ON A SEQUENCE OF \((i, \pm 1, \pm i)\)-TRIDIAGONAL MATRICES, WHOSE DETERMINANTS ARE RELATED TO FIBONACCI NUMBERS

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Abstract: We will generalize a previous result on connection sequence of special tridiagonal matrices to Fibonacci numbers, as we find a new sequence of tridiagonal matrices which are related to Fibonacci numbers.

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1. Introduction

The Fibonacci sequence (or sequence of Fibonacci numbers) \((F_n)_{n \geq 0}\) is the sequence of positive integers satisfying the recurrence \(F_{n+2} = F_{n+1} + F_n\) with the initial conditions \(F_0 = 0\) and \(F_1 = 1\).
Fibonacci numbers turn up in a fantastic variety of interesting applications (see e. g. book [7]), but this paper deals with its connections to determinants of tridiagonal matrices only.

In 1976, Strang [11] included, probably the first example of determinant of \( n \times n \) matrix, which is equal to the Fibonacci number, as he showed that the following holds

\[
\begin{vmatrix}
1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & \ddots & \ddots & \vdots \\
0 & \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 \\
\end{vmatrix} = F_{n+1} \tag{1}
\]

for any \( n \geq 1 \). In 2003, Cahill et al. [1] showed that the following holds

\[
\begin{vmatrix}
1 & i & 0 & \cdots & \cdots & 0 \\
0 & i & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \vdots & \ddots & \ddots & \ddots & i & 0 \\
\vdots & \ddots & \ddots & \ddots & i & 1 & i \\
0 & \cdots & \cdots & 0 & i & 1 \\
\end{vmatrix} = F_{n+1} \tag{2}
\]

for any \( n \geq 1 \) (where \( i = \sqrt{-1} \)). Matrices in (1) and (2) are the special cases of a tridiagonal matrix, what is a square matrix \( A = (a_{jk}) \) of the order \( n \), with entries \( a_{jk} = 0 \) for \( |k - j| > 1 \) and \( 1 \leq j, k \leq n \), i. e.

\[
A(n) = \begin{pmatrix}
a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\
a_{2,1} & a_{2,2} & a_{2,3} & \ddots & \vdots \\
0 & a_{3,2} & a_{3,3} & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \\
\end{pmatrix}.
\]

Many authors derived the similar types of matrices which determinants or permanents are related to Fibonacci numbers or different kinds of their generalizations, e. g. \( k \)-generalized Fibonacci numbers, see [2], [3], [4], [5], [6], [8],
[9], [10], [13], [14], [15] and [16]. Now we turn our attention to the relation of determinants of special tridiagonal matrices with Fibonacci numbers. We show that matrix in (2) can be changed into a matrix, whose determinant is related to Fibonacci numbers too.

2. Preliminary results

Ferguson [12] formulated the following lemma, which can be easily used for finding the recurrence relation for determinants of a sequence of tridiagonal matrices.

**Lemma 1.** (Lemma B1 (a) of [12]) We consider tridiagonal $n \times n$ matrices of the following form

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & c_{n-1} & a_n \end{pmatrix}.$$ 

Let $\det A_n$ denote the determinant of $A_n$. Then

$$\det A_1 = a_1; \quad \det A_2 = a_1a_2 - b_1c_1; \quad \det A_n = a_n \det A_{n-1} - b_{n-1}c_{n-1}A_{n-2}. \quad (3)$$

3. Main result

**Theorem 2.** Let $\{\mathbb{C}^{\alpha,\beta}(n), n = 1, 2, 3, \ldots \} \cap \alpha, \beta \in \{0, 1\}$ be a sequence of tridiagonal matrices in the form

$$c^{\alpha,\beta}_{jk} = \begin{cases} \ (-1)^{j+\beta}i, & j = k - 1; \\ \ (-1)^{j+\alpha}, & j = k; \\ \ i, & j = k + 1; \\ \ 0, & \text{otherwise}, \end{cases}$$
\[ C_{\alpha, \beta}(n) = \begin{pmatrix} (-1)^{1+\alpha} & (-1)^{1+\beta} i & 0 & \cdots & 0 \\ i & (-1)^{2+\alpha} & (-1)^{2+\beta} i & \ddots & \vdots \\ 0 & i & (-1)^{3+\alpha} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (-1)^{n-1+\beta} i \\ 0 & \cdots & 0 & i & (-1)^{n+\alpha} \end{pmatrix}. \]

Then

\[ \det C_{\alpha, \beta}(n) = \begin{cases} (-1)^{\frac{n}{2}} F_{\frac{n}{2}+2-3\beta}, & 2 \mid n; \\ (-1)^{\frac{n+1}{2}} F_{\frac{n+1}{2}}, & 2 \nmid n. \end{cases} \tag{4} \]

**Proof.** There are four cases to consider, however the proof of them follows by the same approach. Then, in order to avoid unnecessary repetitions, we shall prove only the case \( \alpha = \beta = 0 \). Thus we consider matrix

\[ \det C_{0,0}(n) = \begin{pmatrix} (-1)^1 & (-1)^1 i & 0 & \cdots & 0 \\ i & (-1)^2 & (-1)^2 i & \ddots & \vdots \\ 0 & i & (-1)^3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (-1)^{n-1} i \\ 0 & \cdots & 0 & i & (-1)^n \end{pmatrix}. \]

For simplicity of notation, we write \( D(n) \) instead of \( \det C_{0,0}(n) \). Using Lemma 1 we obtain \( D(1) = -1, \ D(2) = -2 \) and for \( n > 2 \) the following recurrence

\[ D(n) = (-1)^n D(n-1) + (-1)^{n-1} D(n-2). \tag{5} \]

We use mathematical induction on \( n \). For \( n = 1 \) and \( n = 2 \) we have \( D(1) = -1 = (-1)^{\frac{1+1}{2}} F_{\frac{1+1}{2}} = -F_1 \) and \( D(2) = -2 = (-1)^{\frac{2}{2}} F_{\frac{2+2}{2}} = -F_3 \), hence relation (4) holds. Suppose that the assertion holds for every \( k, \ 3 \leq k < n \). Then we have to show that the assertion is true for \( n \) too. We use identity (5) in the following two cases

(i) Let \( 2 \mid n \). Then \( 2 \nmid (n-1) \) and \( 2 \mid (n-2) \). We have

\[
D(n) = D(n-1) - D(n-2) \\
= (-1)^{\frac{n}{2}} F_{\frac{n}{2}} - (-1)^{\frac{n}{2}-1} F_{\frac{n-1}{2}+1} \\
= (-1)^{\frac{n}{2}} F_{\frac{n}{2}} + (-1)^{\frac{n}{2}} F_{\frac{n-1}{2}+1} \\
= (-1)^{\frac{n}{2}} (F_{\frac{n}{2}} + F_{\frac{n-1}{2}+1}) = (-1)^{\frac{n}{2}} F_{\frac{n+2}{2}}.
\]
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(ii) Let \(2 \nmid n\). Then \(2 \mid (n - 1)\) and \(2 \nmid (n - 2)\) and we get

\[
D(n) = -D(n-1) + D(n-2) = -(-1)^{\frac{n-1}{2}}F_{\frac{n-1}{2}+1} + (-1)^{\frac{n-1}{2}}F_{\frac{n-1}{2}} = (-1)^{\frac{n+1}{2}}F_{\frac{n+1}{2}+1} = (-1)^{\frac{n+1}{2}}F_{\frac{n+1}{2}}.
\]

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\square
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References


