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EXPLICIT MOORE-PENROSE INVERSE AND GROUP INVERSE OF DOUBLY LESLIE MATRIX

Wiwat Wanicharpichat

Department of Mathematics
Faculty of Science
Phitsanulok 65000, THAILAND
and
Research Center for Academic Excellence in Mathematics
Naresuan University
Phitsanulok 65000, THAILAND

Abstract: A doubly Leslie matrix is a bordered real matrix of the form

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $a_n, b_n \in \mathbb{R}$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-1}$, and $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$ is a diagonal matrix of order $n-1$. The matrix L is a closed form of a doubly companion matrix, a Leslie matrix and a companion matrix. This paper is discussed the explicit formula of the Moore-Penrose inverse and the group inverse of the doubly leslie matrix. In general the Moore-Penrose inverse of a rectangle doubly Leslie matrix is also discussed.

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Key Words: companion matrix, doubly companion matrix, Leslie matrix, doubly Leslie matrix, Moore-Penrose inverse, group inverse

1. Introduction

One of the most popular models of population growth is a matrix-based model, first introduced by P.H. Leslie. In 1945, he published his most famous article in *Biometrika*, a journal. The article was entitled, *On the use of matrices in*

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certain population mathematics [2, pp. 117–120]. The Leslie model describes the growth of the female portion of a population which is assumed to have a maximum lifespan. The females are divided into age classes all of which span an equal number of years. Using data about the average birthrates and survival probabilities of each class, the model is then able to determine the growth of the population over time, [6].

A *Leslie matrix* arises in a discrete, age-dependent model for population growth. It is a matrix of the form

$$\mathbf{L} = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_{n-1} & r_n \\ s_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}, \tag{1}$$

where $r_j \geq 0$, $0 < s_j \leq 1$, $j = 1, 2, \dots, n-1$.

Doubly companion matrices $C \in M_n$ were first introduced by Butcher and Chartier in [4, pp.274–276], given by

$$C = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} & -\alpha_n - \beta_n \\ 1 & 0 & 0 & \dots & 0 & -\beta_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -\beta_2 \\ 0 & 0 & 0 & \dots & 1 & -\beta_1 \end{bmatrix}, \tag{2}$$

that is, a $n \times n$ matrix C with $n > 1$ is called a *doubly companion matrix* if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$.

We define a *doubly Leslie matrix* analogous as the doubly companion matrix by replacing the subdiagonal of the doubly companion matrix by $s_1, s_2, \dots, s_{n-1}$ where s_j, $j = 1, 2, \dots, n-1$, respectively, and denoted by L, that is, a doubly

Leslie matrix is defined to be a matrix as follows:

$$L = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -b_2 \\ 0 & 0 & \cdots & 0 & s_{n-1} & -b_1 \end{bmatrix}, \tag{3}$$

where $a_j, b_j \in \mathbb{R}$, the real numbers, $j = 1, 2, \dots, n$. As the Leslie matrix, we restriction only $s_j > 0$, $j = 1, 2, \dots, n-1$.

For convenience, we can be written the matrix L in a partitioned form as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)} \text{ where } \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \mathbf{q} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix},$$

and $\Lambda = \text{diag}(s_1, s_2, \dots, s_{n-1})$ is a diagonal matrix of order $n-1$.

We also define a rectangular doubly Leslie matrix of order $m \times n$, where $m = n - k$ and $1 < k < n$ as follows:

$$R = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & s_{n-k} & 0 & -b_k \end{bmatrix}_{(m,n)}. \tag{4}$$

For convenience, we can be written the matrix L in a partitioned form as

$$R = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda_k & -\mathbf{q}_k \end{bmatrix}_{(m,n)} \text{ where } \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \mathbf{q}_k = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_k \end{bmatrix}, \tag{5}$$

and $\Lambda_k = [\text{diag}(s_1, s_2, \dots, s_{n-k})|0]$ is a $(n-k) \times (n-1)$ block matrix which the first block is a diagonal matrix $\text{diag}(s_1, s_2, \dots, s_{n-k})$ and the remainder block is a zero matrix of appropriated size.

We abbreviate doubly Leslie matrix to DLM and rectangular doubly Leslie matrix by RDLM.

Let M be a matrix partitioned into four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the submatrix C is assumed to be square and nonsingular. Brezinski in [3, p.232] asserted that, the Schur complement of C in M, denoted by (M/C), is defined by

$$(M/C) = B - AC^{-1}D. \tag{6}$$

As in (6), the Schur complement of Λ in L, denoted by (L/Λ) , is a 1×1 matrix or a scalar

$$\begin{aligned} (L/\Lambda) &= (-a_n - b_n) - (-\mathbf{p}^T)\Lambda^{-1}(-\mathbf{q}) \\ &= -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right). \end{aligned} \tag{7}$$

The author [7] asserted some basic properties of doubly Leslie matrix as in the following lemma.

Lemma 1. *Let L be a doubly Leslie matrix as in* (3) *with partitioned as*

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, *and* $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ *is a diagonal matrix of order* $n-1$, *then*

$$\det L = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \prod_{i=1}^{n-1} s_i. \tag{8}$$

and, if $\det L \neq 0$ *then*

$$L^{-1} = (L/\Lambda)^{-1} \begin{bmatrix} \Lambda^{-1}\mathbf{q} & (L/\Lambda)\Lambda^{-1} + \left(\Lambda^{-1}\mathbf{q}\mathbf{p}^T\Lambda^{-1}\right) \\ 1 & \mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)}, \tag{9}$$

where $(L/\Lambda) = -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right)$, *as in* (7), *and* $\Lambda^{-1} = \operatorname{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_{n-1}})$.

In the present paper we give explicit Moore-Penrose inverse and group inverse formulae for the doubly Leslie matrix and give some related topics.

2. Preliminaries

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the field of real numbers $\mathbb{R}$. The Moore-Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying the four Penrose conditions

$$A = AXA, \quad X = XAX, \quad (AX)^T = AX \quad \text{and} \quad (XA)^T = XA$$

and is denoted by $A^\dagger$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying

$$A = AXA, \quad X = XAX \quad \text{and} \quad AX = XA$$

and is denoted by $A^\sharp$. A well known characterization for the existence of $A^\sharp$ is that $\text{rank}(A) = \text{rank}(A^2)$, [1]. If A is nonsingular, then $A^{-1} = A^\dagger = A^\sharp$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric if $\text{range}(A) = \text{range}(A^T)$. If A is range-symmetric, then $A^\dagger = A^\sharp$.

A system of linear equation $A\mathbf{x} = \mathbf{b}$ need not possess a solution when $\text{rank}(A) \neq \text{rank}[A : \mathbf{b}]$. That is $\mathbf{b}$ is not in the range of A. The Moore-Penrose inverse is most often used to solve least squares systems. It is still desirable to to find a $\mathbf{x}_0$ that is closest to a solution. The residual vector is a key component to solve these systems.

Theorem 2 ([1]). *Let $A \in \mathbb{R}^{m\times n}$ with* $\text{rank}(A) = r \neq 0$, *and suppose $A = FG$ is a full rank factorization of A. Then*

1. $F^\dagger = (F^T F)^{-1} F^T$,
2. $F^\dagger F = I_r$, *the $r \times r$ identity matrix,*
3. $G^\dagger = G^T (GG^T)^{-1}$,
4. $GG^\dagger = I_r$,
5. $A^\dagger = G^\dagger F^\dagger$.

More generally, for any $m \times n$ matrix A of full row rank m, $A = I_m A$ is a full rank factorization of A. Then

$$A^\dagger = A^T (AA^T)^{-1}. \tag{10}$$

The group inverse is very useful and has applications in many fields such as singular differential and difference equations, Markov chains, and iterative methods, see for instance [1].

Theorem 3 ([1]). *Let a square matrix A have the full rank factorization $A = FG$. Then A has a group inverse if and only if GF is nonsingular. In which case,*

$$A^{\sharp} = F(GF)^{-2}G.$$

3. Moore-Penrose Inverse of RDLM

Penrose [5, p.18]. It is possible to calculate $A^{\dagger}$ even when A^*A and AA^* are both singular by the following methods, where A^* is the conjugate transpose of the matrix A.

Any matrix M can be partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $D = CA^{-1}B$, (using a suitable arrangement of rows and columns). A being any non-singular submatrix whose rank is equal to that of the whole matrix. It is then easily verified that

$$M^{\dagger} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger} = \begin{bmatrix} A^*KA^* & A^*KC^* \\ B^*KA^* & B^*KC^* \end{bmatrix}, \tag{11}$$

where $K = (AA^* + BB^*)^{-1}A(A^*A + C^*C)^{-1}$. The matrices $AA^* + BB^*$ and $A^*A + C^*C$ are positive definite, since A is non-singular. Thus the generalized inverse of any matrix can be expressed in terms of ordinary reciprocals of matrices.

We have the following main results.

Lemma 4. *If $A = PB$, where P is a permutation matrix, then*

$$A^{\dagger} = B^{\dagger}P^T. \tag{12}$$

Proof. It is straightforward to verify that $B^{\dagger}P^T$ satisfies the four Penrose conditions. Clearly:

1. $PB(B^{\dagger}P^T)PB = PBB^{\dagger}B = PB$,
2. $(B^{\dagger}P^T)PB(B^{\dagger}P^T) = B^{\dagger}BB^{\dagger}P^T = B^{\dagger}P^T$,
3. $[PB(B^{\dagger}P^T)]^T = [PBB^{\dagger}P^T]^T = P(BB^{\dagger})^TP^T = PBB^{\dagger}P^T = PB(B^{\dagger}P^T)$,

4. $[(B^\dagger P^T)PB]^T = [B^\dagger P^T PB]^T = [B^\dagger B]^T = B^\dagger B = B^\dagger(P^T P)B$
$= (B^\dagger P^T)PB.$ □

Lemma 5. *For an $m \times n$ $\mathbb{R}$-matrix N of rank $r < \min(m,n)$, and N partitioned in the form*

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where C is $r \times r$ nonsingular. Then

$$N^\dagger = \begin{bmatrix} C^T K D^T & C^T K C^T \\ D^T K A^T & D^T K C^T \end{bmatrix}$$

where $K = (CC^T + DD^T)^{-1} C (C^T C + A^T A)^{-1}$. The matrices $CC^T + DD^T$ and $C^T C + A^T A$ are positive definite, since C is non-singular.

Proof. Let $P = \begin{bmatrix} 0 & I_r \\ I_{m-r} & 0 \end{bmatrix}_{(m \times m)}$ be a permutation matrix. Premultiplying the matrix N by P.

$$PN = \begin{bmatrix} C & D \\ A & B \end{bmatrix}.$$

Since P is a unitary matrix and by (12). We have

$$(PN)^\dagger = N^\dagger P^T.$$

Therefore

$$(PN)^\dagger = \begin{bmatrix} C & D \\ A & B \end{bmatrix}^\dagger = N^\dagger P^T.$$

and

$$N^\dagger = \begin{bmatrix} C & D \\ A & B \end{bmatrix}^\dagger P.$$

As in (11), we have

$$\begin{bmatrix} C & D \\ A & B \end{bmatrix}^\dagger = \begin{bmatrix} C^T K C^T & C^T K A^T \\ D^T K C^T & D^T K A^T \end{bmatrix},$$

where $K = (CC^T + DD^T)^{-1} C (C^T C + A^T A)^{-1}$. The matrices $CC^T + DD^T$ and $C^T C + A^T A$ are positive definite, since C is non-singular. Therefore $CC^T + DD^T$ and $C^T C + A^T A$ are also non-singular matrices. We have

$$N^\dagger = \begin{bmatrix} C^T K C^T & C^T K A^T \\ D^T K C^T & D^T K A^T \end{bmatrix} P = \begin{bmatrix} C^T K A^T & C^T K C^T \\ D^T K A^T & D^T K C^T \end{bmatrix}.$$

The proof is complete. □

Theorem 6. *Let L be a doubly Leslie matrix as in (3) with partitioned as*

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, *and* $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ *is a diagonal matrix of order* $n-1$, *then*

$$L^\dagger = \begin{bmatrix} -\Lambda K \mathbf{p} & \Lambda K \Lambda \\ \mathbf{q}^T K \mathbf{p} & -\mathbf{q}^T K \Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1} \Lambda (\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

Proof. If $\det L \neq 0$ then $L^\dagger = L^{-1}$ which appeared in (9).
In general

$$\begin{aligned} L^\dagger &= \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger = \begin{bmatrix} C^T K A^T & C^T K C^T \\ D^T K A^T & D^T K C^T \end{bmatrix} \\ &= \begin{bmatrix} \Lambda^T K (-\mathbf{p}^T)^T & \Lambda^T K \Lambda^T \\ (-\mathbf{q})^T K (-\mathbf{p}^T)^T & (-\mathbf{q})^T K \Lambda^T \end{bmatrix} \\ &= \begin{bmatrix} -\Lambda K \mathbf{p} & \Lambda K \Lambda \\ \mathbf{q}^T K \mathbf{p} & -\mathbf{q}^T K \Lambda \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} K &= (\Lambda \Lambda^T + (-\mathbf{q})(-\mathbf{q})^T)^{-1} \Lambda (\Lambda^T \Lambda + (-\mathbf{p}^T)^T (-\mathbf{p}^T))^{-1} \\ &= (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1} \Lambda (\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}. \end{aligned}$$

□

Corollary 7. *Let R be a rectangle doubly Leslie matrix as in (5) with partitioned as*

$$R = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda_k & -\mathbf{q}_k \end{bmatrix}_{(m,n)},$$

where $m = n - k$,

$$\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_k \end{bmatrix}^T,$$

and $\Lambda_k = [\operatorname{diag}(s_1, s_2, \dots, s_{n-k}) | 0]$ *is a* $(n-k) \times (n-1)$ *block matrix, then*

$$L^\dagger = \begin{bmatrix} -\Lambda_k K \mathbf{p} & \Lambda_k K \Lambda_k \\ \mathbf{q}_k^T K \mathbf{p} & -\mathbf{q}_k^T K \Lambda_k \end{bmatrix}$$

where $K = (\Lambda_k \Lambda_k^T + \mathbf{q}_k \mathbf{q}_k^T)^{-1} \Lambda_k (\Lambda_k^T \Lambda_k + \mathbf{p}\mathbf{p}^T)^{-1}$.

Proof. The proof is an analogous as in Theorem 6. □

Let's consider some examples.

Example

$$L = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} =: \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}$$

where $\mathbf{p} = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, and $\Lambda = \operatorname{diag}(1,2,1)$, is a diagonal matrix of order 3, then

$$L^\dagger = \begin{bmatrix} -\Lambda K \mathbf{p} & \Lambda K \Lambda \\ \mathbf{q}^T K \mathbf{p} & -\mathbf{q}^T K \Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

First we calculate $\mathbf{q}\mathbf{q}^T$ and $\mathbf{p}\mathbf{p}^T$.

$$\begin{aligned} \mathbf{q}\mathbf{q}^T &= \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}, \\ \mathbf{p}\mathbf{p}^T &= \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1} &= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{6} & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}, \\ (\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1} &= \begin{bmatrix} 2 & 2 & -1 \\ 2 & 8 & -2 \\ -1 & -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{16} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{4} \end{bmatrix}, \end{aligned}$$

we have

$$K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1} = \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Finally,

$$
\begin{aligned}
-\Lambda K\mathbf{p} &= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ 0 \end{bmatrix}, \\
\Lambda K\Lambda &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{17}{24} & -\frac{1}{8} & \frac{11}{24} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{q}^T K\mathbf{p} &= \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -\frac{1}{8}, \\
-\mathbf{q}^T K\Lambda &= -\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{24} & \frac{1}{8} & \frac{5}{24} \end{bmatrix}.
\end{aligned}
$$

Therefore

$$
L^\dagger = \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K\Lambda \\ \mathbf{q}^T K\mathbf{p} & -\mathbf{q}^T K\Lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{17}{24} & -\frac{1}{8} & \frac{11}{24} \\ \frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{24} \end{bmatrix}.
$$

This matrix is satisfies the four Penrose conditions. □

EXAMPLE. For a full row rank rectangle doubly Leslie matrix of order 3×4

$$
R = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{bmatrix}.
$$

From (10),

$$\begin{aligned}
R^\dagger &= R^T(RR^T)^{-1} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{11}{6} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{7}{12} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & \frac{7}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}.
\end{aligned}$$

This matrix is also satisfies the four Penrose conditions. $\square$

4. Group Inverse of DLM

As in [1, p.167] we have the following useful result.

Theorem 8. *Let A be a square singular matrix,* $\operatorname{rank} A = \operatorname{rank} A^2$, *and $R(A)$ be the range of A. If the system*

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in R(A)$$

has a solution, it is uniquely given by

$$\mathbf{x} = A^\sharp \mathbf{b}.$$

Proof. Suppose that $\mathbf{x} \in R(A)$ where $R(A)$ is the range of A. There is a vector $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$. Let a solution $\mathbf{x}$ be written as $\mathbf{x} = A\mathbf{y}_1$ for some $\mathbf{y}_1$. We have

$$A\mathbf{x} = AA\mathbf{y}_1 = A^2\mathbf{y}_1,$$

then $A^2\mathbf{y}_1 = \mathbf{b}$. Since $\operatorname{rank} A = \operatorname{rank} A^2$, there is a unique $A^\sharp$ such that

$$AA^\sharp A = A, \quad A^\sharp AA^\sharp = A^\sharp, \quad \text{and} \quad AA^\sharp = A^\sharp A.$$

Therefore

$$\begin{aligned}\mathbf{x} &= A\mathbf{y}_1\\ &= AA^{\sharp}A\mathbf{y}_1\\ &= A^2A^{\sharp}\mathbf{y}_1\\ &= A^{\sharp}A^2\mathbf{y}_1\\ &= A^{\sharp}A\mathbf{x}\\ &= A^{\sharp}\mathbf{b}.\end{aligned}$$

□

Let L be a doubly Leslie matrix as in (3) with partitioned as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}.$$

If $\det L \neq 0$ then $L^{\sharp} = L^{-1}$ which was shown in (9). We interested in study the only case $\operatorname{rank}(L) \neq n$. By the definition of DLM the rank of L is at least $n-1$. Since equivalence matrix has the same rank, we reduce the matrix L to a reduced echelon form as follows:

$$\begin{bmatrix} 0 & \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{s_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{s_{n-1}} \\ 1 & \frac{a_1}{s_1} & \frac{a_2}{s_2} & \cdots & \frac{a_{n-1}}{s_{n-1}} \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & \cdots & 0 & -b_{n-1} \\ 0 & s_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -b_2 \\ 0 & \cdots & 0 & s_{n-1} & -b_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{b_{n-1}}{s_1} \\ 0 & 1 & 0 & \vdots & -\frac{b_{n-2}}{s_2} \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & -\frac{b_1}{s_{n-1}} \\ 0 & 0 & \cdots & 0 & -a_n - b_n - \frac{a_1}{s_1}b_{n-1} - \frac{a_2}{s_2}b_{n-2} - \cdots - \frac{a_{n-2}}{s_{n-2}}b_2 - \frac{a_{n-1}}{s_{n-1}}b_1 \end{bmatrix}.$$

We see that $\operatorname{rank}(L) = n-1$ if and only if

$$-a_n - b_n - \frac{a_1}{s_1}b_{n-1} - \frac{a_2}{s_2}b_{n-2} - \cdots - \frac{a_{n-2}}{s_{n-2}}b_2 - \frac{a_{n-1}}{s_{n-1}}b_1 = 0$$

if and only if

$$-a_n - b_n = \frac{a_1}{s_1}b_{n-1} + \frac{a_2}{s_2}b_{n-2} + \cdots + \frac{a_{n-2}}{s_{n-2}}b_2 + \frac{a_{n-1}}{s_{n-1}}b_1.$$

We factor L to full rank factorization as follows:

$$\begin{aligned} L = FG &= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} \\ s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{b_{n-1}}{s_1} \\ 0 & 1 & \ddots & \vdots & -\frac{b_{n-2}}{s_2} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{b_1}{s_{n-1}} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}, \end{aligned}$$

where $\mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathbf{q}_1 = \begin{bmatrix} \frac{b_{n-1}}{s_1} \\ \frac{b_{n-2}}{s_2} \\ \vdots \\ \frac{b_1}{s_{n-1}} \end{bmatrix}$, and $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$.

Also, by direct computation, we have

$$GF = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{n-2} & -a_{n-1} - b_{n-1}\frac{s_{n-1}}{s_1} \\ s_1 & 0 & 0 & \ldots & 0 & -b_{n-2}\frac{s_{n-1}}{s_2} \\ 0 & s_2 & 0 & \ldots & 0 & -b_{n-3}\frac{s_{n-1}}{s_3} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -b_2\frac{s_{n-1}}{s_{n-2}} \\ 0 & 0 & \cdots & 0 & s_{n-2} & -b_1\frac{s_{n-1}}{s_{n-1}} \end{bmatrix} =: M.$$

The matrix $GF =: M$ is a doubly Leslie matrix of order $(n-1) \times (n-1)$.

$$M = \begin{bmatrix} -\mathbf{p}_1^T & -a_{n-1} - b_{n-1}\frac{s_{n-1}}{s_1} \\ \Lambda_1 & -\mathbf{q}_2 \end{bmatrix}_{(n-1,n-1)},$$

where $\mathbf{p}_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} b_{n-2}\frac{s_{n-1}}{s_2} \\ b_{n-3}\frac{s_{n-1}}{s_3} \\ \vdots \\ b_1\frac{s_{n-1}}{s_{n-1}} \end{bmatrix}$, and

$$\Lambda_1 = \operatorname{diag}(s_1, s_2, \ldots, s_{n-2})$$

is a diagonal matrix of order $n-1$.

By (9), we have

$$M^{-1} = (M/\Lambda_1)^{-1} \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}, \qquad (13)$$

where $(M/\Lambda_1) = -\left((a_{n-1} + b_{n-1}\frac{s_{n-1}}{s_1}) + s_{n-1}\sum_{i=1}^{n-2}\frac{a_i b_{n-i-1}}{s_i s_{i+1}}\right)$, as in (7), and $\Lambda_1^{-1} = \mathrm{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_{n-2}})$.

From Theorem 3, we have

$$\begin{aligned} L^\sharp &= F(GF)^{-2}G \\ &= \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \times \\ &\quad \left((M/\Lambda_1)^{-1}\begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}\right)^2 \times \\ &\quad \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix} \\ &= (M/\Lambda_1)^{-2}\begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \times \\ &\quad \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}^2 \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}. \end{aligned}$$

Let's consider the same example.

Example

$$L = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

where $\mathbf{p} = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, and $\Lambda = \mathrm{diag}(1, 2, 1)$, is a diagonal matrix of order 3.

Since $\det(L) = 0$, we have $\mathrm{rank}(L) = \mathrm{rank}(L^2)$, we know that the unique $L^\sharp$ exists. Now

$$L = FG = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

and

$$GF = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix},$$

$$(GF)^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

$$(GF)^{-2} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

Finally

$$\begin{aligned} L^\sharp &= F(GF)^{-2}G \\ &= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

This matrix is satisfies the three conditions for group inverse. □

5. Conclusion

In this paper, we mainly study about the explicit formula of Moore-Penrose inverse and group inverse of doubly Leslie matrix.

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