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$\mathcal{H}_2$-OPTIMAL DISTURBANCE REJECTION BY MEASUREMENT FEEDBACK: THE SINGULAR CASE

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Abstract: This work concerns a new methodology to solve the $\mathcal{H}_2$-optimal disturbance rejection problem by measurement feedback in the singular case: namely, when the plant has no feedthrough terms from the control input and the disturbance input to the controlled output and the measured output, respectively. A necessary and sufficient condition for problem solvability is expressed as the inclusion of two subspaces — a controlled-invariant subspace and a conditioned-invariant subspace. Such subspaces are directly derived from the Hamiltonian systems associated to the $\mathcal{H}_2$-optimal control problem and, respectively, to the $\mathcal{H}_2$-optimal filtering problem. The proof of sufficiency, which is constructive, provides the computational tools for the synthesis of the feedback regulator. A numerical example is worked out in order to illustrate how to implement the devised procedure.

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1. Introduction

The problem of $\mathcal{H}_2$-optimal disturbance rejection by measurement feedback consists in finding a dynamic feedback regulator such that the closed-loop system is asymptotically stable and the $\mathcal{H}_2$-norm of the transfer function matrix from the disturbance input to the controlled output is minimal. This problem is completely solved and well-settled in the so-called regular case: i.e., when,

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in the plant equations, the linear map from the control input to the controlled output is injective, the linear map from the disturbance input to the measured output is surjective, and the subsystems involved have no invariant zeros on the imaginary axis. Some recent references are, e.g., [1, 2, 3], although this problem has been considered, as the linear quadratic Gaussian optimal control problem, since the early sixties in a huge classic literature. Indeed, the continuous interest that $\mathcal{H}_2$-optimal control has attracted throughout the last sixty years is due not only to its intrinsic theoretic interest, but also to the number and variety of its practical applications [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] as well as to its flexibility in providing the tools to solve more complex control problems [17, 18, 19, 20, 21, 22, 23].

Nonetheless, the treatment of the problem of $\mathcal{H}_2$-optimal rejection by measurement feedback is much more difficult in the so-called singular case: namely, when the assumptions of injectivity and surjectivity mentioned above are dropped. As was pointed out, e.g., in [24], the separation principle, that makes it possible to reduce the regular problem to an optimal control problem by state feedback and an optimal filtering problem, does not hold anymore, in general, and the infimum of the $\mathcal{H}_2$-norm is not always attainable.

As to the solutions of the singular $\mathcal{H}_2$-control problem available from the literature, those presented in [24, 25, 26] are based on an a-priori assumption on the structure of the dynamic feedback regulator: i.e., this is assumed either to have a feedthrough term from the measurement to the control or not to have it. The methodologies developed therein to prove necessary and sufficient conditions for problem solvability exploit tools like linear matrix inequalities [27] and special coordinate basis [28, 29]. In particular, the solution of a pair of linear matrix inequalities leads to an auxiliary system for which the original $\mathcal{H}_2$-optimal control problem by measurement feedback reduces to an exact decoupling problem by measurement feedback with stability. Then, if the latter problem is solvable, a structural decomposition of the system, known as the special coordinate basis, shows how to design the regulator, also pointing out possible degrees of freedom in the eigenvalue assignment.

Instead, the approach introduced in [30] and further developed in this work leads to a synthesis procedure that, first of all, does not need any a-priori assumption on the structure of the dynamic feedback regulator. Actually, the possible absence of the feedthrough term is an outcome of the synthesis procedure, not a postulate. Secondly, the reasoning is completely developed in the framework of the geometric approach [31, 32]. Indeed, the geometric approach has recently shown to provide powerful tools to handle a variety of up-to-date challenging problems [33, 34, 35, 36, 37, 38, 39, 40].

Actually, the methodological approach developed in this work is inspired by those shared by the previous articles [41, 42, 43, 44, 45, 46], where $\mathcal{H}_2$-optimal control problems were solved by elaborating further on the properties of the associated Hamiltonian systems. In particular, the study of the geometric properties of the two Hamiltonian systems related to the $\mathcal{H}_2$-optimal rejection problem by measurement feedback — one associated to the $\mathcal{H}_2$-optimal control problem by state feedback and the other to the $\mathcal{H}_2$-optimal filtering problem — leads to a pair of resolving subspaces for the original problem: a controlled invariant subspace and a conditioned invariant subspace. In fact, the original problem is shown to be solvable if and only if the latter of the subspaces mentioned above is contained in the former. Moreover, the synthesis of the feedback regulator, when the problem admits a solution, consists in the computation of linear maps which are friends of the resolving subspaces and, respectively, projections connected to the latter of the two.

In comparison with the earlier [30], this work is characterized by the reformulation of both the subproblems of $\mathcal{H}_2$-optimal control and $\mathcal{H}_2$-optimal filtering in strict geometric terms, as the search for subspaces and related linear maps enjoying some special properties. The treatment gains tidiness and compactness from this change of perspective. Moreover, this work investigates in detail the computational aspects involved in the synthesis procedure and illustrates the different stages of its implementation by means of a meaningful numerical example.

This work is organized as follows. Section 2 introduces the problem. Sections 3 and 4 deal with the $\mathcal{H}_2$-optimal control problem and the $\mathcal{H}_2$-optimal filtering problem, respectively, in the geometric approach framework. Section 5 provides the necessary and sufficient geometric condition for solvability of the $\mathcal{H}_2$-optimal rejection problem by measurement feedback. Section 6 illustrates a numerical example. Section 7 presents some concluding remarks.

Notation: $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, and $\mathbb{C}^-$ stand for the sets of real numbers, nonnegative real numbers, complex numbers, and complex numbers with negative real part, respectively. Matrices and linear maps are denoted by slanted capital letters, like A. The spectrum, the image, and the kernel of A are denoted by $\mathscr{S}(A)$, $\operatorname{Im} A$, and $\operatorname{Ker} A$, respectively. The trace, the transpose, the inverse, and the Moore-Penrose inverse of A are denoted by $\operatorname{Tr}(A)$, A', A^{-1}, and $A^{\dagger}$, respectively. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The orthogonal complement of $\mathcal{V}$ is denoted by $\mathcal{V}^{\perp}$. The notation $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$ stands for $\mathcal{V} + \mathcal{W} = \mathcal{X}$ and $\mathcal{V} \cap \mathcal{W} = \{0\}$. The symbol $\uplus$ is used to denote union with multiplicity count. The symbol I stands for an identity matrix of suitable dimension. The symbol $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$. The symbol $G^{\mathrm{H}}(s)$ denotes the complex conjugate transpose of the transfer function matrix $G(s)$. The symbol $\|G(s)\|_{\mathcal{H}_2}$ denotes the $\mathcal{H}_2$-norm of $G(s)$. The symbol $\|v(t)\|_{\ell_2}$ denotes the ℓ_2-norm of the deterministic signal $v(t)$. The symbol $\|w(t)\|_{\mathrm{rms}}$ stands for the root mean square norm of the stochastic signal $w(t)$.

2. Problem Statement

The plant Σ is defined as the continuous-time linear time-invariant system

$$\Sigma \equiv \left\{ \begin{array}{rcl} \dot{x}(t) &=& A\,x(t) + B\,u(t) + D\,d(t), \\ y(t) &=& C\,x(t), \\ e(t) &=& E\,x(t), \end{array} \right.$$

where $t \in \mathbb{R}^+$ is the time variable, $x \in \mathcal{X} = \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the control input, $d \in \mathbb{R}^m$ is the to-be-rejected disturbance input, $y \in \mathbb{R}^q$ is the measured output, and $e \in \mathbb{R}^r$ is the to-be-regulated output, with $p, m, q, r \leq n$. The sets of the admissible control inputs and of the admissible disturbance inputs are defined as the sets $\mathcal{U}_f$ and $\mathcal{D}_f$ of all piecewise-continuous functions with finite values in $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively. A, B, D, C, and E are assumed to be constant real matrices. Moreover, B, D, C, and E are assumed to be full-rank. The pair (A, B) is assumed to be stabilizable. The pair (A, C) is assumed to be detectable.

The dynamic feedback regulator Σ_R is defined as the continuous-time linear time-invariant system

$$\Sigma_R \equiv \left\{ \begin{array}{rcl} \dot{x}_R(t) &=& (A + B\,K\,N + L\,C)\,x_R(t) + (B\,K\,M - L)\,y(t), \\ u(t) &=& K\,N\,x_R(t) + K\,M\,y(t), \end{array} \right.$$

where $x_R \in \mathcal{X} = \mathbb{R}^n$ is the state. N, M, L, and K are constant real matrices to be designed. The feedback regulator Σ_R has the dynamic structure of a state observer and provides a feedback control which is a linear combination of the state estimate and of the plant measured output.

The system Σ_L is defined as the closed-loop interconnection of the plant Σ and the dynamic feedback regulator Σ_R: i.e.,

$$\Sigma_L \equiv \left\{ \begin{array}{rcl} \dot{x}_L(t) &=& A_L\,x_L(t) + D_L\,d(t), \\ e(t) &=& E_L\,x_L(t), \end{array} \right.$$

where

$$A_L = \begin{bmatrix} A + B\,K\,M\,C & B\,K\,N \\ (B\,K\,M - L)\,C & A + B\,K\,N + L\,C \end{bmatrix}, \quad D_L = \begin{bmatrix} D \\ 0 \end{bmatrix},$$
$$E_L = \begin{bmatrix} E & 0 \end{bmatrix}.$$

Hence, the problem of the $\mathcal{H}_2$-optimal disturbance rejection by measurement feedback can be stated as follows.

Problem 1 ($\boldsymbol{\mathcal{H}_2}$-optimal disturbance rejection by measurement feedback)**.** *Let the system Σ be given. Find a dynamic feedback regulator Σ_R such that the closed-loop system Σ_L is asymptotically stable and*

$$\|G_L(s)\|^2_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Tr}\,[G_L^{\mathrm{H}}(j\omega)\,G_L(j\omega)]\,\mathrm{d}\omega,$$

where $G_L(s) = E_L\,(s\,I - A_L)^{-1} D_L$, is minimal.

3. Geometric Approach to $\boldsymbol{\mathcal{H}_2}$-Optimal Control

The purpose of this section is to review the basics of the geometric approach to $\mathcal{H}_2$-optimal control, so as to derive a subspace and a linear map that will play a key role in the solution of the $\mathcal{H}_2$-optimal rejection problem by measurement feedback — as will be shown in Section 5. The results concerning the geometric solution to the $\mathcal{H}_2$-optimal control problem in the singular case were first derived for discrete-time systems in [41]. Later, they were extended to the continuous-time case — see, e.g., the more recent [47] and the references therein.

First, the $\mathcal{H}_2$-optimal control problem is stated, in the time domain, in strict geometric terms. To this aim, it is worth recalling the notion of controlled invariant subspace and the related notions of friend and inner asymptotic stabilizability of a controlled invariant subspace — the reader is referred to [31, 32] for more details. A subsystem — henceforth denoted by Σ_C — of the plant Σ, where only the control input and the to-be-regulated output matter, is considered: i.e.,

$$\Sigma_C \equiv \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), \\ e(t) = E\,x(t). \end{cases}$$

The short notation $\mathcal{B}$ is used in place of $\mathrm{Im}\,B$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be an $(A, \mathcal{B})$-controlled invariant subspace if $A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an $(A, \mathcal{B})$-controlled invariant subspace if and only if there exists a linear map

K such that $(A+B\,K)\,\mathcal{V}\subseteq\mathcal{V}$. Any such K is called a *friend* of $\mathcal{V}$. An $(A,\mathcal{B})$-controlled invariant subspace $\mathcal{V}$ is said to be inner stabilizable if there exists a friend K of $\mathcal{V}$ such that the restricted linear map $(A+B\,K)|_{\mathcal{V}}$ is asymptotically stable.

Problem 2 ($\boldsymbol{\mathcal{H}_2}$-Optimal Control)**.** *Let the system Σ_C be given. Find the maximal inner stabilizable $(A,\mathcal{B})$-controlled invariant subpace — henceforth, $\mathcal{V}^*_{\mathcal{H}_2}$ — and an inner stabilizing friend $K_{\mathcal{H}_2}$ such that the output of the compensated system*

$$\Sigma_C^K \equiv \left\{ \begin{array}{rcl} \dot{x}(t) &=& (A+B\,K_{\mathcal{H}_2})\,x(t), \\ e(t) &=& E\,x(t), \end{array} \right.$$

satisfies the condition that $\|e(t)\|_{\ell_2}$ is minimal for all $x_0\in\mathcal{V}^*_{\mathcal{H}_2}$.

The statement above is equivalent to the more usual one, referred to Σ and expressed in the frequency domain (see, e.g., [3, Section 11.2] for the statement in the nonsingular case), by virtue of Parseval's theorem — on the assumptions $d(t)=0$ for all $t\in\mathbb{R}^+$, $u(\cdot)\in\mathcal{U}_f$, and $x_0\in\mathcal{V}^*_{\mathcal{H}_2}$.

The Hamiltonian function associated to Problem 2 is

$$\mathscr{H}(x(t),\lambda(t),u(t)) = x'(t)\,E'\,E\,x(t) + \lambda'(t)\,\left[A\,x(t)+B\,u(t)\right],$$

where $\lambda(t)$ is an undetermined multiplier (called the *costate*). The state and costate equations and the stationarity condition are

$$\dot{x}(t) = \left(\frac{\partial\mathscr{H}}{\partial\lambda}\right)' = A\,x(t)+B\,u(t), \tag{1a}$$

$$\dot{\lambda}(t) = -\left(\frac{\partial\mathscr{H}}{\partial x}\right)' = -2\,E'\,E\,x(t) - A'\,\lambda(t), \tag{1b}$$

$$0 = \left(\frac{\partial\mathscr{H}}{\partial u}\right)' = B'\,\lambda(t). \tag{1c}$$

According to [41], the geometric approach to the solution of the $\mathcal{H}_2$-optimal control problem reduces the latter to a problem of output zeroing for the associated Hamiltonian system — i.e., the dynamical system $\tilde{\Sigma}$ derived from (1a)–(1c) by setting $p(t)=2\,\lambda(t)$ and $\eta(t)=B'\,\lambda(t)$. Hence, let $\tilde{x}(t)=[x'(t)\ p'(t)]'$. Then, $\tilde{\Sigma}$ is described by

$$\tilde{\Sigma} \equiv \left\{ \begin{array}{rcl} \dot{\tilde{x}}(t) &=& \tilde{A}\,\tilde{x}(t)+\tilde{B}\,u(t), \\ \eta(t) &=& \tilde{E}\,\tilde{x}(t), \end{array} \right.$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -E'\,E & -A' \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} 0 & B' \end{bmatrix}. \tag{2}$$

Therefore, Problem 2 is equivalent to the following, which refers to system $\tilde{\Sigma}$. The symbols $\tilde{\mathscr{B}}$ and $\tilde{\mathscr{E}}$ respectively stand for $\operatorname{Im}\tilde{B}$ and $\operatorname{Ker}\tilde{E}$.

Problem 3 (Perfect Decoupling)**.** *Let the system $\tilde{\Sigma}$ be given. Find the maximal inner stabilizable $(\tilde{A},\tilde{\mathscr{B}})$-controlled invariant subspace contained in $\tilde{\mathscr{E}}$ — henceforth, $\tilde{\mathcal{V}}_g^*$ — and an inner stabilizing friend $\tilde{K}_g$.*

Problem 3 is a slight variant of the classic disturbance decoupling problem with stability [31, Section 5.6], [32, Section 4.2], in the sense that, for any initial state $\tilde{x}_0 \in \tilde{\mathcal{V}}_g^*$, the state trajectory, which evolves according to the compensated dynamics

$$\tilde{\Sigma}^K \equiv \begin{cases} \dot{\tilde{x}}(t) = (\tilde{A} + \tilde{B}\,\tilde{K}_g)\,\tilde{x}(t), \\ \eta(t) = \tilde{E}\,\tilde{x}(t), \end{cases}$$

belongs to the null space of the output and converges to the origin as the time approaches ∞. The subspace $\tilde{\mathcal{V}}_g^*$ and a corresponding inner stabilizing friend $\tilde{K}_g$ — note that it may also happen that $\tilde{\mathcal{V}}_g^* = \{0\}$ — can be determined, e.g., by means of the computational algorithms available from [32].

Hence, the last step of this reasoning consists in showing how a pair $(\mathcal{V}_{\mathcal{H}_2}^*, K_{\mathcal{H}_2})$ that solves Problem 2 can be derived from a pair $(\tilde{\mathcal{V}}_g^*, \tilde{K}_g)$ that solves Problem 3.

Proposition 1. *Let the pair $(\tilde{\mathcal{V}}_g^*, \tilde{K}_g)$ solve Problem 3. Let*

$$\tilde{\mathcal{V}}_g^* = \operatorname{Im}\begin{bmatrix} V_X \\ V_P \end{bmatrix}, \tag{3a}$$

$$\tilde{K}_g = \begin{bmatrix} K_X & K_P \end{bmatrix}, \tag{3b}$$

where the partition considered in (3) is consistent with that in (2). Then, a pair $(\mathcal{V}_{\mathcal{H}_2}^*, K_{\mathcal{H}_2})$ that solves Problem 2 is given by

$$\mathcal{V}_{\mathcal{H}_2}^* = \operatorname{Im} V_X, \tag{4a}$$

$$K_{\mathcal{H}_2} = K_X + K_P\,V_P\,V_X^{\dagger}. \tag{4b}$$

Proof. Let $\mathcal{V}_{\mathcal{H}_2}^*$ and $K_{\mathcal{H}_2}$ be defined as (4). First, it will be shown that $\mathcal{V}_{\mathcal{H}_2}^*$ is an inner stable $(A + B\,K_{\mathcal{H}_2})$-invariant subspace. Let $\tilde{V}_g^*$ be the compact notation for the basis matrix of $\tilde{\mathcal{V}}_g^*$ shown in (3a). Since $\tilde{\mathcal{V}}_g^*$ is an inner stable $(\tilde{A} + \tilde{B}\tilde{K}_g)$-invariant subspace, there exists a matrix X such that

$$(\tilde{A} + \tilde{B}\,\tilde{K}_g)\,\tilde{V}_g^* = \tilde{V}_g^*\,X, \tag{5a}$$

$$\mathscr{S}((\tilde{A} + \tilde{B}\,\tilde{K}_g)|_{\tilde{\mathcal{V}}_g^*}) = \mathscr{S}(X) \subset \mathbb{C}^-. \tag{5b}$$

Equation (5a) can also be written as

$$\begin{bmatrix} A+B\,K_X & B\,K_P \\ E'\,E & -A' \end{bmatrix} \begin{bmatrix} V_X \\ V_P \end{bmatrix} = \begin{bmatrix} V_X \\ V_P \end{bmatrix} X, \tag{6}$$

where (2) and (3) have been taken into account. From the first block of rows in (6), one gets $A\,V_X + B\,(K_X\,V_X + K_P\,V_P) = V_X\,X$, which can also be written as

$$(A + B\,K_{\mathcal{H}_2})\,V_X = V_X\,X, \tag{7}$$

since $K_X\,V_X + K_P\,V_P = K_{\mathcal{H}_2}\,V_X$. Equation (7), in light of (4a) and (5b), proves $(A+B\,K_{\mathcal{H}_2})$-invariance and inner stability of $\mathcal{V}^*_{\mathcal{H}_2}$. As to maximality of $\mathcal{V}^*_{\mathcal{H}_2}$ and minimality of $\|e(t)\|_{\ell_2}$, these facts follow from maximality of $\tilde{\mathcal{V}}^*_g$. □

As was shown in [41, Section IV], in the discrete-time case, the matrix V_X is an invertible matrix of dimension n, since the subspace of the admissible initial states is the state space of the original system. Instead, in the continuous-time case, the subspace of the initial states that can be driven asymptotically to the origin along trajectories corresponding to the minimal ℓ_2-norm of the output, by means of a state-feedback, does not match the whole state space, in general (see, e.g., [47] and the references therein, but also [2, Chapter 6]). For this reason, in (4b) — which is the continuous-time counterpart of (18) in [41, Section IV] — the Moore-Penrose inverse of V_X replaces the inverse.

4. Geometric Approach to $\mathcal{H}_2$-Optimal Filtering

The aim of this section is to state and solve the $\mathcal{H}_2$-optimal filtering problem in the geometric framework, in order to derive the second pair, made up of a subspace and a linear map, needed to solve the $\mathcal{H}_2$-optimal rejection problem by measurement feedback — refer to Section 5. The lines followed by the reasoning developed herein are similar to those drawn in Section 4. Indeed, duality arguments are extensively used.

First, the $\mathcal{H}_2$-optimal filtering problem is given a time-domain formulation in pure geometric terms. To this aim, the notion of conditioned invariant subspace and the joint notions of friend and outer stabilizability of a conditioned invariant subspace are to be retrieved [32, Section 4.1]. The system Σ_E is derived from the plant Σ by only considering the disturbance input and the measured output, so that

$$\Sigma_E \equiv \begin{cases} \dot{x}(t) = A\,x(t) + D\,d(t), \\ y(t) = C\,x(t). \end{cases}$$

The input $d(\cdot)$ is assumed to be a zero-mean white Gaussian noise. The symbol $\mathcal{C}$ stands for $\operatorname{Ker} C$. A subspace $\mathcal{W} \subseteq \mathcal{X}$ is said to be an $(A, \mathcal{C})$-conditioned invariant subspace if $A\,(\mathcal{W} \cap \mathcal{C}) \subseteq \mathcal{W}$. A subspace $\mathcal{W} \subseteq \mathcal{X}$ is an $(A, \mathcal{C})$-conditioned invariant subspace if and only if a there exists a linear map L such that $(A + L\,C)\,\mathcal{W} \subseteq \mathcal{W}$. Any such L is called a friend[1] of $\mathcal{W}$. An $(A, \mathcal{C})$-conditioned invariant subspace $\mathcal{W}$ is said to be outer stabilizable if there exists a friend L such that the induced linear map $(A + L\,C)|_{\mathcal{X}/\mathcal{W}}$ is asymptotically stable.

Problem 4 (**$\boldsymbol{\mathcal{H}_2}$-**Optimal Filtering)**.** *Let the system Σ_E be given. Find the minimal outer stabilizable $(A, \mathcal{C})$-conditioned invariant subspace — henceforth, $\mathcal{W}^*_{\mathcal{H}_2}$ — and an outer stabilizing friend $L_{\mathcal{H}_2}$ such that the state of the system*

$$\Sigma_E^L \equiv \left\{ \; \dot{x}(t) \, = \, (A + L\,C)\,x(t) + D\,d(t), \right.$$

satisfies the condition that $\|x(t)\|_{\text{rms}}$ is minimal for all $x_0 \in \mathcal{X}/\mathcal{W}^*_{\mathcal{H}_2}$.

The solution of Problem 4 can be derived from that of Problem 2 by duality arguments. Namely, let $(\mathcal{V}^*_{\mathcal{H}_2(A', C', D')}, K_{\mathcal{H}_2(A', C', D')})$ be a solution of Problem 2 where the triple (A, B, E) has been replaced by the triple $(A',\, C',\, D')$. Then, a pair $(\mathcal{W}^*_{\mathcal{H}_2}, L_{\mathcal{H}_2})$ that solves Problem 4 is given by $\mathcal{W}^*_{\mathcal{H}_2} = (\mathcal{V}^*_{\mathcal{H}_2(A', C', D')})^{\perp}$ and $L_{\mathcal{H}_2} = K'_{\mathcal{H}_2(A', C', D')}$.

5. Problem Solution

In this section, a necessary and sufficient condition for the solution of Problem 1 is derived by exploiting the properties of the subspaces $\mathcal{V}^*_{\mathcal{H}_2}$ and $\mathcal{W}^*_{\mathcal{H}_2}$ and of their corresponding friends $K_{\mathcal{H}_2}$ and $L_{\mathcal{H}_2}$, respectively introduced in Sections 3 and 4. The proof of sufficiency is constructive, so that it outlines the procedure for the synthesis of the dynamic feedback regulator introduced in Section 2.

Beforehand, it is worth reviewing the notions of outer stabilizability of an $(A, \mathcal{B})$-controlled invariant subspace and inner stabilizability of an $(A, \mathcal{C})$-conditioned invariant subspace, along with their respective relations with stabilizability of the pair (A, B) and detectability of (A, C). Indeed, these definitions are the obvious complement of those respectively given in Sections 3 and 4.

An $(A, \mathcal{B})$-controlled invariant subspace $\mathcal{V}$ is said to be outer stabilizable if there exists a friend K of $\mathcal{V}$ such that the induced linear map $(A + B\,K)|_{\mathcal{X}/\mathcal{V}}$ is asymptotically stable. Similarly, an $(A, \mathcal{C})$-conditioned invariant subspace

[1] Note that the name is the same as that used for controlled invariant subspaces. However, the different meaning is clear from the fact the state feedback is related to a controlled invariant subspace, while the output injection is related to a conditioned invariant subspace.

$\mathcal{W}$ is said to be inner stabilizable if there exists a friend L of $\mathcal{W}$ such that the restricted linear map $(A+L\,C)|_{\mathcal{W}}$ is asymptotically stable. Moreover, as can be easily shown, if (A,B) is stabilizable, any $(A,\mathcal{B})$-controlled invariant subspace is outer stabilizable. Likewise, if (A,C) is detectable, any $(A,\mathcal{C})$-conditioned invariant subspace is inner stabilizable.

It is also worth mentioning that inner and outer stabilizability of an $(A,\mathcal{B})$-controlled invariant subspace are indipendent of each other — the same is true for an $(A,\mathcal{C})$-conditioned invariant subspace. This fact can be shown by considering a friend K of an $(A,\mathcal{B})$-controlled invariant subspace $\mathcal{V}$ and performing a state space basis transformation $T=[T_1\,T_2]$, where $\operatorname{Im} T_1=\mathcal{V}$. In fact, with respect to the new coordinates,

$$\hat{A}+\hat{B}\,\hat{K}=T^{-1}(A+B\,K)\,T=\begin{bmatrix}\hat{A}_{11}+\hat{B}_1\,\hat{K}_1 & \hat{A}_{12}+\hat{B}_1\,\hat{K}_2\\ 0 & \hat{A}_{22}+\hat{B}_2\,\hat{K}_2\end{bmatrix}.$$

The upper-block triangular structure of $\hat{A}+\hat{B}\,\hat{K}$ shows the separation between the inner and outer dynamics of $\mathcal{V}$.

In particular, from the previous considerations, it follows that the subspace $\mathcal{V}^*_{\mathcal{H}_2}$ can be rendered outer stable without affecting the inner dynamics, assigned through $K_{\mathcal{H}_2}$. More formally, there exists a friend K of $\mathcal{V}^*_{\mathcal{H}_2}$ such that

$$(A+B\,K)|_{\mathcal{V}^*_{\mathcal{H}_2}}=(A+B\,K_{\mathcal{H}_2})|_{\mathcal{V}^*_{\mathcal{H}_2}},\quad \mathscr{S}((A+B\,K)|_{\mathcal{X}/\mathcal{V}^*_{\mathcal{H}_2}})\subset\mathbb{C}^-. \tag{8}$$

Similarly, there exists a friend L of $\mathcal{W}^*_{\mathcal{H}_2}$ such that

$$(A+L\,C)|_{\mathcal{X}/\mathcal{W}^*_{\mathcal{H}_2}}=(A+L_{\mathcal{H}_2}\,C)|_{\mathcal{X}/\mathcal{W}^*_{\mathcal{H}_2}},\quad \mathscr{S}((A+L\,C)|_{\mathcal{W}^*_{\mathcal{H}_2}})\subset\mathbb{C}^-. \tag{9}$$

The necessary and sufficient condition for the solution of Problem 1 — Theorem 1 below — is preceded by a lemma concerning the matrices M and N of the to-be-designed feedback regulator. In particular, Lemma 1 establishes how to derive the matrices M and N, that define the linear combination between the state estimate and the measured output in the feedback control law, by exploiting the features of the subspace $\mathcal{W}^*_{\mathcal{H}_2}$.

Lemma 1. *Let the system Σ be given. Let the subspace $\mathcal{Q}\subseteq\mathcal{X}$ be such that*

$$\mathcal{Q}\oplus(\mathcal{W}^*_{\mathcal{H}_2}\cap\mathcal{C})=\mathcal{W}^*_{\mathcal{H}_2}. \tag{10}$$

Then, there exist linear maps M and N, such that

$$M\,C+N=I, \tag{11a}$$

$$\operatorname{Ker} N=\mathcal{Q}. \tag{11b}$$

Proof. The proof is constructive. First, note that $\mathcal{Q}\cap\mathcal{C}=\{0\}$ owing to (10). Let the subspace $\mathcal{P}\subseteq\mathcal{X}$ be such that

$$\mathcal{P}\oplus\mathcal{Q}=\mathcal{X},\quad \mathcal{P}\supseteq\mathcal{C}. \tag{12}$$

Then, let N be the projection on $\mathcal{P}$ along $\mathcal{Q}$. Consequently, (11b) holds. Furthermore, the linear map $I-N$ is the complementary projection — i.e., the projection on $\mathcal{Q}$ along $\mathcal{P}$ — which implies that $\operatorname{Ker}(I-N)=\mathcal{P}\supseteq\mathcal{C}$. Then, the matrix equation (11a) in the unknown M is solved by $M=(I-N)\,C^{\dagger}$. □

Theorem 1. *Problem 1 is solvable if and only if*

$$\mathcal{W}^*_{\mathcal{H}_2}\subseteq\mathcal{V}^*_{\mathcal{H}_2}. \tag{13}$$

Proof. If. Let the to-be-designed matrices K, L, M, and N of Σ_R be picked so as to satisfy (8), (9), and (11), respectively. Then, in order to show that the regulator Σ_R thus determined solves Problem 1, consider the closed-loop system Σ_L and apply the state space basis transformation $T_L=\begin{bmatrix} I & 0\\ I & -I\end{bmatrix}$. Hence, in the new coordinates,

$$\begin{aligned}\hat{A}_L &= T_L^{-1}\,A_L\,T_L=\begin{bmatrix} A+B\,K & -B\,K\,N\\ 0 & A+L\,C\end{bmatrix},\quad \hat{D}_L=T_L^{-1}\,D_L=\begin{bmatrix} D\\ D\end{bmatrix},\\ \hat{E}_L &= E_L\,T_L=\begin{bmatrix} E & 0\end{bmatrix},\end{aligned}$$

where (11a) has been taken into account. Then, it is sufficient to show that the subspace

$$\mathcal{R}=\left\{\begin{bmatrix} x\\ z\end{bmatrix} : x\in\mathcal{V}^*_{\mathcal{H}_2},\ z\in\mathcal{W}^*_{\mathcal{H}_2}\right\}$$

is an inner and outer asymptotically stable A_L-invariant subspace. Since, $\mathcal{V}^*_{\mathcal{H}_2}$ is an inner and outer asymptotically stable $(A+B\,K)$-invariant subspace and, likewise, $\mathcal{W}^*_{\mathcal{H}_2}$ is an inner and outer asymptotically stable $(A+L\,C)$-invariant subspace, showing that $\mathcal{R}$ is an inner and outer asymptotically stable A_L-invariant subspace reduces to showing that

$$B\,K\,N\,\mathcal{W}^*_{\mathcal{H}_2}\subseteq\mathcal{V}^*_{\mathcal{H}_2}. \tag{14}$$

Equation (13) along with $(A+B\,K)$-invariance of $\mathcal{V}^*_{\mathcal{H}_2}$ imply $(A+B\,K)\,\mathcal{W}^*_{\mathcal{H}_2}\subseteq\mathcal{V}^*_{\mathcal{H}_2}$, which, in turn, implies $(A+B\,K)\,(\mathcal{W}^*_{\mathcal{H}_2}\cap\mathcal{C})\subseteq\mathcal{V}^*_{\mathcal{H}_2}$, where the latter can be written as

$$(A+B\,K\,M\,C+B\,K\,N)\,(\mathcal{W}^*_{\mathcal{H}_2}\cap\mathcal{C})\subseteq\mathcal{V}^*_{\mathcal{H}_2}, \tag{15}$$

in light of (11a). Note that $A\,(\mathcal{W}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{W}^*_{\mathcal{H}_2}$, since $\mathcal{W}^*_{\mathcal{H}_2}$ is an $(A, \mathcal{C})$-conditioned invariant subspace. Moreover, note that $(\mathcal{W}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{C}$ implies $B\,K\,M\,C\,(\mathcal{W}^*_{\mathcal{H}_2} \cap \mathcal{C}) = \{0\}$. Therefore, from (15) it follows that

$$B\,K\,N\,(\mathcal{W}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{V}^*_{\mathcal{H}_2}, \tag{16}$$

Finally, (16), (10), and (11b) imply (14).

Only if. It is direct consequence of minimality of $\mathcal{W}^*_{\mathcal{H}_2}$ and maximality of $\mathcal{V}^*_{\mathcal{H}_2}$ as resolving subspaces of the associated $\mathcal{H}_2$-optimal filtering problem and $\mathcal{H}_2$-optimal control problem, respectively. □

In light of the constructive proof of the if-part of Theorem 1, it is worth noting that, if $\mathcal{W}^*_{\mathcal{H}_2} \subseteq \mathcal{C}$, the feedthrough term from the measured output to the control input in the feedback regulator Σ_R is zero, while the feedback involves the whole state estimate. In fact, in this case, $\mathcal{W}^*_{\mathcal{H}_2} \cap \mathcal{C} = \mathcal{W}^*_{\mathcal{H}_2}$. Consequently, according to (10), $\mathcal{Q} = \{0\}$ and, according to (12), $\mathcal{P} = \mathcal{X}$. Finally, according to (11), $M = 0$ and $N = I$. In few words, one can say that, in this case, the separation principle holds.

6. An Illustrative Example

In this section, a numerical example is worked out to illustrate the synthesis procedure discussed so far. The computational support consists of the Matlab files implementing the geometric approach algorithms [32]. The variables will be displayed in scaled fixed point format with five digits, although computations are made in floating point precision.

Let the system Σ be defined by the matrices

$$A = \begin{bmatrix} -5 & 1 & 2 & 3 & 4 \\ 0 & -8.95 & -6.45 & 0 & 0 \\ 0 & 2.15 & -0.35 & 0 & 0 \\ 0 & -10.89 & -40.94 & -16.10 & -7.95 \\ 0 & 8.17 & 28.87 & 7.07 & -0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 5 & 6 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 10 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.2765 \\ -0.1573 \\ -0.9282 \\ 0.0905 \\ 0.1296 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 5 & 6 \\ 2 & 3 & 0 & 0 & 0 \end{bmatrix}.$$

The pair (A, B) is controllable, while (A, C) is observable. The Hamiltonian system $\tilde{\Sigma}$ associated to Problem 2 is determined according to (2). The subspace $\tilde{\mathcal{V}}_g^*$ and a linear map $\tilde{K}_g$ solving Problem 3 are

$$\tilde{\mathcal{V}}_g^* = \mathrm{Im}\begin{bmatrix} V_X \\ V_P \end{bmatrix} = \left[\begin{array}{ccc} 0.7723 & -0.2765 & 0.0517 \\ -0.5243 & 0.1573 & -0.0346 \\ 0.3305 & 0.9282 & 0.0081 \\ 0.0068 & -0.0905 & 0.7715 \\ 0.1047 & -0.1296 & -0.6331 \\ \hline 0.0796 & -0.0195 & 0.0035 \\ 0.0002 & 0 & 0 \\ 0.0459 & 0.1088 & 0.0007 \\ -0.0022 & -0.0002 & -0.0001 \\ -0.0002 & 0 & 0 \end{array}\right],$$

$$\tilde{K}_g = \begin{bmatrix} K_X & K_P \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 0.2391 & -0.2822 & 4.2301 & 0.0635 & -0.6714 & 0.0590 & 0.0005 & 0.5079 & -0.0046 & -0.0005 \\ -0.2058 & 0.1345 & 0.0904 & -0.0520 & -0.0143 & -0.0196 & 0 & 0.0092 & 0.0004 & 0 \end{array}\right].$$

Note that $\mathscr{S}((\tilde{A}+\tilde{B}\,\tilde{K}_g)|_{\tilde{\mathcal{V}}_g^*}) = \{-1.7025, -3.4785, -9.5746\} \subset \mathbb{C}^-$. The subspace $\mathcal{V}_{\mathcal{H}_2}^*$ is determined according to (4a), while the linear map $K_{\mathcal{H}_2}$, given by (4b), is

$$K_{\mathcal{H}_2} = \begin{bmatrix} 0.2459 & -0.2885 & 4.2905 & 0.0591 & -0.6759 \\ -0.2070 & 0.1353 & 0.0913 & -0.0522 & -0.0145 \end{bmatrix}.$$

Since (A, B) is controllable, the outer eigenvalues of $\mathcal{V}_{\mathcal{H}_2}^*$ are assignable. By picking those eigenvalues, e.g., equal to -2.2 and -3.3, one gets that a linear map K which satisfies (8) is

$$K = \begin{bmatrix} -0.1985 & 0.9970 & 5.3050 & 5.2079 & 5.5045 \\ 0.1626 & 0.4522 & -0.0185 & -0.6794 & -0.7672 \end{bmatrix}.$$

The subspace $\mathcal{W}_{\mathcal{H}_2}^*$ and a linear map $L_{\mathcal{H}_2}$ that solve Problem 4 are

$$\mathcal{W}_{\mathcal{H}_2}^* = \mathrm{Im}\begin{bmatrix} 0.2782 \\ -0.1583 \\ -0.9339 \\ 0.0910 \\ 0.1304 \end{bmatrix}, \quad L_{\mathcal{H}_2} = \begin{bmatrix} 7.3499 & 0.2692 \\ -25.6113 & -1.2143 \\ 7.0753 & 0.3259 \\ -135.4649 & -3.2703 \\ 98.48 & 2.5692 \end{bmatrix}.$$

Since (A, C) is observable, the inner eigenvalue of $\mathcal{W}^*_{\mathcal{H}_2}$ is assignable. By choosing this eigenvalue, e.g., equal to -8.8, one gets that a linear map L meeting (9) is

$$L = \begin{bmatrix} 0.6533 & 0.5811 \\ -21.8015 & -1.3917 \\ 29.5546 & -0.7209 \\ -137.6561 & -3.1682 \\ 95.3415 & 2.7154 \end{bmatrix}.$$

Then, the design of the feedback regulator requires that the linear maps M and N are computed in accordance with Lemma 1. Since $\mathcal{W}^*_{\mathcal{H}_2} \cap \mathcal{C} = \{0\}$, a subspace $\mathcal{Q}$ such that (10) holds is $\mathcal{W}^*_{\mathcal{H}_2}$. A subspace $\mathcal{P}$ such that (12) holds, is

$$\mathcal{P} = \mathrm{Im} \begin{bmatrix} 0 & 0 & 0 & 0.0057 \\ -0.9692 & 0 & 0 & 0.2462 \\ 0.0313 & -0.9919 & 0 & 0.1231 \\ 0.1563 & 0.0813 & -0.7682 & 0.6154 \\ 0.1876 & 0.0976 & 0.6402 & 0.7385 \end{bmatrix}.$$

Consequently, the linear maps M and N are

$$M = \begin{bmatrix} 1 & -0.0007 \\ -0.5689 & 0.0004 \\ -3.3567 & 0.0024 \\ 0.3272 & -0.0002 \\ 0.4687 & -0.0003 \end{bmatrix},$$

$$N = \begin{bmatrix} 0 & 0.0014 & 0.0007 & 0.0035 & 0.0042 \\ 0.5689 & 0.9992 & -0.0004 & -0.0020 & -0.0024 \\ 3.3567 & -0.0047 & 0.9976 & -0.0118 & -0.0142 \\ -0.3272 & 0.0005 & 0.0002 & 1.0012 & 0.0014 \\ -0.4687 & 0.0007 & 0.0003 & 0.0017 & 1.0020 \end{bmatrix}.$$

Thus, the design of Σ_R is complete. Correspondingly, the value of the $\mathcal{H}_2$-norm of the transfer function matrix of the closed-loop system is $\|G(j\omega)\|_{\mathcal{H}_2} = 0.3263$.

7. Conclusions

A methodology to solve the singular $\mathcal{H}_2$-optimal rejection problem by measurement feedback has been completely developed in the framework of the geometric approach. A necessary and sufficient condition for the existence of a solution to the stated problem has been expressed in terms of the inclusion between two subspaces — a controlled-invariant subspace and a conditioned-invariant subspace — respectively derived from the Hamiltonian systems associated to the $\mathcal{H}_2$-optimal control problem and to the $\mathcal{H}_2$-optimal filtering problem. The if-part of the proof, which is constructive, shows the procedure for synthesizing the dynamic feedback regulator. No a-priori choices on the structure of the regulator (either with or without the feedthrough term) are required. Nonetheless, the method at issue retrieves the regulator without the feedthrough term when this is able to attain the minimum of the $\mathcal{H}_2$-norm or, equivalently, when the separation principle holds. A worked-out example illustrates how to apply the discussed techniques.

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