**ON NEW SUBCLASSES
OF BI-UNIVALENT FUNCTIONS ASSOCIATED
WITH AL-OBOUDI DIFFERENTIAL OPERATOR**

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Abstract: In the present paper we introduce two new subclasses of the class of bi-univalent functions defined on the open unit disk $\mathbb{U}$, which are associated with the Al-Oboudi differential operator. Also we obtain the estimates of the coefficients $|a_2|$ and $|a_3|$ for the functions of these classes. Relevant connections with the various well known results are indicated.

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1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the open unit disk,

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$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}.$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of the functions which are also univalent in $\mathbb{U}$ (for details, see [8]). It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where,

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \end{aligned} \quad (2)$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$. The study of coefficient problems involving bi-univalent functions was revived recently by Srivastava et. al. [17]. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates of $|a_2|$ and $|a_3|$ of the functions in these subclasses were found in some recent investigations (see [5, 6, 7, 9, 11, 14, 16] etc.). For the brief history and examples of functions in the class Σ , see [17].

For functions in the class Σ , Lewin [10] proved that $|a_2| < 1.51$, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$ and Netanyahu [12] proved that $\max_{f \in \Sigma} |a_2| = 4/3$. However the coefficient estimate problem for each $|a_n|$, ($n = 3, 4, \dots$) is still an open problem.

Brannan and Taha [4], (see also [3, 18]) introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor - Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

Definition 1. [4] A function $f(z)$ given by (1) is said to be in the class $\mathcal{S}_\Sigma^*[\alpha]$ where $0 < \alpha \leq 1$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{U})$$

and

$$\left| \arg \left\{ \frac{wg'(w)}{g(w)} \right\} \right| < \frac{\alpha\pi}{2}, \quad (w \in \mathbb{U})$$

where the function g is defined by (2).

The subclass $\mathcal{S}_{\Sigma}^*[\alpha]$ is known as the class of strongly bi-starlike functions of order α .

Definition 2. [4] A function $f(z)$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^*(\beta)$ where $0 \leq \beta < 1$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad (z \in \mathbb{U})$$

and

$$\Re \left\{ \frac{wg'(w)}{g(w)} \right\} > \beta, \quad (w \in \mathbb{U})$$

where the function g is defined by (2).

The subclass $\mathcal{S}_{\Sigma}^*(\beta)$ is known as the class of bi-starlike functions of order β .

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator :

$$D_{\delta}^0 f(z) = f(z)$$

$$D_{\delta}^1 f(z) = (1 - \delta)f(z) + \delta zf'(z) = D_{\delta} f(z); \quad (\delta \geq 0) \quad (3)$$

$$D_{\delta}^n f(z) = D_{\delta}(D_{\delta}^{n-1} f(z)); \quad (n \in \mathbb{N}) \quad (4)$$

If f is given by (1) then from (3) and (4) we see that,

$$D_{\delta}^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k; \quad (n \in \mathbb{N}_0)$$

with $D_{\delta}^n f(0) = 0$ and when $\delta = 1$, we get the Sălăgean's differential operator $D_1^n = D^n$ [15].

The object of the present paper is to introduce two new subclasses of the function class Σ defined by using Al-Oboudi differential operator and to find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Firstly, in order to prove our main results, we need the following lemma [13].

Lemma 3. [13] If $p(z) \in \mathcal{P}$, the class of functions analytic in $\mathbb{U}$ with

$$\Re(p(z)) > 0,$$

then $|c_n| \leq 2$ for each $n \in \mathbb{N}$, where

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (z \in \mathbb{U}).$$

2. Coefficient Bounds for the Function Class $\mathcal{B}_\Sigma(\delta, n, \alpha, \lambda)$

Definition 4. A function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\delta, n, \alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left| \arg \left\{ \frac{(1-\lambda)D_\delta^n f(z) + \lambda D_\delta^{n+1} f(z)}{z} \right\} \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U}) \quad (5)$$

and

$$\left| \arg \left\{ \frac{(1-\lambda)D_\delta^n g(w) + \lambda D_\delta^{n+1} g(w)}{w} \right\} \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U}) \quad (6)$$

where the function g is given by (2).

We note that for $\delta = 1$ the class $\mathcal{B}_\Sigma(\delta, n, \alpha, \lambda)$ reduces to the class $\mathcal{B}_\Sigma(n, \alpha, \lambda)$ introduced by S. Porwal and M. Darus [14] and for $\delta = 1, n = 0, \lambda = 1$ the class $\mathcal{B}_\Sigma(\delta, n, \alpha, \lambda)$ reduces to the class $\mathcal{H}_\Sigma^\alpha$ introduced by Srivastava et.al. [17]. Also for $\delta = 1, n = 0$ this class reduces to the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ introduced by B.A. Frasin and M.K. Aouf [9].

Theorem 5. Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma(\delta, n, \alpha, \lambda)$. Then,

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(1+2\delta)^n(1+2\lambda\delta) - (\alpha-1)(1+\delta)^{2n}(1+\lambda\delta)^2}} \quad (7)$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+\delta)^{2n}(1+\lambda\delta)^2} + \frac{2\alpha}{(1+2\delta)^n(1+2\lambda\delta)}. \quad (8)$$

Proof. From (5) and (6), we have

$$\frac{(1 - \lambda)D_{\delta}^n f(z) + \lambda D_{\delta}^{n+1} f(z)}{z} = [p(z)]^{\alpha} \quad (9)$$

and

$$\frac{(1 - \lambda)D_{\delta}^n g(w) + \lambda D_{\delta}^{n+1} g(w)}{w} = [q(w)]^{\alpha} \quad (10)$$

where $p(z), q(w) \in \mathcal{P}$ with

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (z \in \mathbb{U}) \quad (11)$$

and

$$q(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots, \quad (w \in \mathbb{U}). \quad (12)$$

Equating the coefficients in (9) and (10), we get

$$(1 + \delta)^n (1 + \lambda \delta) a_2 = \alpha c_1 \quad (13)$$

$$(1 + 2\delta)^n (1 + 2\lambda \delta) a_3 = \alpha c_2 + \frac{\alpha(\alpha - 1)}{2} c_1^2 \quad (14)$$

$$-(1 + \delta)^n (1 + \lambda \delta) a_2 = \alpha d_1 \quad (15)$$

$$(1 + 2\delta)^n (1 + 2\lambda \delta) (2a_2^2 - a_3) = \alpha d_2 + \frac{\alpha(\alpha - 1)}{2} d_1^2 \quad (16)$$

From (13) and (15), we obtain

$$c_1 = -d_1 \quad (17)$$

and

$$2(1 + \delta)^{2n} (1 + \lambda \delta)^2 a_2^2 = \alpha^2 (c_1^2 + d_1^2) \quad (18)$$

Adding (14) and (16) and then using (18), we obtain

$$a_2^2 = \frac{\alpha^2 (c_2 + d_2)}{2\alpha(1 + 2\delta)^n (1 + 2\lambda \delta) - (\alpha - 1)(1 + \delta)^{2n} (1 + \lambda \delta)^2}$$

Using Lemma 3, we get

$$a_2^2 \leq \frac{4\alpha^2}{2\alpha(1 + 2\delta)^n (1 + 2\lambda \delta) - (\alpha - 1)(1 + \delta)^{2n} (1 + \lambda \delta)^2}$$

which gives the desired result (7).

Next, subtracting (16) from (14) and then using (17), we obtain

$$a_3 - a_2^2 = \frac{\alpha(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

By (18), we get

$$a_3 = \frac{\alpha^2(c_1^2 + d_1^2)}{2(1 + \delta)^{2n}(1 + \lambda\delta)^2} + \frac{\alpha(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

Using Lemma 3, we get

$$|a_3| \leq \frac{4\alpha^2}{(1 + \delta)^{2n}(1 + \lambda\delta)^2} + \frac{2\alpha}{(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

which is the desired result (8).

This completes the proof of Theorem 5. $\square$

3. Coefficient Bounds for the Function Class $\mathcal{H}_\Sigma(\delta, n, \beta, \lambda)$

Definition 6. A function $f(z)$ given by (1) is said to be in the class $\mathcal{H}_\Sigma(\delta, n, \beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \Re \left\{ \frac{(1 - \lambda)D_\delta^n f(z) + \lambda D_\delta^{n+1} f(z)}{z} \right\} > \beta$$

$$(0 \leq \beta < 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U}) \quad (19)$$

and

$$\Re \left\{ \frac{(1 - \lambda)D_\delta^n g(w) + \lambda D_\delta^{n+1} g(w)}{w} \right\} > \beta$$

$$(0 \leq \beta < 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U}), \quad (20)$$

where the function g is given by (2).

We note that for $\delta = 1$ the class $\mathcal{H}_\Sigma(\delta, n, \beta, \lambda)$ reduces to the class $\mathcal{H}_\Sigma(n, \beta, \lambda)$ introduced by S. Porwal and M. Darus [14] and for $\delta = 1, n = 0$ it reduces to the class $\mathcal{H}_\Sigma(\beta, \lambda)$ introduced by B.A. Frasin and M.K. Aouf [9]. Also for $\delta = 1, n = 0, \lambda = 1$ this class reduces to the class $\mathcal{H}_\Sigma(\lambda)$ introduced by Srivastava et. al. [17].

Theorem 7. Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_\Sigma(\delta, n, \beta, \lambda)$. Then,

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(1+2\delta)^n(1+2\lambda\delta)}} \quad (21)$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\delta)^{2n}(1+\lambda\delta)^2} + \frac{2(1-\beta)}{(1+2\delta)^n(1+2\lambda\delta)}. \quad (22)$$

Proof. Eqns. (19) and (20) implies that there exists $p(z), q(w) \in \mathcal{P}$ such that

$$\frac{(1-\lambda)D_\delta^n f(z) + \lambda D_\delta^{n+1} f(z)}{z} = \beta + (1-\beta)p(z) \quad (23)$$

and

$$\frac{(1-\lambda)D_\delta^n g(w) + \lambda D_\delta^{n+1} g(w)}{w} = \beta + (1-\beta)q(w) \quad (24)$$

where $p(z)$ and $q(w)$ are given by (11) and (12) respectively. Equating the coefficients in (23) and (24), we get

$$(1+\delta)^n(1+\lambda\delta)a_2 = (1-\beta)c_1 \quad (25)$$

$$(1+2\delta)^n(1+2\lambda\delta)a_3 = (1-\beta)c_2 \quad (26)$$

$$-(1+\delta)^n(1+\lambda\delta)a_2 = (1-\beta)d_1 \quad (27)$$

$$(1+2\delta)^n(1+2\lambda\delta)(2a_2^2 - a_3) = (1-\beta)d_2 \quad (28)$$

Using (25) and (27), we obtain

$$c_1 = -d_1 \quad (29)$$

and

$$2(1+\delta)^{2n}(1+\lambda\delta)^2 a_2^2 = (1-\beta)^2(c_1^2 + d_1^2) \quad (30)$$

Adding (26) and (28), we obtain

$$2(1+2\delta)^n(1+2\lambda\delta)a_2^2 = (1-\beta)(c_2 + d_2)$$

Or

$$a_2^2 = \frac{(1-\beta)(c_2 + d_2)}{2(1+2\delta)^n(1+2\lambda\delta)}$$

Using Lemma 3, we get

$$|a_2^2| \leq \frac{2(1-\beta)}{(1+2\delta)^n(1+2\lambda\delta)}$$

which gives the desired result (21).

Next, for the estimates of $|a_3|$, subtracting (28) from (26), we obtain

$$2(1 + 2\delta)^n(1 + 2\lambda\delta)(a_3 - a_2^2) = (1 - \beta)(c_2 - d_2)$$

Or

$$a_3 = a_2^2 + \frac{(1 - \beta)(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

Using (30), we obtain

$$a_3 = \frac{(1 - \beta)^2(c_1^2 + d_1^2)}{2(1 + \delta)^{2n}(1 + \lambda\delta)^2} + \frac{(1 - \beta)(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

Using Lemma 3, we get

$$|a_3| \leq \frac{4(1 - \beta)^2}{(1 + \delta)^{2n}(1 + \lambda\delta)^2} + \frac{2(1 - \beta)}{(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

which is the desired result (22).

This completes the proof of Theorem 7. □

4. Remarks

Remark 8. If we put $\delta = 1$ in Theorem 5 and Theorem 7, we obtain the corresponding results due to S. Porwal and M. Darus [14].

Remark 9. If we put $\delta = 1$ and $n = 0$ in Theorem 5 and Theorem 7, we obtain the corresponding results due to B.A. Frasin and M.K. Aouf [9].

Remark 10. If we put $\delta = 1, n = 0$ and $\lambda = 1$ in Theorem 5 and Theorem 7, we obtain the corresponding results due to Srivastava et. al. [17].

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