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A COUPLED FIXED POINT THEOREM IN *b*-METRIC SPACES

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Abstract: In this paper, we utilize the notion of coupled fixed point in sense of Bhaker and Lakshmikantham [T. Bhakar, V. Lakshmikantham, Fixed point theorems in partially orderd metric spaces and applications, *Nonlinear Anal.*, **65** (2006), 1379-1393] to introduce and prove a coupled fixed point theorem in a *b*-metric space. Our contractive condition is the most general contractive form in linear form. Our results modified and generalized many exciting results in the literature.

AMS Subject Classification: 47H10, 54H25

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1. Introduction

Bhaskar and LakshmiKantham [2] initiated the notion of coupled fixed point

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and they proved some coupled fixed point theorems in standard metric spaces. Then after, many authors formulated and studied some coupled fixed point theorems, for examples see [2, 3, 4, 5, 6]. The notion of b-metric spaces was introduced by Czerwik [1] as a generalization of standard metric spaces.

We begin with the definition of b-metric spaces:

Definition 1.1. [1] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric iff $\forall x, y, z \in X$ the following conditions hold
(b1) $d(x, y) = 0$ iff $x = y$,
(b2) $d(x, y) = d(y, x)$,
(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Definition 1.2. [7] Let (X, d) be a metric space. Then the sequence (x_n) in X converges to an element $x \in X$, $if\ \forall \epsilon > 0, \exists k \in N$ such that: $d(x_n, x) < \epsilon, \forall n \geq k$; that is, $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 1.3. [7] A sequence x_n in a metric space (X, d) is called a Cauchy sequence, if $\forall \epsilon > 0$, $\exists k \in N$ such that: $d(x_n, x_m) < \epsilon, \forall m, n \geq k$; that is, $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$.

Definition 1.4. [7] A metric space is called complete if every Cauchy sequence
converges to an element in the same metric space.

Definition 1.5. [2] An element (x,y) $\in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \longrightarrow X$ if
$F(x, y) = x$ and $F(y, x) = y$.

Definition 1.6. [4] An element (x,y) $\in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if
$F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.7. [4] An element (x,y) $\in X \times X$ is called a coupled fixed point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if
$F(x, y) = gx = x$ and $F(y, x) = gy = y$.

Definition 1.8. [4] Let X be a nonempty set. Then we say that the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are commutative if
$gF(x, y) = F(gx, gy)$.

2. Main Results

We start our results with the following coupled fixed point theorem.

Theorem 2.1. *Let (X, d) be a complete b-metric space with constant $s \geq 1$. Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings on X. Suppose there exist nonnegative constants $\alpha_i : i = 1, 2, ..., 10$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \quad & \alpha_1 d(gx,gu) + \alpha_2\, d(gy,gv) \\ & +\alpha_3\, d(F(x,y),gx) + \alpha_4\, d(F(y,x),gy) \\ & +\alpha_5\, d(F(u,v),gu) + \alpha_6\, d(F(v,u),gv) \\ & +\alpha_7\, d(F(x,y),gu) + \alpha_8\, d(F(y,x),gv) \\ & +\alpha_9\, d(F(u,v),gx) + \alpha_{10}\, d(F(v,u),gy), \end{aligned}$$

holds for all x, y, u , $v \in X$. Also suppose the following conditions:

i) $s\alpha_1 + s\alpha_2 + s\alpha_3 + s\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + (s^2 + s)\alpha_9 + (s^2 + s)\alpha_{10} < 1.$

ii) $g(X)$ is a complete subspace of X.
iii) $F(X \times X) \subseteq g(X)$.
Then F and g have a coupled coincidence point. In addition, if F and g are weakly compatible, then F and g have a unique coupled fixed point. Moreover, for any $x_1 \in X$, the sequence $\{x_n\}$ defined by $g(x_{n+1}) = F(x_n, x_n)$ converges to the common fixed point of F and g.

Proof. Let $x_0 \in X$. Since $F(X \times X) \subseteq g(X)$, then $\exists x_1 \in X$ such that $F(x_0, x_0) = gx_1$. Again, since $F(X \times X) \subseteq g(X)$, then $\exists x_2 \in X$ such that $F(x_1, x_1) = gx_2$. Continuing this process we obtain a sequence (y_n) in $g(X)$, such that

$$y_n = gx_{n+1} = F(x_n, x_n).$$

Now, we have

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}) &= d(F(x_n, x_n), F(x_{n+1}, x_{n+1})) \\ &\leq \alpha_1 d(gx_n, gx_{n+1}) + \alpha_2 d(gx_n, gx_{n+1}) \\ &+ \alpha_3 d(F(x_n, x_n), gx_n) + \alpha_4 d(F(x_n, x_n), gx_n) \\ &+ \alpha_5 d(F(x_{n+1}, x_{n+1}), gx_{n+1}) + \alpha_6 d(F(x_{n+1}, x_{n+1}), gx_{n+1}) \\ &+ \alpha_7 d(F(x_n, x_n), gx_{n+1}) + \alpha_8 d(F(x_n, x_n), gx_{n+1}) \\ &+ \alpha_9 d(F(x_{n+1}, x_{n+1}), gx_n) + \alpha_{10} d(F(x_{n+1}, x_{n+1}), gx_n). \end{aligned}$$

Therefore

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}) \leq &\alpha_1 d(gx_n, gx_{n+1}) + \alpha_2 d(gx_n, gx_{n+1}) \\ &+ \alpha_3 d(gx_{n+1}, gx_n) + \alpha_4 d(gx_{n+1}, gx_n) \\ &+ \alpha_5 d(gx_{n+2}, gx_{n+1}) + \alpha_6 d(gx_{n+2}, gx_{n+1}) \\ &+ \alpha_7 d(gx_{n+1}, gx_{n+1}) + \alpha_8 d(gx_{n+1}, gx_{n+1}) \\ &+ \alpha_9 d(gx_{n+2}, gx_n) + \alpha_{10} d(gx_{n+2}, gx_n) \\ =&(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(gx_n, gx_{n+1}) \\ &+ (\alpha_5 + \alpha_6) d(gx_{n+1}, gx_{n+2}) \\ &+ (\alpha_9 + \alpha_{10}) d(gx_n, gx_{n+2}), \end{aligned}$$

which implies that

$$\begin{aligned} [1 - \alpha_5 - \alpha_6] d(gx_{n+1}, gx_{n+2}) \leq &(\alpha_9 + \alpha_{10}) d(gx_n, gx_{n+2}) \\ &+ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(gx_n, gx_{n+1}). \end{aligned}$$

Since

$$d(gx_n, gx_{n+2}) \leq s d(gx_n, gx_{n+1}) + s d(gx_{n+1}, gx_{n+2}),$$

we have

$$\begin{aligned} [1 - \alpha_5 - \alpha_6] d(gx_{n+1}, gx_{n+2}) \leq &s(\alpha_9 + \alpha_{10})[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2})] \\ &+ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(gx_n, gx_{n+1}). \end{aligned}$$

or

$$\begin{aligned} &[1 - \alpha_5 - \alpha_6 - s\alpha_9 - s\alpha_{10}] d(gx_{n+1}, gx_{n+2}) \\ &\qquad \leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + s\alpha_9 + s\alpha_{10}) d(gx_n, gx_{n+1}). \end{aligned}$$

Therefore, we obtain

$$d(gx_{n+1}, gx_{n+2}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + s\alpha_9 + s\alpha_{10}}{[1 - \alpha_5 - \alpha_6 - s\alpha_9 - s\alpha_{10}]} d(gx_n, gx_{n+1}).$$

Let us set

$$r = \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + s\alpha_9 + s\alpha_{10})}{(1 - \alpha_5 - \alpha_6 - s\alpha_9 - s\alpha_{10})}.$$

Since

$$sr = \frac{(s\alpha_1 + s\alpha_2 + s\alpha_3 + s\alpha_4 + s^2\alpha_9 + s^2\alpha_{10})}{(1 - \alpha_5 - \alpha_6 - s\alpha_9 - s\alpha_{10})} < 1$$

and $r \leq sr$, we get $r < 1$, then

$$d(gx_{n+1}, gx_{n+2}) \leq rd(gx_n, gx_{n+1}). \tag{1}$$

Again,

$$\begin{aligned} d(gx_n, gx_{n+1}) =& d(F(x_{n-1}, x_{n-1}), F(x_n, x_n)) \\ & \leq \alpha_1 d(gx_{n-1}, gx_n) + \alpha_2 d(gx_{n-1}, gx_n) \\ & + \alpha_3 d(F(x_{n-1}, x_{n-1}), gx_{n-1}) + \alpha_4 d(F(x_{n-1}, x_{n-1}), gx_{n-1}) \\ & + \alpha_5 d(F(x_n, x_n), gx_n) + \alpha_6 d(F(x_n, x_n), gx_n) \\ & + \alpha_7 d(F(x_{n-1}, x_{n-1}), gx_n) + \alpha_8 d(F(x_{n-1}, x_{n-1}), gx_n) \\ & + \alpha_9 d(F(x_n, x_n), gx_{n-1}) + \alpha_{10} d(F(x_n, x_n), gx_{n-1}). \end{aligned}$$

So, we get

$$\begin{aligned} d(gx_n, gx_{n+1}) \leq& \alpha_1 d(gx_{n-1}, gx_n) + \alpha_2 d(gx_{n-1}, gx_n) \\ & + \alpha_3 d(gx_n, gx_{n-1}) + \alpha_4 d(gx_n, gx_{n-1}) \\ & + \alpha_5 d(gx_{n+1}, gx_n) + \alpha_6 d(gx_{n+1}, gx_n) \\ & + \alpha_7 d(gx_n, gx_n) + \alpha_8 d(gx_n, gx_n) \\ & + \alpha_9 d(gx_{n+1}, gx_{n-1}) + \alpha_{10} d(gx_{n+1}, gx_{n-1}), \end{aligned}$$

which leads us to

$$d(gx_n, gx_{n+1}) \leq rd(gx_{n-1}, gx_n). \tag{2}$$

Repeating (1) and (2), n times, we get

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}) \quad & \leq r^2 d(gx_{n-1}, gx_n) \\ & \vdots \\ & \leq r^{n+1} d(gx_0, gx_1). \end{aligned} \tag{3}$$

Given $n, m \in N$, with $m > n$. Then

$$\begin{aligned} d(gx_n, gx_m) \leq& sd(gx_n, gx_{n+1}) + sd(gx_{n+1}, gx_m) \\ \leq& sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^2 d(gx_{n+2}, gx_m) \\ \leq& sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\ & + s^{m-n} d(gx_{m-1}, gx_m). \end{aligned}$$

From (3), we get

$$d(gx_n, gx_m) \leq sr^n d(gx_0, gx_1) + sr^{n+1} d(gx_0, gx_1)$$

$$\begin{aligned}
&+ ... + s^{m-n}r^{m-1}d(gx_0, gx_1)\\
\leq& sr^n d(gx_0, gx_1)[1 + sr + s^2r^2 + s^3r^3 + ...]\\
\leq& sr^n d(gx_0, gx_1)\sum_{i=0}^{+\infty}(sr)^i\\
=& \frac{sr^n}{1-sr}d(gx_0, gx_1).
\end{aligned}$$

Since $r^n \longrightarrow 0$ as $n \longrightarrow +\infty$, we get that

$$\lim_{n,m\longrightarrow+\infty} d(gx_n, gx_m) = 0.$$

So (gx_n) is a b-Cauchy sequence in $(g(X), d)$. By completeness of $g(X)$ $\exists$ p $\in g(X)$ such that

$$\lim_{n\longrightarrow+\infty} gx_n = gu = p.$$

We shall show that (u, u) is a coupled coincidence point of F and g.

$$\begin{aligned}
d(gx_{n+1}, F(u, u)) \quad &= d(F(x_n, x_n), F(u, u))\\
&\leq \alpha_1 d(gx_n, gu) + \alpha_2 d(gx_n, gu)\\
&+\alpha_3 d(F(x_n, x_n), gx_n) + \alpha_4 d(F(x_n, x_n), gx_n)\\
&+\alpha_5 d(F(u, u), gu) + \alpha_6 d(F(u, u), gu)\\
&+\alpha_7 d(F(x_n, x_n), gu) + \alpha_8 d(F(x_n, x_n), gu)\\
&+\alpha_9 d(F(u, u), gx_n) + \alpha_{10} d(F(u, u), gx_n),
\end{aligned}$$

which implies that

$$\begin{aligned}
d(gx_{n+1}, F(u, u)) \quad &\leq \alpha_1 d(gx_n, gu) + \alpha_2 d(gx_n, gu)\\
&+\alpha_3 d(gx_{n+1}, gx_n) + \alpha_4 d(gx_{n+1}, gx_n)\\
&+\alpha_5 d(F(u, u), gu) + \alpha_6 d(F(u, u), gu)\\
&+\alpha_7 d(gx_{n+1}, gu) + \alpha_8 d(gx_{n+1}, gu)\\
&+\alpha_9 d(F(u, u), gx_n) + \alpha_{10} d(F(u, u), gx_n).
\end{aligned} \tag{4}$$

Note that

$$d(gu, F(u, u)) \quad \leq sd((gu, gx_{n+1}) + d(gx_{n+1}, F(u, u))).$$

Letting $n \longrightarrow +\infty$ in above inequality we get

$$\frac{1}{s}d(gu, F(u, u)) \leq \lim_{n\longrightarrow\infty} d(gx_{n+1}, F(u, u)). \tag{5}$$

Also,

$$d(gx_{n+1}, gx_n) \quad \leq s(d(gx_{n+1}, gu) + d(gu, gx_n)).$$

Letting $n \longrightarrow +\infty$ in above inequality we get

$$\lim_{n\longrightarrow\infty}(gx_{n+1}, gx_n) = 0. \tag{6}$$

Again, note that

$$d(F(u,u), gx_n) \quad \leq s(d(F(u,u), gu) + d(gu, gx_n))$$

Letting $n \longrightarrow +\infty$ in above inequality, we get

$$\lim_{n\longrightarrow\infty} d(F(u,u), gx_n) \leq sd(F(u,u), gu). \tag{7}$$

Letting $n \longrightarrow +\infty$ in (4) and using (5), (6) and (7), we get

$$\begin{aligned}\frac{1}{s}d(gu, F(u,u)) \quad &\leq \alpha_5 d(F(u,u), gu) + \alpha_6 d(F(u,u), gu)\\ &+s\alpha_9 d(F(u,u), gu) + s\alpha_{10} d(F(u,u), gu),\end{aligned}$$

which implies that

$$d(gu, F(u,u)) \quad \leq (s\alpha_5 + s\alpha_6 + s^2\alpha_9 + s^2\alpha_{10})d(F(u,u), gu).$$

Since $(s\alpha_5 + s\alpha_6 + s^2\alpha_9 + s^2\alpha_{10} < 1)$, we get $d(gu, F(u,u)) = 0$.
Hence $gu = F(u,u)$.
Hence (u,u) is a coupled coincidence point of F and g.
To prove the uniqueness of the coupled coincidence point (u,u), assume that (v,v) is a coupled coincidence point of F and g. Then

$$\begin{aligned}d(gu, gv) \quad &= d(F(u,u), F(v,v))\\ &\leq \alpha_1 d(gu, gv) + \alpha_2 d(gu, gv)\\ &+\alpha_3 d(F(u,u), gu) + \alpha_4 d(F(u,u), gu)\\ &+\alpha_5 d(F(v,v), gv) + \alpha_6 d(F(v,v), gv)\\ &+\alpha_7 d(F(u,u), gv) + \alpha_8 d(F(u,u), gv)\\ &+\alpha_9 d(F(v,v), gu) + \alpha_{10} d(F(v,v), gu)\end{aligned}$$

which implies that
$d(gu, gv) \leq (\alpha_1 + \alpha_2 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10})d(gu, gv)$.
Since $(\alpha_1 + \alpha_2 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) < 1$, then
$d(gu, gv) = 0$. So $gu = gv$.
Hence (u,u) is the unique coupled coincidence point of F and g.
Now, suppose that F and g are weakly compatible. Then
$g(F(u,u)) = F(gu, gu)$.
Put $w = gu$. Then

$$gw = g(gu) = g(F(u,u)) = F(gu,gu) = F(w,w).$$

So, (w,w) is a coupled coincidence point of F and g. Therefore $gw = gu$.

Hance $gw = w$ and $F(w,w) = gw = w$.

So (w,w) is a coupled fixed point of F and g. □

Corollary 2.1. *Let (X,d) be a complete b-metric space with constant $s \geq 1$. Let $F : X \times X \longrightarrow X$ be a mapping. Suppose there exist non−negative constants $\alpha_i : i = 1,2,...,10$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq\ & \alpha_1 d(x,u) + \alpha_2 d(y,v) \\ & +\alpha_3 d(F(x,y),x) + \alpha_4 d(F(y,x),y) \\ & +\alpha_5 d(F(u,v),u) + \alpha_6 d(F(v,u),v) \\ & +\alpha_7 d(F(x,y),u) + \alpha_8 d(F(y,x),v) \\ & +\alpha_9 d(F(u,v),x) + \alpha_{10} d(F(v,u),y). \end{aligned}$$

holds for all x, y, u , $v \in X$.
If $s\alpha_1 + s\alpha_2 + s\alpha_3 + s\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + (s^2+s)\alpha_9 + (s^2+s)\alpha_{10} < 1$, then F has a unique coupled fixed point.

Proof. It follows from Theorem 2.1 by taking $g = \mathrm{Id_x}$ the identity mapping. □

Corollary 2.2. *Let (X,d) be a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping. Suppose there exist non−negative constants $\alpha_i : i = 1,2,...,10$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq\ & \alpha_1 d(x,u) + \alpha_2 d(y,v) \\ & +\alpha_3 d(F(x,y),x) + \alpha_4 d(F(y,x),y) \\ & +\alpha_5 d(F(u,v),u) + \alpha_6 d(F(v,u),v) \\ & +\alpha_7 d(F(x,y),u) + \alpha_8 d(F(y,x),v) \\ & +\alpha_9 d(F(u,v),x) + \alpha_{10} d(F(v,u),y). \end{aligned}$$

holds for all x, y, u , $v \in X$.
If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + 2\alpha_9 + 2\alpha_{10} < 1$, then F has a unique coupled fixed point.

Proof. It follows from Corollary 2.1 by noting that d is a b-metric space with $s = 1$. □

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