ON NEW SUBCLASSES
OF BI-UNIVALENT FUNCTIONS ASSOCIATED
WITH AL-OBBOUDI DIFFERENTIAL OPERATOR

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Abstract: In the present paper we introduce two new subclasses of the class of bi-univalent functions defined on the open unit disk \( \mathbb{U} \), which are associated with the Al-Oboudi differential operator. Also we obtain the estimates of the coefficients \( |a_2| \) and \( |a_3| \) for the functions of these classes. Relevant connections with the various well known results are indicated.

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1. Introduction

Let \( \mathcal{A} \) denote the class of all functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
which are analytic in the open unit disk,
\[ \mathbb{U} = \{ z : z \in \mathbb{C}, |z| < 1 \} . \]

Let \( \mathcal{S} \) denote the subclass of \( \mathcal{A} \) consisting of the functions which are also univalent in \( \mathbb{U} \) (for details, see [8]). It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), defined by
\[
f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})
\]
and
\[
f(f^{-1}(w)) = w, \quad \left( |w| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right),
\]
where,
\[
g(w) = f^{-1}(w)
\]
\[
= w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{2}
\]

A function \( f(z) \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). We denote by \( \Sigma \) the class of all bi-univalent functions in \( \mathbb{U} \). The study of coefficient problems involving bi-univalent functions was revived recently by Srivastava et. al. [17]. Various subclasses of the bi-univalent function class \( \Sigma \) were introduced and non-sharp estimates of \( |a_2| \) and \( |a_3| \) of the functions in these subclasses were found in some recent investigations (see [5, 6, 7, 9, 11, 14, 16] etc.). For the brief history and examples of functions in the class \( \Sigma \), see [17].

For functions in the class \( \Sigma \), Lewin [10] proved that \( |a_2| < 1.51 \), Brannan and Clunie [2] conjectured that \( |a_2| \leq 2 \) and Netanyahu [12] proved that \( \max_{f \in \Sigma}|a_2| = 4/3 \). However the coefficient estimate problem for each \( |a_n|, \ (n = 3, 4, \ldots) \) is still an open problem.

Brannan and Taha [4], (see also [3, 18]) introduced the following two subclasses of the bi-univalent function class \( \Sigma \) and obtained non-sharp estimates on the first two Taylor–Maclaurin coefficients \( |a_2| \) and \( |a_3| \) of functions in each of these subclasses.

**Definition 1.** [4] A function \( f(z) \) given by (1) is said to be in the class \( \mathcal{S}_{\Sigma}^*[\alpha] \) where \( 0 < \alpha \leq 1 \) if the following conditions are satisfied:
\[
f \in \Sigma, \quad \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\alpha \pi}{2}, \quad (z \in \mathbb{U})
\]
and
\[ \left| \arg \left\{ \frac{w g'(w)}{g(w)} \right\} \right| < \frac{\alpha \pi}{2}, \quad (w \in \mathbb{U}) \]
where the function \( g \) is defined by (2).

The subclass \( S^*_\Sigma [\alpha] \) is known as the class of strongly bi-starlike functions of order \( \alpha \).

**Definition 2.** [4] A function \( f(z) \) given by (1) is said to be in the class \( S^*_\Sigma (\beta) \) where \( 0 \leq \beta < 1 \) if the following conditions are satisfied:
\[ f \in \Sigma, \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad (z \in \mathbb{U}) \]
and
\[ \Re \left\{ \frac{w g'(w)}{g(w)} \right\} > \beta, \quad (w \in \mathbb{U}) \]
where the function \( g \) is defined by (2).

The subclass \( S^*_\Sigma (\beta) \) is known as the class of bi-starlike functions of order \( \beta \).

For \( f \in \mathcal{A} \), Al-Oboudi [1] introduced the following operator :
\[ D_0^\delta f(z) = f(z) \]
\[ D_\delta^1 f(z) = (1 - \delta)f(z) + \delta zf'(z) = D_\delta f(z); \quad (\delta \geq 0) \] (3)
\[ D_\delta^n f(z) = D_\delta (D_\delta^{n-1} f(z)); \quad (n \in \mathbb{N}) \] (4)
If \( f \) is given by (1) then from (3) and (4) we see that,
\[ D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k - 1)\delta]^n a_k z^k; \quad (n \in \mathbb{N}_0) \]
with \( D_\delta^n f(0) = 0 \) and when \( \delta = 1 \), we get the Sălăgean’s differential operator \( D_1^n = D^n \) [15].

The object of the present paper is to introduce two new subclasses of the function class \( \Sigma \) defined by using Al-Oboudi differential operator and to find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in these new subclasses. Firstly, in order to prove our main results, we need the following lemma [13].
Lemma 3. [13] If \( p(z) \in \mathcal{P} \), the class of functions analytic in \( \mathbb{U} \) with
\[
\Re(p(z)) > 0,
\]
then \( |c_n| \leq 2 \) for each \( n \in \mathbb{N} \), where
\[
p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots, \quad (z \in \mathbb{U}).
\]

2. Coefficient Bounds for the Function Class \( B_\Sigma (\delta, n, \alpha, \lambda) \)

Definition 4. A function \( f(z) \) given by (1) is said to be in the class \( B_\Sigma (\delta, n, \alpha, \lambda) \) if the following conditions are satisfied:
\[
f \in \Sigma, \quad \left| \arg \left\{ \frac{(1 - \lambda)D_\delta^n f(z) + \lambda D_\delta^{n+1} f(z)}{z} \right\} \right| < \frac{\alpha \pi}{2}
\]
(0 \( < \alpha \leq 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U} \) (5)
and
\[
\left| \arg \left\{ \frac{(1 - \lambda)D_\delta^n g(w) + \lambda D_\delta^{n+1} g(w)}{w} \right\} \right| < \frac{\alpha \pi}{2}
\]
(0 \( < \alpha \leq 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U} \) (6)
where the function \( g \) is given by (2).

We note that for \( \delta = 1 \) the class \( B_\Sigma (\delta, n, \alpha, \lambda) \) reduces to the class \( B_\Sigma (n, \alpha, \lambda) \) introduced by S. Porwal and M. Darus [14] and for \( \delta = 1, n = 0, \lambda = 1 \) the class \( B_\Sigma (\delta, n, \alpha, \lambda) \) reduces to the class \( H_\alpha^\lambda \) introduced by Srivastava et.al. [17]. Also for \( \delta = 1, n = 0 \) this class reduces to the class \( B_\Sigma (\alpha, \lambda) \) introduced by B.A. Frasin and M.K. Aouf [9].

Theorem 5. Let the function \( f(z) \) given by (1) be in the class \( B_\Sigma (\delta, n, \alpha, \lambda) \). Then,
\[
|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(1 + 2\delta)^n(1 + 2\lambda\delta) - (\alpha - 1)(1 + \delta)^{2n}(1 + \lambda\delta)^2}}
\]
(7)
and
\[
|a_3| \leq \frac{4\alpha^2}{(1 + \delta)^{2n}(1 + \lambda\delta)^2} + \frac{2\alpha}{(1 + 2\delta)^n(1 + 2\lambda\delta)}.
\]
(8)
Proof. From (5) and (6), we have
\[
\frac{(1 - \lambda)D_\delta^n f(z) + \lambda D_\delta^{n+1} f(z)}{z} = [p(z)]^\alpha
\]
and
\[
\frac{(1 - \lambda)D_\delta^n g(w) + \lambda D_\delta^{n+1} g(w)}{w} = [q(w)]^\alpha
\]
where \( p(z), q(w) \in P \) with
\[
p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots , \ (z \in \mathbb{U})
\]
and
\[
q(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots , \ (w \in \mathbb{U}).
\]
Equating the coefficients in (9) and (10), we get
\[
(1 + \delta)^n(1 + \lambda \delta)a_2 = \alpha c_1
\]
\[
(1 + 2\delta)^n(1 + 2\lambda \delta)a_3 = \alpha c_2 + \frac{\alpha(\alpha - 1)}{2}c_1^2
\]
\[
-(1 + \delta)^n(1 + \lambda \delta)a_2 = \alpha d_1
\]
\[
(1 + 2\delta)^n(1 + 2\lambda \delta)(2a_2^2 - a_3) = \alpha d_2 + \frac{\alpha(\alpha - 1)}{2}d_1^2
\]
From (13) and (15), we obtain
\[
c_1 = -d_1
\]
and
\[
2(1 + \delta)^{2n}(1 + \lambda \delta)^2a_2^2 = \alpha^2(c_1^2 + d_1^2)
\]
Equating the coefficients in (9) and (10), we get
\[
a_2^2 = \frac{\alpha^2(c_2 + d_2)}{2\alpha(1 + 2\delta)^n(1 + 2\lambda \delta) - (\alpha - 1)(1 + \delta)^{2n}(1 + \lambda \delta)^2}
\]
Using Lemma 3, we get
\[
a_2^2 \leq \frac{4\alpha^2}{2\alpha(1 + 2\delta)^n(1 + 2\lambda \delta) - (\alpha - 1)(1 + \delta)^{2n}(1 + \lambda \delta)^2}
\]
which gives the desired result (7).
Next, subtracting (16) from (14) and then using (17), we obtain

\[ a_3 - a_2^2 = \frac{\alpha(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda \delta)} \]

By (18), we get

\[ a_3 = \frac{\alpha^2(c_1^2 + d_1^2)}{2(1 + \delta)^{2n}(1 + \lambda \delta)^2} + \frac{\alpha(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda \delta)} \]

Using Lemma 3, we get

\[ |a_3| \leq \frac{4\alpha^2}{(1 + \delta)^{2n}(1 + \lambda \delta)^2} + \frac{2\alpha}{(1 + 2\delta)^n(1 + 2\lambda \delta)} \]

which is the desired result (8).

This completes the proof of Theorem 5.

3. Coefficient Bounds for the Function Class \( \mathcal{H}_\Sigma (\delta, n, \beta, \lambda) \)

Definition 6. A function \( f(z) \) given by (1) is said to be in the class \( \mathcal{H}_\Sigma (\delta, n, \beta, \lambda) \) if the following conditions are satisfied:

\[ f \in \Sigma, \ \Re \left\{ \frac{(1 - \lambda) D^n_{\delta} f(z) + \lambda D^{n+1}_{\delta} f(z)}{z} \right\} > \beta \]

\[ (0 \leq \beta < 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U}) \]  (19)

and

\[ \Re \left\{ \frac{(1 - \lambda) D^n_{\delta} g(w) + \lambda D^{n+1}_{\delta} g(w)}{w} \right\} > \beta \]

\[ (0 \leq \beta < 1, \delta \geq 0, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U}) \],  (20)

where the function \( g \) is given by (2).

We note that for \( \delta = 1 \) the class \( \mathcal{H}_\Sigma (\delta, n, \beta, \lambda) \) reduces to the class \( \mathcal{H}_\Sigma (n, \beta, \lambda) \) introduced by S. Porwal and M. Darus [14] and for \( \delta = 1, n = 0 \) it reduces to the class \( \mathcal{H}_\Sigma (\beta, \lambda) \) introduced by B.A. Frasin and M.K. Aouf [9]. Also for \( \delta = 1, n = 0, \lambda = 1 \) this class reduces to the class \( \mathcal{H}_\Sigma (\lambda) \) introduced by Srivastava et. al. [17].
Theorem 7. Let the function \( f(z) \) given by (1) be in the class \( \mathcal{H}_\Sigma (\delta, n, \beta, \lambda) \). Then,

\[
|a_2| \leq \sqrt{\frac{2(1-\beta)}{(1+2\delta)^n(1+2\lambda\delta)}}
\]

and

\[
|a_3| \leq \frac{4(1-\beta)^2}{(1+\delta)^2n(1+\lambda\delta)^2} + \frac{2(1-\beta)}{(1+2\delta)^n(1+2\lambda\delta)}.
\]

Proof. Eqns. (19) and (20) implies that there exists \( p(z), q(w) \in \mathcal{P} \) such that

\[
(1-\lambda)D_\delta^n f(z) + \lambda D_\delta^{n+1} f(z) = \beta + (1-\beta)p(z)
\]

and

\[
(1-\lambda)D_\delta^n g(w) + \lambda D_\delta^{n+1} g(w) = \beta + (1-\beta)q(w)
\]

where \( p(z) \) and \( q(w) \) are given by (11) and (12) respectively. Equating the coefficients in (23) and (24), we get

\[
(1+\delta)^n(1+\lambda\delta)a_2 = (1-\beta)c_1
\]

(25)

\[
(1+2\delta)^n(1+2\lambda\delta)a_3 = (1-\beta)c_2
\]

(26)

\[-(1+\delta)^n(1+\lambda\delta)a_2 = (1-\beta)d_1
\]

(27)

\[(1+2\delta)^n(1+2\lambda\delta)(2a_2^2 - a_3) = (1-\beta)d_2
\]

(28)

Using (25) and (27), we obtain

\[
c_1 = -d_1
\]

(29)

and

\[
2(1+\delta)^{2n}(1+\lambda\delta)^2a_2^2 = (1-\beta)^2(c_1^2 + d_1^2)
\]

(30)

Adding (26) and (28), we obtain

\[
2(1+2\delta)^n(1+2\lambda\delta)a_2^2 = (1-\beta)(c_2 + d_2)
\]

Or

\[
\frac{a_2^2}{c_2 + d_2} = \frac{(1-\beta)(c_2 + d_2)}{2(1+2\delta)^n(1+2\lambda\delta)}
\]

Using Lemma 3, we get

\[
|a_2^2| \leq \frac{2(1-\beta)}{(1+2\delta)^n(1+2\lambda\delta)}
\]
which gives the desired result (21).

Next, for the estimates of $|a_3|$, subtracting (28) from (26), we obtain

$$2(1 + 2\delta)^n(1 + 2\lambda\delta)(a_3 - a_2^2) = (1 - \beta)(c_2 - d_2)$$

Or

$$a_3 = a_2^2 + \frac{(1 - \beta)(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

Using (30), we obtain

$$a_3 = \frac{(1 - \beta)^2(c_1^2 + d_1^2)}{2(1 + \delta)^2n(1 + \lambda\delta)^2} + \frac{(1 - \beta)(c_2 - d_2)}{2(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

Using Lemma 3, we get

$$|a_3| \leq \frac{4(1 - \beta)^2}{(1 + \delta)^{2n}(1 + \lambda\delta)^2} + \frac{2(1 - \beta)}{(1 + 2\delta)^n(1 + 2\lambda\delta)}$$

which is the desired result (22).

This completes the proof of Theorem 7.

4. Remarks

Remark 8. If we put $\delta = 1$ in Theorem 5 and Theorem 7, we obtain the corresponding results due to S. Porwal and M. Darus [14].

Remark 9. If we put $\delta = 1$ and $n = 0$ in Theorem 5 and Theorem 7, we obtain the corresponding results due to B.A. Frasin and M.K. Aouf [9].

Remark 10. If we put $\delta = 1$, $n = 0$ and $\lambda = 1$ in Theorem 5 and Theorem 7, we obtain the corresponding results due to Srivastava et. al. [17].

References


ON NEW SUBCLASSES...


