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# *J*-CLASS SEMIGROUP OPERATORS

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**Abstract:** A $C_0$-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on an infinite-dimensional separable complex Banach space $X$ is called subspace-hypercyclic for a subspace $M$, if $Orb(\mathcal{T}, x) \bigcap M$ is dense in $M$ for a vector $x \in M$. In this paper, we localize the notion of M-extended semigroup(resp. M-extended semigroup mixing) limit set of $x$ under $\mathcal{T}$ and We give sufficient conditions of being $M$-hypercyclic for this semigroup. Then by this result, we prove that $(T_t^{-1})_{t\geq 0}$ is a $M$-hypercyclic. This result is an answer of the question of B. F. Madore and R. A. Martnez-Avendano for $C_0$-semigroup.

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**Key Words:** $C_0$-semigroup, subspace-hypercyclic, subspace-topologically transitive, $J-$class semigroup, $J^{mix}-$class semigroup

## 1. Introduction

For an infinite-dimensional separable complex Banach space $X$, $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on $X$. For $x \in X$, the orbit of $x$ under $(T_n)_n \subset \mathcal{B}(X)$ is the set $Orb(T_n, x) = \{T_n x : n \in \mathbb{Z}_+\}$. A sequence $(T_n)_n$ of operators is called a hypercyclic, if there is some $x$ whose orbit under $(T_n)_n$ is dense in $X$. In such a case, $x$ is called a hypercyclic or universal vector for $(T_n)_n$. A sequence $(T_n)_n$ of operators is called topologically transitive, if for every nonempty open subsets $U$ and $V$ of $X$, there is some $n \geq 0$ such that

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$T_n(U) \cap V \neq \emptyset$. For some sources on these topics, see [1], [3], [8], [10], [12], [16], [18].

The notions of the limit and extended limit sets are well known in the theory of topological dynamics, see [4]. It is not difficult to show that $T$ is topologically transitive if and only if $J(x) = X$ for every $x \in X$ and that $T$ is topologically mixing if, and only if $J^{mix}(x) = X$ for every $x \in X$, see [6]. For more information on the $J$-class set, see [2], [5], [6].

The first example of a hypercyclic operator was shown by Birkhoff for the translation operator, Rolewicz constructed the first example of operators on Banach space considering $B$ the backward shift on $l^p$, he showed that $\lambda B$ is hypercyclic if, and only if $|\lambda| > 1$ (see [3]). Ansari [1] showed that hypercyclic operators exist on infinite-dimensional separable Banach space.

Recall that a one-parameter family $(T_t)_{t\geq 0}$ of operators on $X$ is called a strongly continuous semigroup (or $C_0$-semigroup) of operators, if $T_0 = I$, $T_{t+s} = T_t T_s$ for all $t, s \geq 0$ and $\lim_{t \longrightarrow s} T_t(x) = T_s(x)$ for all $s \geq 0$ and $x \in X$, see [7], [10], [14].

In 2011, B. F. Madore and R. A. Martnez-Avendano in [13] introduced and studied the concept of subspace-hypercyclicity for an operator. An operator $T$ is subspace-hypercyclic or $M$-hypercyclic for a subspace $M$ of $X$, if there exists $x \in X$ such that $Orb(T, x) \bigcap M$ is dense in $M$. Such a vector $x$ is called a $M$-hypercyclic vector for $T$, they showed that there are operators which are $M$-hypercyclic but not hypercyclic. They introduced analogously the concept of subspace-transitivity. Let $T \in \mathcal{B}(X)$ and $M$ be a closed subspace of $X$, we say that $T$ is $M$-transitive, if for any non-empty open sets $U, V$ in $M$, there exists $n \geq 0$ such that $T^{-n}(U) \bigcap V$ contain a non-empty open subset of $M$.The authors showed that $M$-transitivity implies $M$-hypercyclicity. Note that the converse is not true, this is proven recently by C. M. Le in [11]; for more information see [9], [15]. In 2013 S.Talebi, M.Asadipour localized the notion of subspace-transitivity and gave the answer of the question asked by B.F. Madore and R.A. Martnez-Avendano, see [19].

Recently, in 2015 Abdelaziz Tajmouati , Abdeslam El Bakkali and Ahmed Toukmati introduced and studied the $M$-Hypercyclicity of $C_0$-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on an infinite-dimensional separable complex Banach space $X$ and gave sufficient conditions of being $M$-hypercyclic for this semigroup. Moreover, some proprieties and analogous results for the notion of $M$-transitive, see [17].

In this paper, we will localize at first the notion of M-extended semigroup(resp. $M$-extended semigroup mixing) limit set of $x$ under $\mathcal{T}$ and We will give sufficient conditions of being $M$-hypercyclic for this semigroup. Then by this result, we will prove that $(T_t^{-1})_{t\geq 0}$ is a $M$-hypercyclic. This result is an answer of the question $(i)$ of B. F. Madore and R. A. Martnez-Avendano for $C_0$-semigroup, see [13].

## 2. Main Results

We will assume that the subspace $\mathcal{M} \subset X$ is topologically closed. We start with our main definitions.

**Definition 2.1.** *[17] Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. We say that $\mathcal{T}$ is $M-$hypercyclic if there exists a vector $x \in X$ such that $Orb(\mathcal{T}, x) \bigcap M$ is dense in $M$ with $Orb(\mathcal{T}, x) = \{T_t x : t \geq 0\}$*

**Definition 2.2.** *[17] Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. We call that $\mathcal{T}$ is $M$-transitive if for every tow open, non-empty subsets $U, V$ of $M$ there is $t \geq 0$ such that $T_t^{-1}(U) \bigcap V$ contains a non-empty open set of $M$.*

**Definition 2.3.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. We call that $\mathcal{T}$ is $M$-mixing if for every tow open, non-empty subsets $U, V$ of $M$ there is $t_0 \geq 0$ such that for all $t \geq t_0, T_t^{-1}(U) \bigcap V$ contains a non-empty open set of $M$*

**Theorem 2.1.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. Then the following conditions are equivalent*

1. *$\mathcal{T}$ is $M$-mixing.*
2. *For every non-empty open $U$ and $V$ of $M$, there is $t_0 \geq 0$ such that for all $t \geq t_0, T_t^{-1}(U) \bigcap V$ is a non-empty open of $M$.*
3. *For every non-empty open $U$ and $V$ of $M$, there is $t_0 \geq 0$ such that for all $t \geq t_0, T_t^{-1}(U) \bigcap V$ is non-empty and $T_t(M) \subset M$.*

*Proof.* (2) $\Leftrightarrow$ (1) is clair.

(3) $\Rightarrow$ (2). Let $U$ and $V$ be nonempty open subsets of $M$, by (3) there is $t_0 \geq 0$ such that for all $t \geq t_0, T_t^{-1}(U) \bigcap V$ is non-empty and $T_t(M) \subset M$.

Since $T_{t|M} : M \longrightarrow M$ is continuous, then $T_t^{-1}(U)$ is open in $M$, therefore $T_t^{-1}(U) \cap V$ is nonempty open of $M$.

(1) $\Rightarrow$ (3). Let $U$ and $V$ be tow nonempty open subsets of $M$. By (1) there exists $t_0 \geq 0$ such that for all $t \geq t_0, T_t^{-1}(U) \bigcap V$ contains a nonempty open $W$ of $M$, it follows that $W \subset T_t^{-1}(U) \cap V$ and $T_t^{-1}(U) \cap V \neq \phi$.

Next, We prove that $T_t(M) \subset M$.

Let $x \in M$, we have $W \subset T_t^{-1}(U) \cap V$, this implies that$T_t(W) \subset U \subset M$. Let $x_0 \in W$, since $W$ is open of $M$ then for all $r$ enough small we have $x_0+rx \in W$, therefore $T_t(x_0+rx) = T_tx_0+rT_tx \in T_t(W) \in M$. From $T_tx_0 \in M$ it follows that $T_tx \in M$.

We then conclude that $T_t(M) \subset M$. □

**Corollairy 2.1.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. If $\mathcal{T}$ is $M$-mixing then $\mathcal{T}$ is $M$-hypercyclic.*

**Definition 2.4.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$.*

*$J(T_t, M, x) = \{y \in X$ : for every relatively open neighborhoods $U, V$ of $x, y$ in $M$ respectively, and there exists $t \geq 0$ such that , $T_t(U) \cap V \neq \emptyset$ and $T_t(M) \subset M\}$, $J^{mix}(T_t, M, x) = \{y \in X$ : for every relatively open neighborhoods $U, V$ of $x, y$ in $M$ respectively, and every $t \geq t_0$, $T_t(U) \cap V \neq \emptyset$ and $T_t(M) \subset M$ for some $t_0 \geq 0\}$ denote the M-extended semigroup(resp M-extended semigroup mixing) limit set of x under $(T_t)_{t\geq 0}$.*

**Theorem 2.2.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. Then the following conditions are equivalent*

1. *$\mathcal{T}$ is $M$-mixing.*
2. *For every $x \in M$, $J^{mix}(\mathcal{T}, M, x) = M$.*

*Proof.* We prove at first that (1) implies (2). Let $x \in U$ and $y \in V$ and $U, V$ be nonempty relatively open subsets of $M$ . Then by [3, Theorem 2.1] there is $t_0 \geq 0$ such that for all $t \geq t_0, U \bigcap T_t(V)$ is non-empty and $T_t(M) \subset M$. Thus $y \in J^{mix}(\mathcal{T}, M, x)$, and consequently $J^{mix}(\mathcal{T}, M, x) = M$.

Now we prove that (2) $\Rightarrow$ (1). Let $U \subset M, V \subset M$, both nonempty and relatively open. We consider $x_0 \in U, y_0 \in V$, and since $J^{mix}(\mathcal{T}, M, x_0) = M$, then there exists $t_0 \geq 0$ such that for all $t \geq t_0$ $T_{t_0}^{-1}(V) \cap U \neq \emptyset$ and $T_{t_0}(M) \subset M$. Theorem 2.1 implies that $\mathcal{T}$ is $M$-mixing. □

**Theorem 2.3.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M$ be a nonzero subspace of $X$. Then the following conditions are equivalent*

1. *$\mathcal{T}$ is $M$-transitive.*
2. *For every $x \in M$, $J(\mathcal{T}, M, x) = M$.*

*Proof.* We first prove that (1) implies (2). Let $U$ and $V$ be nonempty relatively open subsets of M. Then there exists $t_0 \geq 0$ such that $T_{t_0}^{-1}(U) \cap V$ contains a nonempty open $W$ of $M$, it follows that $W \subset T_{t_0}^{-1}(U) \cap V$ and $T_{t_0}^{-1}(U) \cap V \neq \phi$.

Next, We prove that $T_{t_0}(M) \subset M$.

Let $x \in M$, we have $W \subset T_{t_0}^{-1}(U) \cap V$, this implies that$T_{t_0}(W) \subset U \subset M$. Let $x_0 \in W$, since $W$ is open of $M$ then for all $r$ enough small we have $x_0 + rx \in W$, therefore $T_{t_0}(x_0 + rx) = T_{t_0}x_0 + rT_{t_0}x \in T_{t_0}(W) \in M$. From $T_{t_0}x_0 \in M$ it follows that $T_{t_0}x \in M$.

We then conclude that $T_{t_0}(M) \subset M$. Therefore $J(\mathcal{T}, M, x) = M$, For every $x \in M$.

Now we prove that (2) $\Rightarrow$ (1). Let $U \subset M, V \subset M$, both nonempty and relatively open. We consider $x_0 \in U, y_0 \in V$, and since $J(\mathcal{T}, M, y_0) = M$, then there exists $t_0 \geq 0$ such that $T_{t_0}^{-1}(U) \cap V \neq \phi$ and $T_{t_0}(M) \subset M$.

Since $T_{t_0|M} : M \longrightarrow M$ is continuous, then $T_{t_0}^{-1}(U)$ is open in $M$, therefore $T_{t_0}^{-1}(U) \cap V$ is a relatively open nonempty subset of $M$. □

**Theorem 2.4.** *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup and $M-$mixing. Let $M$ be a nonzero subspace of $X$ and $T_t,\ t \geq 0$ be an invertible operators . Then $\mathcal{T}^{-1} = (T_t^{-1})_{t\geq 0}$ is M-hypercyclic.*

*Proof.* Let $x, y \in M$ and $U, V$ are relatively open subsets of $M$ such that contain $x, y$ respectively. Then by Theorem 2.2 and the invertibility of $\mathcal{T} = (T_t)_{t\geq 0}$ imply that there exists $t_0 \geq 0$ such that for all $t \geq t_0$,

$$U \cap T_t^{-1}(V) \neq \emptyset \textit{ and } T_t^{-1}(M) \subset M.$$

Hence for every $x \in M, x \in J^{mix}(T_t^{-1}, M, y)$ and consequently;

$$\forall y \in M, M = J^{mix}(T_t^{-1}, M, y)$$

or equivalently $\mathcal{T}^{-1}$ is M-mixing. by corollary 2.1 ,$\mathcal{T}^{-1}$ is $M-$hypercyclic. □

**Definition 2.5.** *Let $(T_n)_{n\geq 0} \subset \mathcal{B}(X)$ be a sequence of operators and $M$ be a nonzero subspace of $X$.*

$J(T_n, M, x) = \{y \in X :$ *there exist a strictly increasing sequence of positive integers $k_n$ and a sequence $x_n \subset X$ such that $x_n \to x$ and $T_{k_n}x_n \to y$ and for every $n, T_{k_n}(M) \subset M\}$*

$J(T_n, M, x) = \{y \in X :$ *for every relatively open neighborhoods $U, V$ of $x, y$ in $M$ respectively, and every positive integer $N \geq 0$, there exists $n > N$ such that , $T_n(U) \cap V \neq \emptyset$ and $T_n(M) \subset M\}$*

*$J^{mix}(T_n, M, x) = \{y \in X$ : for every relatively open neighborhoods $U, V$ of $x, y$ in $M$ respectively, and every positive integer $n > N$, $T_n(U) \cap V \neq \emptyset$ and $T_n(M) \subset M$ for some $N \in \mathbb{N}\}$*

*denote the M-extended sequence(resp M-extended sequence mixing) limit set of $x$ under $(T_n)_{n\in\mathbb{N}}$.*

**Proposition 2.1.** *An equivalent definition for the sets $J(T_n, M, x)$ is the following:*

*$J(T_n, M, x) = \{y \in X$ : there exist a strictly increasing sequence of positive integers $k_n$ and a sequence $x_n \subset X$ such that $x_n \to x$ and $T_{k_n} x_n \to y$ and for every $n, T_{k_n}(M) \subset M\}$*

*Proof.* Let $y \in J(T_n, M, x)$ and consider the open balls $U_n = B(x, \frac{1}{n}) \cap M, V_n = B(y, \frac{1}{n}) \cap M$ centered at $x, y \in X$ and radius $1/n$ for $n = 1, 2, ....$ and $N = k_{n-1}, k_0 = 1$. Then there exists $k_n > N = k_{n-1}$ such that

$$T_{k_n}(U_n) \cap V_n \neq \emptyset \text{ and } T_{k_n}(M) \subset M.$$

Hence there exists $x_n \in U_n$ such that $T_{k_n} x_n \in V_n$ and $T_{k_n}(M) \subseteq M$. Therefore $(k_n)$ is an strictly increasing sequence of positive integers and $(x_n)$ is a sequence in $M$ such that $x_n \to x$ and $T_{k_n} x_n \to y$ and for every $n, T_{k_n}(M) \subset M$. The converse is obvious. □

**Theorem 2.5.** *Let $(T_n)_{n\geq 1} \subset \mathcal{B}(X)$ be a sequence of operators and $M$ be a nonzero subspace of $X$. Then the following conditions are equivalent:*

1. *$(T_n)_{n\geq 1}$ is $M$-transitive( resp $M$-mixing) .*
2. *For every $x \in M, J(T_n, M, x) = M$(resp $J^{mix}(T_n, M, x) = M$).*

*Proof.* In the proof , we use a proofs method of the theorem 2.3(resp theorem 2.1) □

The next example will show that $(T_n)_{n\geq 0} \subset \mathcal{B}(X)$ is not hypercyclic but that $J^{mix}(T_n, M, x) = M$ for every $x \in M$, i.e $(T_n)_{n\geq 0}$ is $M-$mixing.

**Example 1.** Let $T = 2B$ such that $B$ is the backward shift on $l^2$, i.e. for every $x = (x_1, x_2, x_3, ...) \in l^2$;

$$T^n(x_1, x_2, x_3, ...) = 2^n(x_{n+1}, x_{n+2}, ...), n = 1, 2, ....$$

It is well known that $T_n = T^n \oplus I : l^2 \times l^2 \to l^2 \times l^2$ is $M-$hypercyclic where $M = l^2 \times \{0\}$, but $T_n = T^n \oplus I$ is not hypercyclic, see ([17], and [13]). Suppose that $U \subseteq l^2, V \subseteq l^2$, both relatively open, and $x = (x_1, x_2, x_3, ...) \in U, y = (y_1, y_2, y_3, ...) \in V$, Then $U \times \{0\}$ and $V \times \{0\}$ are two relatively open sets of $M$. Now for all $n \in \mathbb{N}^*$ set;

$$z_n = (x_1, x_2, ...x_{n-1}, \frac{y_1}{2^n}, \frac{y_2}{2^n}, ..., \frac{y_{n-1}}{2^n}, \frac{y_1}{2^{2n}}, \frac{y_2}{2^{2n}}, ..., \frac{y_{n-1}}{2^{2n}} ...),$$

then $z_n \to x$ and $T^n z_n \to y$. Hence there exists a non-negative integer $N$ such that;

$$\forall n \geq N, (T^n \oplus I)(U \times \{0\}) \cap (V \times \{0\}) \neq \emptyset.$$

Since $T_n(M) \subseteq M$, then by using a similar argument as in Theorem 2.5 we conclude that the operator $(T_n)_n = (T^n \oplus I)_n$ is $M-$mixing. □

The next example will show that $(T_n)_{n\geq 0}$ subspace-hypercyclicity does not imply $J^{mix}(T_n, M, x) = M$ , i.e $(T_n)_{n\geq 0}$ does not imply subspace-mixing with respect to $M$.

**Example 2.** Let $\lambda \in \mathbb{C}$ be of modulus greater than 1 and let $B$ be the backward shift on $l^2$. Let $m$ be a positive integer and $M$ be the subspace of $l^2$ consisting of all sequences with zero on the first $m$ entries, i.e.

$$M = \{\{a_n\}_{n=0}^{\infty} : a_n \ = \ 0 \ for \ n \leq m\},$$

then $T_n = \lambda^n B, n = 1, 2, ....$ is $M-$hypercyclic, see [13]. Now consider

$$V = \{\{a_n\}_{n=0}^{\infty} \in l^2 : a_n = 0 \ for \ n \leq m \ and \ |a_n| > 0 \, for \ n > m\},$$

so $V$ is relatively open subset of $M$. If $N = m+1$, then for every $n > N, T_n(V) \cap M = \emptyset$. Thus for every $x \in M, J^{mix}(T_n, M, x) \neq M$ and consequently the operator $(T_n)_{n\geq 1}$ is not a $M-$mixing. □

**Theorem 2.6.** *Let $T = (T_n)_n$ be a sequence of $\mathcal{B}(X)$ and $M-$mixing. Let $M$ be a nonzero subspace of $X$ and $T_n$, $n = 1, 2, ...$ , be an invertible operators. Then $T^{-1} = (T_n^{-1})_n$, is $M-$hypercyclic.*

*Proof.* Similar to theorem 2.4 □

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