A NEW FAMILY OF SECANT-LIKE METHOD WITH SUPER-LINEAR CONVERGENCE

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Abstract: Another new family of Secant-like method is proposed in this paper. Analysis of convergence shows that the method has a super-linear convergence as Secant method. Numerical experiments show that the efficiency of the method is depended on the value of its parameter.

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Key Words: secant-like method, secant method, order of convergence, iterative method

1. Introduction

In this study, we are concerned of finding an approximated solution of a non-linear equation of the form

$$f(x) = 0. \quad (1)$$

A classic known method without employing derivative to solve (1) is Secant method derived by passing a line through two point, says $(x_0, f(x_0))$ and $(x_1, f(x_1))$. The equation of this line is

$$p(x) = f(x_1) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1). \quad (2)$$

The intersection of (2) with x -axis is the first approximation to the root of

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(1). Repeating this proses, we obtain Secant iteration,

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n \geq 1. \quad (3)$$

Secant method satisfies the equation error

$$e_{n+1} = c_2 e_{n-1} e_n, \quad (4)$$

where $c_2 = \frac{f^{(2)}(x^*)}{2!f'(x^*)}$ and its order is $(1 + \sqrt{5})/2$ (super linear) [3]. Many works have been done developing new Secant-like methods, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] for example.

Recently V. Kanwar, J.R. Sharma, and Mamta [10] introduces a variation of Secant method, derived by creating a parabola passing a point $(x_1, 0)$. The parabola's equation is

$$y = \alpha(x - x_1)^2, \quad (5)$$

where α is a parameter. Parabola (5) intercepts $p(x)$ at $(x_2, p(x_2))$, so that

$$p(x_2) = f(x_1) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1), \quad (6)$$

and

$$p(x_2) = \alpha(x_2 - x_1)^2. \quad (7)$$

From (6) and (7), we obtain

$$\alpha(x_2 - x_1)^2 - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1) - f(x_1) = 0. \quad (8)$$

Solving the quadratic equation (8) yields

$$x_2 = x_1 - \frac{2(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0) \pm \sqrt{(f(x_1) - f(x_0))^2 + 4\alpha f(x_1)(x_1 - x_0)^2}}. \quad (9)$$

Equation (9) suggests the iteration formula

$$x_{n+1} = x_n - \frac{2(x_n - x_{n-1})f(x_n)}{K + \text{sign}(K)\sqrt{K^2 + 4\alpha f(x_n)(x_n - x_{n-1})^2}}, \quad (10)$$

where

$$K = f(x_n) - f(x_{n-1}),$$

α is a parameter and

$$\text{sign}(K) = \begin{cases} 1 & , \text{ untuk } K \geq 0, \\ -1 & , \text{ untuk } K < 0, \end{cases}$$

for $n = 1, 2, 3, \dots$. Kanwar shows that the convergence of iterative method (10) is the same as Secant method.

Chen[4] rearranges (8) as

$$\left(\alpha(x_2 - x_1) - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x_2 - x_1) - f(x_1) = 0. \quad (11)$$

Equation (11) can be expressed as

$$x_2 - x_1 = \frac{f(x_1)(x_1 - x_0)}{\alpha(x_2 - x_1)(x_1 - x_0) - (f(x_1) - f(x_0))}. \quad (12)$$

By approximating x_2 in the right hand side of (12) with Secant method and simplifying, Chen obtains a super linearly convergent iterative method

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})(f(x_n) - f(x_{n-1}))}{(f(x_n) - f(x_{n-1}))^2 + \alpha f(x_n)(x_n - x_{n-1})^2}. \quad (13)$$

In this article we discuss another variation of Secant method and its convergent analysis in Section 2. Then, in Section 3 we compare the proposed method with the methods given in (10) and (13) using some test functions.

2. Proposed Method

Dividing both side equation (8) with $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$ yields

$$\alpha(x_2 - x_1)^2 \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right) - (x_2 - x_1) - f(x_1) \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right) = 0. \quad (14)$$

Rearranging the terms on equation (14), we obtain

$$x_2 - x_1 = \alpha(x_2 - x_1)^2 \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right) - f(x_1) \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right). \quad (15)$$

By approximating x_2 in the right-hand side equation (15) with Secant method yields

$$x_2 = x_1 - f(x_1) \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right) + \alpha f^2(x_1) \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right)^3. \quad (16)$$

From (16), we propose the iterative formula

$$x_{n+1} = x_n - f(x_n)L(1 - \alpha f(x_n)L^2), \quad (17)$$

where

$$L = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}. \quad (18)$$

If we take $\alpha = 0$, then equation (17) becomes Secant method.

Theorem 1. Assume that the function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D has a simple root $x^* \in D$ that is $f'(x^*) \neq 0$. Let $f(x)$ have first, and second derivatives in the interval D , then the order of convergence of the method defined by (17) is $(1 + \sqrt{5})/2$.

Proof. Let $e_n = x_n - x^*$, then

$$L = \frac{e_n - e_{n-1}}{f(x_n) - f(x_{n-1})}. \quad (19)$$

Taylor expansion of $f(x_n)$ about $x_n = x^*$ is given by

$$f(x_n) = f'(x^*)(e_n + c_2 e_n^2) + \cdots, \quad (20)$$

where $c_j = \frac{f^{(j)}(x^*)}{j!f'(x^*)}$. Similarly, we obtain

$$f(x_{n-1}) = f'(x^*)(e_{n-1} + c_2 e_{n-1}^2) + \cdots. \quad (21)$$

Subtracting (21) from (20) yields

$$f(x_n) - f(x_{n-1}) = f'(x^*)((e_n - e_{n-1}) + c_2(e_n^2 - e_{n-1}^2) + \cdots). \quad (22)$$

Since e_n is small enough, then substituting (22) into (19) we obtain

$$L = \frac{1}{f'(x^*)(1 + c_2(e_n + e_{n-1}))}. \quad (23)$$

Using geometry series, (23) can be written as

$$L = \frac{1}{f'(x^*)} (1 - c_2 e_n - c_2 e_{n-1} + c_2^2 e_n^2 + 2c_2^2 e_n e_{n-1} + c_2^2 e_{n-1}^2). \quad (24)$$

By ignoring the terms containing $e_n^j e_{n-1}^k$, with $k + j > 2$ yields

$$L^2 = \frac{1}{(f'(x^*))^2} (1 + 6c_2^2 e_n e_{n-1} - 2c_2 e_n - 2c_2 e_{n-1} + 3c_2^2 e_n^2 + 3c_2^2 e_{n-1}^2). \quad (25)$$

Then, using (21), (24), dan (25) we have

$$f(x_n)L(1 - \alpha f(x_n)L^2) = e_n - c_2 e_n e_{n-1} - \alpha \frac{e_n^2}{f'(x^*)}. \quad (26)$$

On substituting (26) into (17), and recalling $e_{n+1} = x_{n+1} - x^*$, we get

$$e_{n+1} = c_2 e_{n-1} e_n + \alpha \frac{e_n^2}{f'(x^*)}. \quad (27)$$

Equation (27) is the same as equation (4), so the order of convergence the iterative method (17) is $(1 + \sqrt{5})/2$. $\square$

3. Numerical Experiments

In this section we compare the proposed method (17) with the methods introduced by Kanwar (10) and Chen (13) using the following test functions

$$\begin{aligned} f_1(x) &= \log(x), & x^* &= 1.0, \\ f_2(x) &= x^3 - 2x - 5, & x^* &= 2.094551481542327, \\ f_3(x) &= x \exp(x) - 1, & x^* &= 0.5671432904097839. \\ f_4(x) &= x \exp(x) - \cos(x), & x^* &= 0.5177573636824583. \end{aligned}$$

We use different initial guesses and stop the iteration proseses by the following condition

- (a) Maximum number of iteration=100,
- (b) If $|f(x_{n+1})| < 1.0 \times 10^{-14}$,
- (c) If $|x_{n+1} - x_n| < 1.0 \times 10^{-14}$.

The results are shown in Table 1. In this table, NC indicates that the methods are not convergent or exceed the number iterations allowed or blow up.

From computational experiments, we can see that the proposed method (PM) is comparable with Kanwar's method (KM) and Chen's method (CM), especially for small enough α . Hence, we can conclude that the proposed method can be used as an alternative method for Secant-like methods.

Table 1: The number iterations needed to obtain the solution of $f(x) = 0$ by varying parameter α

Function	x_0	x_1	Method					
				10	1	0.1	0.01	0.001
$\log(x)$	3	4	KM	18	9	8	14	17
			CM	84	15	8	13	17
			PM	NC	CN	NC	14	17
	0.5	2	KM	11	7	8	7	7
			CM	22	8	8	7	7
			PM	NC	NC	7	7	7
	2	3	KM	15	8	7	9	9
			CM	49	11	8	9	9
			PM	NC	NC	8	9	9
Function	x_0	x_1	Method	10	1	0.1	0.01	0.001
$x^3 - 2x - 5$	2	3	KM	8	6	6	6	6
			CM	8	6	6	6	6
			PM	8	6	6	6	6
	-3	2	KM	7	6	6	6	6
			CM	7	6	6	6	6
			PM	7	6	6	6	6
	1	4	KM	8	9	10	10	10
			CM	9	9	10	10	10
			PM	12	9	10	10	10
$x \exp(x) - 1$	1	2	KM	11	9	8	8	8
			CM	11	9	8	8	8
			PM	13	9	8	8	8
	-1	3	KM	11	14	NC	NC	NC
			CM	13	NC	NC	NC	NC
			PM	17	9	NC	NC	NC
	-3	1	KM	7	9	15	17	18
			CM	9	7	13	17	18
			PM	NC	10	11	17	18
$x \exp(x) - \cos(x)$	2	3	KM	12	11	11	11	11
			CM	13	11	11	11	11
			PM	13	11	11	11	11
	0	2	KM	10	7	9	9	9
			CM	11	7	9	9	9
			PM	15	7	9	9	9
	-1	1	KM	7	8	11	11	11
			CM	9	8	11	11	11
			PM	12	8	10	11	11

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