded ideal of R. Then I is a graded 2-absorbing primary ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq Gr(I)$ or $I_1I_3 \subseteq Gr(I)$.*

Proof. It can be easily obtained by the similar argument in the proof of Theorem 2.19 in [8]. □

Now we present the general form of Theorem 4 just under the condition that M is a graded multiplication.

Theorem 8. *Let M be a graded multiplication module over graded ring R and N be a graded submodule of M. If $(N :_R M)$ is a graded 2-absorbing primary ideal of R, then N is a graded 2-absorbing primary submodule of M.*

Proof. Suppose that $(N :_R M)$ is a graded 2-absorbing primary ideal of R and $I_1I_2K \subseteq N$ for some graded ideals I_1, I_2 of R and some graded

submodule K of M and $I_1I_2 \nsubseteq (N :_R M)$. Here, there exists a graded ideal I_3 of R with $K = I_3M$ as M is graded multiplication. Then $I_1I_2I_3 \subseteq$ $\not(N :_R M)$. Since $(N :_R M)$ is graded 2-absorbing primary, we conclude that either $I_2I_3 \subseteq Gr(N :_R M)$ or $I_1I_3 \subseteq Gr(N :_R M)$ by Lemma 7. Without loss generality assume that $I_2I_3 \subseteq Gr(N :_R M)$. Therefore $I_2I_3M = I_2K \subseteq (Gr(N :_R M))M = M\text{-}Gr(N)$, we are done. □

Note that the restriction on M is necessary in Theorem 8 as one can see by the following example.

Example 9. Let $R = \mathbb{Z} = R_0$ as $\mathbb{Z}$-graded ring and $M = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ as $\mathbb{Z}$-graded R-module with $M_0 = \{0\} \times \mathbb{Z} \times \mathbb{Z}$, $M_2 = \mathbb{Z} \times \{0\} \times \mathbb{Z}$ and $M_3 = \mathbb{Z} \times \mathbb{Z} \times \{0\}$. Now consider a graded submodule $N = 2\mathbb{Z} \times 3\mathbb{Z} \times \{0\}$. Observe that $(N :_R M) = 0$ is a graded 2-absorbing primary ideal of R. However N is not a graded 2-absorbing primary submodule of M. To see this, first observe that $2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z} \times \{0\}$ are graded prime submodules of M containing N. Then $M\text{-}Gr(N) \subseteq 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cap \mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z} \cap \mathbb{Z} \times \mathbb{Z} \times \{0\} = N$. Since $2.3.(1,1,0) \in N$ but neither $2.3 \in (N :_R M)$ nor $2.(1,1,0) \in M\text{-}Gr(N)$ nor $3.(1,1,0) \in M\text{-}Gr(N)$, N is not graded 2-absorbing primary submodule of M.

Theorem 10. *Let M be a graded multiplication R-module and N_1 and N_2 be graded primary submodules of M. Then $N_1 \cap N_2$ is a graded 2-absorbing primary submodule of M.*

Proof. Suppose that N_1 and N_2 are graded primary submodules of M. Then $(N_1 :_R M)$ and $(N_2 :_R M)$ are graded primary ideals of R by Proposition 2.5 in [6]. Hence $(N_1 :_R M) \cap (N_2 :_R M) = (N_1 \cap N_2 :_R M)$ is a graded 2-absorbing primary ideal of R by Theorem 2 in [15]. Thus $N_1 \cap N_2$ is a graded 2-absorbing primary submodule of M by Theorem 4. □

Theorem 11. *Let M be a graded multiplication R-module and $N_1, N_2, ...,$ N_n be graded 2-absorbing primary submodules of M with the same M- graded radical. Then $N = \cap_{i=1}^{n} N_i$ is a graded 2-absorbing primary submodule of M.*

Proof. Suppose that $abm \in N$ for some $a, b \in h(R)$ and $m \in h(M)$ and $ab \notin (N :_R M)$. Then $ab \notin (N_i :_R M)$ for some $1 \leq i \leq n$. Hence $am \in M\text{-}\mathrm{Gr}(N_i) = \cap_{i=1}^{n} M - \mathrm{Gr}(N_i) = M\text{-}Gr(N)$ or $bm \in M\text{-}\mathrm{Gr}(N_i) = \cap_{i=1}^{n} M\text{-}\mathrm{Gr}(N_i) = M\text{-}Gr(N)$. □

Theorem 12. *Let R be a graded ring and N be a proper graded submodule of a graded R-module M. Then N is graded 2-absorbing primary if and only if $(N :_M ab) \subseteq (M\text{-}Gr(N) :_M a)$ or $(N :_M ab) \subseteq (M\text{-}Gr(N) :_M b)$ for all $a, b \in h(R)$ with $ab \notin (N :_R M)$.*

Proof. Suppose that N is a graded 2-absorbing primary submodule of M, $ab \notin (N :_R M)$ for some $a, b \in h(R)$ and $m \in (N :_M ab)$. Hence $abm \in N$, which implies either $am \in M\text{-}Gr(N)$ or $bm \in M\text{-}Gr(N)$. This means that $m \in (M\text{-}Gr(N) :_M a)$ or $m \in (M\text{-}Gr(N) :_M b)$, so we are done. The converse part is clear. □

Theorem 13. *Let R be a graded ring and N be a proper graded submodule of a graded R-module M. Then N is a graded 2-absorbing primary submodule of M if and only if whenever $I_1 I_2 K \subseteq N$ for some graded ideals I_1, I_2 of R and some graded submodule K of M, then $I_1 I_2 \subseteq (N :_R M)$ or $I_1 K \subseteq M\text{-}Gr(N)$ or $I_2 K \subseteq M\text{-}Gr(N)$.*

Proof. Suppose that N is a graded 2-absorbing primary submodule of M and $I_1 I_2 K \subseteq N$ for some graded ideals I_1, I_2 of R and some graded submodule K of M and $I_1 I_2 \nsubseteq (N :_R M)$. We show that $I_1 K \subseteq M\text{-}Gr(N)$ or $I_2 K \subseteq M\text{-}Gr(N)$. Assume on the contrary that $I_1 K \nsubseteq M\text{-}Gr(N)$ and $I_2 K \nsubseteq M\text{-}Gr(N)$. Then $a_1 K \nsubseteq M\text{-}Gr(N)$ and $a_2 K \nsubseteq M\text{-}Gr(N)$ for some $a_1 \in I_1$ and $a_2 \in I_2$. Since $a_1 a_2 K \subseteq N$ and neither $a_1 K \subseteq M\text{-}Gr(N)$ nor $a_2 K \subseteq M\text{-}Gr(N)$, we have $a_1 a_2 \in (N :_R M)$ by Theorem 3.
Since $I_1 I_2 \nsubseteq (N :_R M)$, we have $b_1 b_2 \notin (N :_R M)$ for some $b_1 \in I_1$ and $b_2 \in I_2$. Since $b_1 b_2 K \subseteq N$ and $b_1 b_2 \notin (N :_R M)$, we have $b_1 K \subseteq M\text{-}Gr(N)$ or $b_2 K \subseteq M\text{-}Gr(N)$ by Theorem 3. We consider three cases.

Case 1. Suppose that $b_1 K \subseteq M\text{-}Gr(N)$ but $b_2 K \nsubseteq M\text{-}Gr(N)$. Since $a_1 b_2 K \subseteq N$ and neither $b_2 K \subseteq M\text{-}Gr(N)$ nor $a_1 K \subseteq M\text{-}Gr(N)$, we conclude that $a_1 b_2 \in (N :_R M)$ by Theorem 3. Since $b_1 K \subseteq M\text{-}Gr(N)$ but $a_1 K \nsubseteq M\text{-}Gr(N)$, we conclude that $(a_1 + b_1)K \nsubseteq M\text{-}Gr(N)$. Since $(a_1 + b_1) b_2 K \subseteq N$ and neither $b_2 K \subseteq M\text{-}Gr(N)$ nor $(a_1 + b_1)K \subseteq M\text{-}Gr(N)$, we conclude that $(a_1 + b_1) b_2 \in (N :_R M)$ by Theorem 3. Since $(a_1 + b_1) b_2 = a_1 b_2 + b_1 b_2 \in (N :_R M)$ and $a_1 b_2 \in (N :_R M)$, we conclude $b_1 b_2 \in (N :_R M)$, a contradiction.

Case 2. Suppose that $b_2 K \subseteq M\text{-}Gr(N)$ but $b_1 K \nsubseteq M\text{-}Gr(N)$. Similar to the previous case we reach a contradiction.

Case 3. Suppose that $b_1 K \subseteq M\text{-}Gr(N)$ and $b_2 K \subseteq M\text{-}Gr(N)$. Since $b_2 K \subseteq M\text{-}Gr(N)$ and $a_2 K \nsubseteq M\text{-}Gr(N)$, we conclude that $(a_2 + b_2)K \nsubseteq M\text{-}Gr(N)$. Since $a_1(a_2 + b_2)K \subseteq N$ and neither $a_1 K \subseteq M\text{-}Gr(N)$ nor $(a_2 + b_2)K \subseteq M\text{-}Gr(N)$, we conclude that $a_1(a_2 + b_2) = a_1 a_2 + a_1 b_2 \in (N :_R M)$ by Theorem 3. Since $a_1 a_2 \in (N :_R M)$ and $a_1 a_2 + a_1 b_2 \in (N :_R M)$, we conclude that $a_1 b_2 \in (N :_R M)$. Since $b_1 K \subseteq M\text{-}Gr(N)$ and $a_1 K \nsubseteq M\text{-}Gr(N)$, we conclude that $(a_1 + b_1)K \nsubseteq M\text{-}Gr(N)$. Since $(a_1 + b_1) a_2 K \subseteq N$ and neither $a_2 K \subseteq M\text{-}Gr(N)$ nor $(a_1 + b_1)K \subseteq M\text{-}Gr(N)$, we obtain that $(a_1 + b_1) a_2 = a_1 a_2 + b_1 a_2 \in (N :_R M)$ by Theorem 3. Since $a_1 a_2 \in (N :_R M)$ and $a_1 a_2 + b_1 a_2 \in (N :_R M)$, we have $b_1 a_2 \in (N :_R M)$. Now, since $(a_1 + b_1)(a_2 + b_2)K \subseteq N$ and neither $(a_1 + b_1)K \subseteq M\text{-}Gr(N)$ nor $(a_2 + b_2)K \subseteq M\text{-}Gr(N)$, we conclude that $(a_1 + b_1)(a_2 + b_2) = a_1 a_2 + a_1 b_2 + b_1 a_2 + b_1 b_2 \in (N :_R M)$ by Theorem 3. Since $a_1 a_2, a_1 b_2, b_1 a_2 \in (N :_R M)$, we get $b_1 b_2 \in (N :_R M)$, a contradiction. Consequently $I_1 K \subseteq M\text{-}Gr(N)$ or $I_2 K \subseteq M\text{-}Gr(N)$, so we are done. □

Let R be a graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring. Indeed, $S^{-1}R = \oplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{\frac{r}{s} : r \in R,\ s \in S \text{ and } g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module if $S^{-1}M = \oplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{\frac{m}{s} : m \in M,\ s \in S$ and $g = (\deg s)^{-1}(\deg m)\}$. Here $h(S^{-1}R) = \cup_{g\in G}(S^{-1}R)_g$ and $h(S^{-1}M) = \cup_{g\in G}(S^{-1}M)_g$.

Theorem 14. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded 2-absorbing primary submodule of M and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a graded 2-absorbing primary submodule of $S^{-1}M$.*

Proof. Let $\frac{a}{s_1}, \frac{b}{s_2} \in h(S^{-1}R)$ and $\frac{m}{s_3} \in h(S^{-1}M)$ such that $\frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}N$. Then there exists $u \in S$ with $uabm \in N$. This implies that $ab \in (N :_R M)$ or $uam \in M\text{-}Gr(N)$ or $ubm \in M\text{-}Gr(N)$. If $ab \in (N :_R M)$, then $\frac{a}{s_1}\frac{b}{s_2} = \frac{ab}{s_1s_2} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. Suppose that $uam \in M\text{-}Gr(N)$. Then $\frac{a}{s_1}\frac{m}{s} = \frac{uam}{us_1s} \in S^{-1}(M\text{-}Gr(N)) \subseteq (S^{-1}M)\text{-Gr}(S^{-1}N)$. Similarly, if $ubm \in M\text{-}Gr(N)$ then we have $\frac{b}{s_2}\frac{m}{s} = \frac{ubm}{us_2s} \in (S^{-1}M)\text{-Gr}(S^{-1}N)$. □

Let M and M' be graded modules over a G-graded ring R. A module homomorphism $f : M \to M'$ is said to be graded if $f(M_g) \subseteq M'_g$. f is a graded isomorphism if it is a graded module homomorphism and an isomorphism.

Lemma 15. *([10, Corollary 1.3]) Let M and M' be graded multiplication R-modules with $f : M \to M'$ an graded R-module epimorphism. If N is a graded submodule of M containing $Ker(f)$, then $f(M\text{-}Gr(N)) = M'\text{-}$*Gr$(f(N))$.

Theorem 16. *Let M and M' be graded multiplication R-modules and $f : M \to M'$ be a graded R-module homomorphism.*

1. If N' is a graded 2-absorbing primary submodule of M', then $f^{-1}(N')$ is a graded 2-absorbing primary submodule of M.

2. If f is epimorphism and N is a graded 2-absorbing primary submodule of M containing $Ker(f)$, then $f(N)$ is a graded 2-absorbing primary submodule of M'.

Proof. (1) Let $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in f^{-1}(N')$. Then $abf(m) \in N'$. Hence $ab \in (N' :_R M')$ or $af(m) \in M'\text{-Gr}(N')$ or $bf(m) \in M'\text{-Gr}(N')$, and thus $ab \in (f^{-1}(N') :_R M)$ or $am \in f^{-1}(M'\text{-Gr}(N'))$ or $bm \in$

$f^{-1}(M'\text{-Gr}(N'))$. By using the inclusion $f^{-1}(M'\text{-Gr}(N')) \subseteq M\text{-Gr}(f^{-1}(N'))$, we conclude that $f^{-1}(N')$ is a graded 2-absorbing primary submodule of M.
(2) Let $a, b \in h(R)$, $m' \in h(M')$ and $abm' \in f(N)$. Since f is epimorphism there exists $m \in h(M)$ such that $m' = f(m)$ and so $f(abm) \in f(N)$. Since $Ker(f) \subseteq N$, we have $abm \in N$. It implies that $ab \in (N :_R M)$ or $am \in M$-$Gr(N)$ or $bm \in M$-$Gr(N)$. Hence $ab \in (f(N) :_R M')$ or $am' \in f(M\text{-}Gr(N)) = M'\text{-Gr}(f(N))$ or $bm' \in f(M\text{-}Gr(N)) = M'\text{-Gr}(f(N))$. Consequently $f(N)$ is a graded 2-absorbing primary submodule of M'. □

As an immediate consequence of Theorem 16 (2) we have the following Corollary.

Corollary 17. *Let M be a graded multiplication R-module and $L \subseteq N$ be graded submodules of M. If N is a graded 2-absorbing primary submodule of M, then N/L is a graded 2-absorbing primary submodule of M/L.*

Let R_i be a graded commutative ring with identity and M_i be a graded R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded R-module and each graded submodule of M is of the form $N = N_1 \times N_2$ for some graded submodules N_1 of M_1 and N_2 of M_2. In addition, if M_i is a graded multiplication R_i-module, for $i = 1, 2$, then M is a graded multiplication R-module. In this case, for each graded submodule $N = N_1 \times N_2$ of M we have $M\text{-}Gr(N) = M_1\text{-Gr}(N_1) \times M_2\text{-Gr}(N_2)$.

Theorem 18. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where M_1 is a graded multiplication R_1-module and M_2 is a graded multiplication R_2-module.*

1. A proper graded submodule K_1 of M_1 is a graded 2-absorbing primary submodule if and only if $N = K_1 \times M_2$ is a graded 2-absorbing primary submodule of M.

2. A proper graded submodule K_2 of M_2 is a graded 2-absorbing primary submodule if and only if $N = M_1 \times K_2$ is a graded 2-absorbing primary submodule of M.

3. Assume that M_1 is a graded cyclic R_1-module and M_2 is a graded cyclic R_2-module. If K_1 and K_2 are graded primary submodules of M_1 and M_2, respectively, then $N = K_1 \times K_2$ is a graded 2-absorbing primary submodule of M.

Proof. (1) Suppose that $N = K_1 \times M_2$ is a graded 2-absorbing primary submodule of M. From our hypothesis, N is proper, so $K_1 \neq M_1$. Set $M' = \frac{M}{\{0\} \times M_2}$. Hence $N' = \frac{N}{\{0\} \times M_2}$ is a graded 2-absorbing primary submodule of

M' by Corollary 17. Also observe that $M' \cong M_1$ and $N' \cong K_1$. Thus K_1 is a graded 2-absorbing primary submodule of M_1. Conversely, if K_1 is a graded 2-absorbing primary submodule of M_1, then it is clear that $N = K_1 \times M_2$ is a graded 2-absorbing primary submodule of M.

(2) It can be easily verified similar to (1).

(3) Assume that $N = K_1 \times K_2$ where K_1 and K_2 are graded primary submodules of M_1 and M_2, respectively. Hence $(K_1 \times M_2) \cap (M_1 \times K_2) = K_1 \times K_2 = N$ is a graded 2-absorbing primary submodule of M, by parts (1) and (2) and Theorem 10. □

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