**ITERATIVE ALGORITHM FOR SOLVING THE NEW SYSTEM
OF GENERALIZED VARIATIONAL INEQUALITIES
IN HILBERT SPACES**

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Abstract: In this paper, we introduce an iterative method to approximate a common solution of a new general system of variational inequalities, a mixed equilibrium problem and a fixed point problem for a nonexpansive mapping in real Hilbert spaces. We prove that the iterative sequence converges strongly to a common solution of the three problems in the framework of Hilbert spaces. Our main results extend and improve some results in the literature.

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1. Introduction

The study of variational inequality problem is an interesting and fascinating branch of applicable mathematics with a wide range of applications industry, finance, economics, optimization, social, regional, pure and applied science. A closely related subject of current interest is the problem of finding common elements in the fixed point set of nonlinear operators and in the solution set of monotone variational inequalities; see [1, 2, 3] and the references therein. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been extended and generalized in various directions using and innovative techniques; see [4, 5, 6, 7] and the references therein.

Motivated by recent work going in this direction. In this paper, we introduce a new iterative scheme for finding a common element of the set of solutions of a new general system of variational inequalities, the set of solutions of a mixed equilibrium problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Furthermore, we prove that the sequence generated by the iterative scheme converges strongly to a common element of those three sets under some control conditions. The results presented in this paper extend and improve the corresponding results of [6] and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let C be a nonempty closed convex subset of H . A mapping $T : C \rightarrow C$ is said to be *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$. A mapping $A : C \rightarrow H$ is called *α -inverse-strongly monotone*, if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let $A_i : C \rightarrow H$ for all $i = 1, 2, 3$ be three mappings, then we consider the *new general system of variational inequalities* of finding $(x, y, z) \in C \times C \times C$ such that

$$\begin{cases} \langle \lambda_1 A_1 y + x - y, x - x \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 A_2 z + y - z, x - y \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_3 A_3 x + z - x, x - z \rangle \geq 0, & \forall x \in C, \end{cases} \quad (2.1)$$

where $\lambda_i > 0$ for all $i = 1, 2, 3$.

Some special cases:

(I) If $A_3 = 0$ and $z = x$, then problem (2.1) reduces to find $(x, y) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 A_1 y + x - y, x - x \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 A_2 x + y - x, x - y \rangle \geq 0, & \forall x \in C, \end{cases} \quad (2.2)$$

which is called *a general system of variational inequalities* and defined by the authors in [6]. The set of solutions of problem (2.2) denoted by $GVI(C, A_1, A_2)$.

(II) If $A_3 = 0$, $z = x$ and $A_1 = A_2 := A$, then problem (2.2) reduces to find $(x, y) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 A y + x - y, x - x \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 A x + y - x, x - y \rangle \geq 0, & \forall x \in C, \end{cases} \quad (2.3)$$

which is called *the new system of variational inequalities*, and defined by the author in [7].

(III) If $A_3 = A_2 = 0$, $z = y = x$, $A_1 := A$ and $\lambda_1 = 1$, then problem (2.3) reduces to find $x \in C$ such that

$$\langle Ax, x - x \rangle \geq 0, \quad \forall x \in C,$$

which is called *the variational inequality problem*.

Let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. In 2008, Ceng and Yao [8], introduced *the mixed equilibrium problem* which is to find $x \in C$ such that

$$F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (2.4)$$

The set of solution of problem (2.4) is denoted by $MEP(F, \varphi)$. It is easy to see that x is a solution of problem (2.4) implies that $x \in \text{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi = 0$, then the problem (2.4) reduces to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C, \quad (2.5)$$

which is called *the equilibrium problem*. The set of solution of (2.5) is denoted by $EP(F)$. In recent years, the equilibrium problem has been intensively studied by many authors (see, for example [1, 9, 10] and references therein).

We recall the well-known results and give some useful lemmas that are used in the next section.

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$. P_C is called the *metric*

projection of H onto C . It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H. \quad (2.6)$$

Obviously, this immediately implies that

$$\|(x - y) - (P_Cx - P_Cy)\|^2 \leq \|x - y\|^2 - \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H. \quad (2.7)$$

Recall that, P_Cx is characterized by the following properties: $P_Cx \in C$,

$$\langle x - P_Cx, y - P_Cx \rangle \leq 0 \text{ and } \|x - y\|^2 \geq \|x - P_Cx\|^2 + \|P_Cx - y\|^2, \quad (2.8)$$

for all $x \in H$ and $y \in C$.

For solving the mixed equilibrium problem, let us assume the following assumptions for the bifunction F, φ and the set C :

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
- (A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;
- (A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
- (B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z).$$

- (B2) C is a bounded set.

In the sequel we shall need to use the following lemma.

Lemma 2.1. ([11]) *Let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows.*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \quad \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:

- (1) For each $x \in H$, $T_r(x) \neq \emptyset$;
- (2) T_r is single-valued;
- (3) T_r is firmly nonexpansive, i.e. for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_rx - T_ry, x - y \rangle;$$

- (4) $F(T_r) = \text{MEP}(F, \varphi)$;
- (5) $\text{MEP}(F, \varphi)$ is closed and convex.

Lemma 2.2. ([12]) *Let H be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ &\quad - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \end{aligned}$$

Lemma 2.3. *In a real Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 2.4. ([13]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (ii) $\limsup_n \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_n a_n = 0$.

Lemma 2.5. ([14]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_n b_n \leq \limsup_n b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_nx_n$ for all integers $n \geq 1$ and $\limsup_n (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_n \|y_n - x_n\| = 0$.*

Lemma 2.6. ([15]) *Demi-closedness principle. Assume that T is a non-expansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H . If T has a fixed point, then $I - T$ is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \rightarrow y$), it follows that $(I - T)x = y$. Here I is the identity operator of H .*

Lemma 2.7. ([16]) *Let C be a nonempty closed and convex subset of a real Hilbert space H and $A_i : C \rightarrow H$ be three possibly nonlinear mappings, for $i = 1, 2, 3$. Define a mapping $G : C \rightarrow C$ as follows:*

$$\begin{aligned} G(x) &= P_C[P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x)) \\ &\quad - \lambda_1 A_1 P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x))], \quad \forall x \in C. \end{aligned}$$

For given $x, y, z \in C$, (x, y, z) is a solution of problem (2.1) if and only if $x \in F(G)$, $y = P_C(z - \lambda_2 A_2 z)$ and $z = P_C(x - \lambda_3 A_3 x)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $GVI(C, A_1, A_2, A_3)$.

3. Main Results

In this section, we prove our strong convergence theorem. The next lemma is crucial for proving the main theorem.

Lemma 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $A_i : C \rightarrow H$ be α_i -inverse-strongly monotone mappings, for $i = 1, 2, 3$. If $\lambda_i \in (0, 2\alpha_i]$, for all $i = 1, 2, 3$, then $G : C \rightarrow C$ is nonexpansive, where G is the mapping defined as in Lemma 2.7.*

Proof. For all $x, y \in C$, we have

$$\begin{aligned}
 \|G(x) - G(y)\| &= \|P_C[P_C(P_C(I - \lambda_3 A_3)x - \lambda_2 A_2 P_C(I - \lambda_3 A_3)x) \\
 &\quad - \lambda_1 A_1 P_C(P_C(I - \lambda_3 A_3)x - \lambda_2 A_2 P_C(I - \lambda_3 A_3)x)) \\
 &\quad - P_C[P_C(P_C(I - \lambda_3 A_3)y - \lambda_2 A_2 P_C(I - \lambda_3 A_3)y) \\
 &\quad - \lambda_1 A_1 P_C(P_C(I - \lambda_3 A_3)y - \lambda_2 A_2 P_C(I - \lambda_3 A_3)y))] \| \\
 &\leq \|P_C(P_C(I - \lambda_3 A_3)x - \lambda_2 A_2 P_C(I - \lambda_3 A_3)x) \\
 &\quad - \lambda_1 A_1 P_C(P_C(I - \lambda_3 A_3)x - \lambda_2 A_2 P_C(I - \lambda_3 A_3)x) \\
 &\quad - [P_C(P_C(I - \lambda_3 A_3)y - \lambda_2 A_2 P_C(I - \lambda_3 A_3)y) \\
 &\quad - \lambda_1 A_1 P_C(P_C(I - \lambda_3 A_3)y - \lambda_2 A_2 P_C(I - \lambda_3 A_3)y)] \| \\
 &= \|(I - \lambda_1 A_1)P_C(I - \lambda_2 A_2)P_C(I - \lambda_3 A_3)x \\
 &\quad - (I - \lambda_1 A_1)P_C(I - \lambda_2 A_2)P_C(I - \lambda_3 A_3)y\|. \tag{3.1}
 \end{aligned}$$

It is well known that if $A : C \rightarrow H$ be α -inverse-strongly monotone, then $I - \lambda A$ is nonexpansive for all $\lambda \in (0, 2\alpha]$. By our assumption, we obtain $I - \lambda_i A_i$ is nonexpansive for all $i = 1, 2, 3$. It follows that $(I - \lambda_1 A_1)P_C(I - \lambda_2 A_2)P_C(I - \lambda_3 A_3)$ is nonexpansive. Therefore, from (3.1), we obtain immediately that the mapping G is nonexpansive. $\square$

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $A_i : C \rightarrow H$ be α_i -inverse-strongly monotone, for all $i = 1, 2, 3$ and T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap GVI(C, A_1, A_2, A_3) \cap MEP(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . Let $x_1 \in C$ and*

$\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ be the sequences generated by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_3 A_3 u_n), \\ y_n = P_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T P_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1, \end{cases}$$

where $\lambda_i \in (0, 2\alpha_i)$, for all $i = 1, 2, 3$ and $\{a_n\}, \{b_n\}$ are two sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfying

$$\begin{aligned} \text{(C1)} \quad & \lim_n a_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty; \\ \text{(C2)} \quad & 0 < \liminf_n b_n \leq \limsup_n b_n < 1; \\ \text{(C3)} \quad & \liminf_n r_n > 0 \text{ and } \lim_n |r_{n+1} - r_n| = 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{\Omega}v$ and $(\bar{x}, \bar{y}, \bar{z})$ is a solution of problem (2.1), where $\bar{y} = P_C(\bar{z} - \lambda_2 A_2 \bar{z})$ and $\bar{z} = P_C(\bar{x} - \lambda_3 A_3 \bar{x})$.

Proof. Step 1. We claim that $\{x_n\}$ is bounded.

Let $x \in \Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. It follows from Lemma 2.7 that

$$\begin{aligned} x &= P_C [P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x)) \\ &\quad - \lambda_1 A_1 P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x))]. \end{aligned}$$

Put $y = P_C(z - \lambda_2 A_2 z)$, $z = P_C(x - \lambda_3 A_3 x)$ and $t_n = P_C(y_n - \lambda_1 A_1 y_n)$. Then $x = P_C(y - \lambda_1 A_1 y)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T t_n.$$

By nonexpansiveness of $I - \lambda_i$ ($i = 1, 2, 3$), we have

$$\begin{aligned} \|t_n - x\| &= \|P_C(I - \lambda_1 A_1)y_n - P_C(I - \lambda_1 A_1)y\| \\ &\leq \|y_n - y\| = \|P_C(I - \lambda_2 A_2)z_n - P_C(I - \lambda_2 A_2)z\| \\ &\leq \|z_n - z\| = \|P_C(I - \lambda_3 A_3)u_n - P_C(I - \lambda_3 A_3)x\| \\ &\leq \|u_n - x\| = \|T_{r_n}x_n - T_{r_n}x\| \leq \|x_n - x\|, \end{aligned} \tag{3.2}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x\| &= \|a_n v + b_n x_n + (1 - a_n - b_n) T t_n - x\| \\ &\leq a_n \|v - x\| + b_n \|x_n - x\| + (1 - a_n - b_n) \|t_n - x\| \\ &\leq a_n \|v - x\| + b_n \|x_n - x\| + (1 - a_n - b_n) \|x_n - x\| \end{aligned}$$

$$\leq \max\{\|v - x\|, \|x_1 - x\|\}.$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{A_1y_n\}$, $\{A_2z_n\}$, $\{A_3u_n\}$ and $\{Tt_n\}$ are also bounded.

Step 2. We claim that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

By nonexpansiveness of P_C and $I - \lambda_i A_i$ ($i = 1, 2, 3$), we have

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|P_C(y_{n+1} - \lambda_1 A_1 y_{n+1}) - P_C(y_n - \lambda_1 A_1 y_n)\| \leq \|y_{n+1} - y_n\| \\ &= \|P_C(z_{n+1} - \lambda_2 A_2 z_{n+1}) - P_C(z_n - \lambda_2 A_2 z_n)\| \leq \|z_{n+1} - z_n\| \\ &= \|P_C(u_{n+1} - \lambda_3 A_3 u_{n+1}) - P_C(u_n - \lambda_3 A_3 u_n)\| \\ &\leq \|u_{n+1} - u_n\|. \end{aligned} \quad (3.3)$$

On the other hand, from $u_n = T_{r_n} x_n \in \text{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}} x_{n+1} \in \text{dom}\varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (3.4)$$

and

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \quad (3.5)$$

Putting $y = u_{n+1}$ in (3.4) and $y = u_n$ in (3.5), we have

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0,$$

and

$$F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

From the monotonicity of F , we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0,$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Then, we have

$$\|u_{n+1} - u_n\|^2 \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle$$

$$\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \|u_{n+1} - x_{n+1}\| \right\},$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \quad (3.6)$$

It follows from (3.3) and (3.6) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \quad (3.7)$$

Let $x_{n+1} = b_n x_n + (1 - b_n) w_n$ for all $n \geq 1$. Then, we obtain

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - b_{n+1} x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\ &= \frac{a_{n+1} v + (1 - a_{n+1} - b_{n+1}) T t_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n) T t_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}} (v - T t_{n+1}) + \frac{a_n}{1 - b_n} (T t_n - v) + T t_{n+1} - T t_n. \end{aligned} \quad (3.8)$$

By (3.7) and (3.8), we have

$$\begin{aligned} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| &\leq \frac{a_{n+1}}{1 - b_{n+1}} \|v - T t_{n+1}\| + \frac{a_n}{1 - b_n} \|T t_n - v\| \\ &\quad + \|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \\ &\leq \frac{a_{n+1}}{1 - b_{n+1}} \|v - T t_{n+1}\| + \frac{a_n}{1 - b_n} \|T t_n - v\| \\ &\quad + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \end{aligned}$$

This together with (C1)–(C3), we obtain that

$$\limsup_n \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq 0.$$

Hence, by Lemma 2.5, we get $\|x_n - w_n\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently,

$$\lim_n \|x_{n+1} - x_n\| = \lim_n (1 - b_n) \|w_n - x_n\| = 0. \quad (3.9)$$

Step 3. We claim that $\|T t_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Since

$$x_{n+1} - x_n = a_n (v - x_n) + (1 - a_n - b_n) (T t_n - x_n),$$

therefore

$$\|Tt_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.10)$$

Next, we prove that $\lim_n \|x_n - u_n\| = 0$. From Lemma 2.1(3), we have

$$\begin{aligned} \|u_n - x\|^2 &= \|T_{r_n}x_n - T_{r_n}x\|^2 \leq \langle T_{r_n}x_n - T_{r_n}x, x_n - x \rangle \\ &= \langle u_n - x, x_n - x \rangle = \frac{1}{2} \{ \|u_n - x\|^2 + \|x_n - x\|^2 - \|x_n - u_n\|^2 \}. \end{aligned}$$

Hence

$$\|u_n - x\|^2 \leq \|x_n - x\|^2 - \|x_n - u_n\|^2. \quad (3.11)$$

From Lemma 2.2, (3.2) and (3.11), we have

$$\begin{aligned} \|x_{n+1} - x\|^2 &\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 + (1 - a_n - b_n) \|t_n - x\|^2 \\ &\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 + (1 - a_n - b_n) \|u_n - x\|^2 \\ &\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 \\ &\quad + (1 - a_n - b_n) [\|x_n - x\|^2 - \|x_n - u_n\|^2] \\ &\leq a_n \|v - x\|^2 + \|x_n - x\|^2 - (1 - a_n - b_n) \|x_n - u_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - a_n - b_n) \|x_n - u_n\|^2 &\leq a_n \|v - x\|^2 + \|x_n - x\|^2 - \|x_{n+1} - x\|^2 \\ &\leq a_n \|v - x\|^2 + (\|x_n - x\| + \|x_{n+1} - x\|) \|x_{n+1} - x_n\|. \end{aligned}$$

From the conditions (C1), (C2) and (3.9), we obtain

$$\lim_n \|x_n - u_n\| = 0. \quad (3.12)$$

By (3.10) and (3.12), we have

$$\|Tt_n - u_n\| \leq \|Tt_n - x_n\| + \|x_n - u_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (3.13)$$

Next, we show that $\|A_1y_n - A_1y\| \rightarrow 0$, $\|A_2z_n - A_2z\| \rightarrow 0$ and $\|A_3u_n - A_3x\| \rightarrow 0$, as $n \rightarrow \infty$.

From (3.2) and A_1 is α_1 -inverse-strongly monotone mapping, we have

$$\begin{aligned} \|x_{n+1} - x\|^2 &\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 + (1 - a_n - b_n) \|t_n - x\|^2 \\ &= a_n \|v - x\|^2 + b_n \|x_n - x\|^2 \\ &\quad + (1 - a_n - b_n) \|P_C(y_n - \lambda_1 A_1 y_n) - P_C(y - \lambda_1 A_1 y)\|^2 \\ &\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 \end{aligned}$$

$$\begin{aligned}
& + (1 - a_n - b_n) \|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y)\|^2 \\
& \leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 \\
& \quad + (1 - a_n - b_n) [\|y_n - y\|^2 + \lambda_1(\lambda_1 - 2\alpha_1) \|A_1 y_n - A_1 y\|^2] \\
& \leq a_n \|v - x\|^2 + \|x_n - x\|^2 \\
& \quad + (1 - a_n - b_n) \lambda_1(\lambda_1 - 2\alpha_1) \|A_1 y_n - A_1 y\|^2. \tag{3.14}
\end{aligned}$$

Similarly, since A_i are α_i -inverse-strongly monotone mappings for $i = 2, 3$, $\|t_n - x\| \leq \|y_n - y\|$ and $\|y_n - y\| \leq \|z_n - z\|$, we can show that

$$\begin{aligned}
\|x_{n+1} - x\|^2 & \leq a_n \|v - x\|^2 + \|x_n - x\|^2 \\
& \quad + (1 - a_n - b_n) \lambda_2(\lambda_2 - 2\alpha_2) \|A_2 z_n - A_2 z\|^2 \tag{3.15}
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x\|^2 & \leq a_n \|v - x\|^2 + \|x_n - x\|^2 \\
& \quad + (1 - a_n - b_n) \lambda_3(\lambda_3 - 2\alpha_3) \|A_3 u_n - A_3 x\|^2. \tag{3.16}
\end{aligned}$$

From (3.14), (3.15) and (3.16), we have

$$\begin{aligned}
-(1 - a_n - b_n) \lambda_1(\lambda_1 - 2\alpha_1) \|A_1 y_n - A_1 y\|^2 & \leq a_n \|v - x\|^2 \\
& \quad + (\|x_n - x\| + \|x_{n+1} - x\|) \|x_{n+1} - x_n\|, \\
-(1 - a_n - b_n) \lambda_2(\lambda_2 - 2\alpha_2) \|A_2 z_n - A_2 z\|^2 & \leq a_n \|v - x\|^2 \\
& \quad + (\|x_n - x\| + \|x_{n+1} - x\|) \|x_{n+1} - x_n\|
\end{aligned}$$

and

$$\begin{aligned}
-(1 - a_n - b_n) \lambda_3(\lambda_3 - 2\alpha_3) \|A_3 u_n - A_3 x\|^2 & \leq a_n \|v - x\|^2 \\
& \quad + (\|x_n - x\| + \|x_{n+1} - x\|) \|x_{n+1} - x_n\|.
\end{aligned}$$

This together with (C1), (C2) and (3.9), we obtain that

$$\lim_n \|A_1 y_n - A_1 y\| = \lim_n \|A_2 z_n - A_2 z\| = \lim_n \|A_3 u_n - A_3 x\| = 0. \tag{3.17}$$

Next, we prove that $\|Tt_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (2.6), (3.2) and nonexpansiveness of $I - \lambda_2 A_2$ and $I - \lambda_3 A_3$, we get

$$\begin{aligned}
\|y_n - y\|^2 & = \|P_C(z_n - \lambda_2 A_2 z_n) - P_C(z - \lambda_2 A_2 z)\|^2 \\
& \leq \langle (z_n - \lambda_2 A_2 z_n) - (z - \lambda_2 A_2 z), y_n - y \rangle \\
& = \frac{1}{2} [\|(z_n - \lambda_2 A_2 z_n) - (z - \lambda_2 A_2 z)\|^2 + \|y_n - y\|^2]
\end{aligned}$$

$$\begin{aligned}
& - \|(z_n - \lambda_2 A_2 z_n) - (z - \lambda_2 A_2 z) - (y_n - y)\|^2] \\
& \leq \frac{1}{2} [\|z_n - z\|^2 + \|y_n - y\|^2 \\
& \quad - \|(z_n - y_n) - (z - y) - \lambda_2 (A_2 z_n - A_2 z)\|^2] \\
& \leq \frac{1}{2} [\|x_n - x\|^2 + \|y_n - y\|^2 - \|(z_n - y_n) - (z - y)\|^2 \\
& \quad + 2\lambda_2 \langle (z_n - y_n) - (z - y), A_2 z_n - A_2 z \rangle - \lambda_2^2 \|A_2 z_n - A_2 z\|^2],
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - z\|^2 &= \|P_C(u_n - \lambda_3 A_3 u_n) - P_C(x - \lambda_3 A_3 x)\|^2 \\
&\leq \langle (u_n - \lambda_3 A_3 u_n) - (x - \lambda_3 A_3 x), z_n - z \rangle \\
&= \frac{1}{2} [\|(u_n - \lambda_3 A_3 u_n) - (x - \lambda_3 A_3 x)\|^2 + \|z_n - z\|^2 \\
&\quad - \|(u_n - \lambda_3 A_3 u_n) - (x - \lambda_3 A_3 x) - (z_n - z)\|^2] \\
&\leq \frac{1}{2} [\|u_n - x\|^2 + \|z_n - z\|^2 \\
&\quad - \|(u_n - z_n) - (x - z) - \lambda_3 (A_3 u_n - A_3 x)\|^2] \\
&\leq \frac{1}{2} [\|x_n - x\|^2 + \|z_n - z\|^2 - \|(u_n - z_n) - (x - z)\|^2 \\
&\quad + 2\lambda_3 \langle (u_n - z_n) - (x - z), A_3 u_n - A_3 x \rangle - \lambda_3^2 \|A_3 u_n - A_3 x\|^2].
\end{aligned}$$

Therefore

$$\begin{aligned}
\|y_n - y\|^2 &\leq \|x_n - x\|^2 - \|(z_n - y_n) - (z - y)\|^2 \\
&\quad + 2\lambda_2 \langle (z_n - y_n) - (z - y), A_2 z_n - A_2 z \rangle
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
\|z_n - z\|^2 &\leq \|x_n - x\|^2 - \|(u_n - z_n) - (x - z)\|^2 \\
&\quad + 2\lambda_3 \langle (u_n - z_n) - (x - z), A_3 u_n - A_3 x \rangle.
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we have

$$\begin{aligned}
\|x_{n+1} - x\|^2 &\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 + (1 - a_n - b_n) \|y_n - y\|^2 \\
&\leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 \\
&\quad + (1 - a_n - b_n) [\|x_n - x\|^2 - \|(z_n - y_n) - (z - y)\|^2 \\
&\quad + 2\lambda_2 \langle (z_n - y_n) - (z - y), A_2 z_n - A_2 z \rangle] \\
&\leq a_n \|v - x\|^2 + \|x_n - x\|^2
\end{aligned}$$

$$\begin{aligned}
& - (1 - a_n - b_n) \|(z_n - y_n) - (z - y)\|^2 \\
& + (1 - a_n - b_n) 2\lambda_2 \|(z_n - y_n) - (z - y)\| \|A_2 z_n - A_2 z\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x\|^2 & \leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 + (1 - a_n - b_n) \|z_n - z\|^2 \\
& \leq a_n \|v - x\|^2 + b_n \|x_n - x\|^2 \\
& \quad + (1 - a_n - b_n) [\|x_n - x\|^2 - \|(u_n - z_n) - (x - z)\|^2 \\
& \quad + 2\lambda_3 \langle (u_n - z_n) - (x - z), A_3 u_n - A_3 x \rangle] \\
& \leq a_n \|v - x\|^2 + \|x_n - x\|^2 \\
& \quad - (1 - a_n - b_n) \|(u_n - z_n) - (x - z)\|^2 \\
& \quad + (1 - a_n - b_n) 2\lambda_3 \|(u_n - z_n) - (x - z)\| \|A_3 u_n - A_3 x\|.
\end{aligned}$$

Hence

$$\begin{aligned}
& (1 - a_n - b_n) \|(z_n - y_n) - (z - y)\|^2 \\
& \leq a_n \|v - x\|^2 + (1 - a_n - b_n) 2\lambda_2 \|(z_n - y_n) - (z - y)\| \|A_2 z_n - A_2 z\| \\
& \quad + (\|x_n - x\| + \|x_{n+1} - x\|) \|x_{n+1} - x_n\|
\end{aligned}$$

and

$$\begin{aligned}
& (1 - a_n - b_n) \|(u_n - z_n) - (x - z)\|^2 \\
& \leq a_n \|v - x\|^2 + (1 - a_n - b_n) 2\lambda_3 \|(u_n - z_n) - (x - z)\| \|A_3 u_n - A_3 x\| \\
& \quad + (\|x_n - x\| + \|x_{n+1} - x\|) \|x_{n+1} - x_n\|.
\end{aligned}$$

This together with (C1), (C2), (3.9) and (3.17), we obtain

$$\lim_n \|(z_n - y_n) - (z - y)\| = \lim_n \|(u_n - z_n) - (x - z)\| = 0. \quad (3.20)$$

Therefore

$$\begin{aligned}
\|(u_n - y_n) - (x - y)\| & \leq \|(z_n - y_n) - (z - y)\| \\
& \quad + \|(u_n - z_n) - (x - z)\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.21)
\end{aligned}$$

From Lemma 2.3 and (2.7), it follows that

$$\begin{aligned}
\|(y_n - t_n) + (x - y)\|^2 & = \|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y)\| \\
& \quad - [P_C(y_n - \lambda_1 A_1 y_n) - P_C(y - \lambda_1 A_1 y)] + \lambda_1 \|A_1 y_n - A_1 y\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y) - [P_C(y_n - \lambda_1 A_1 y_n) - P_C(y - \lambda_1 A_1 y)]\|^2 \\
&\quad + 2\lambda_1 \langle A_1 y_n - A_1 y, (y_n - t_n) + (x - y) \rangle \\
&\leq \|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y)\|^2 - \|P_C(y_n - \lambda_1 A_1 y_n) - P_C(y - \lambda_1 A_1 y)\|^2 \\
&\quad + 2\lambda_1 \|A_1 y_n - A_1 y\| \|(y_n - t_n) + (x - y)\| \\
&\leq \|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y)\|^2 - \|TP_C(y_n - \lambda_1 A_1 y_n) \\
&\quad - TP_C(y - \lambda_1 A_1 y)\|^2 \\
&\quad + 2\lambda_1 \|A_1 y_n - A_1 y\| \|(y_n - t_n) + (x - y)\| \\
&\leq \|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y) \\
&\quad - (Tt_n - x)\| [\|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y)\| + \|Tt_n - x\|] \\
&\quad + 2\lambda_1 \|A_1 y_n - A_1 y\| \|(y_n - t_n) + (x - y)\| \\
&= \|u_n - Tt_n + x - y - (u_n - y_n) \\
&\quad - \lambda_1 (A_1 y_n - A_1 y)\| [\|(y_n - \lambda_1 A_1 y_n) - (y - \lambda_1 A_1 y)\| + \|Tt_n - x\|] \\
&\quad + 2\lambda_1 \|A_1 y_n - A_1 y\| \|(y_n - t_n) + (x - y)\|.
\end{aligned}$$

This together with (3.13), (3.17) and (3.21), we obtain $\|(y_n - t_n) + (x - y)\| \rightarrow 0$ as $n \rightarrow \infty$. This together with (3.13) and (3.20), we obtain that

$$\begin{aligned}
\|Tt_n - t_n\| &\leq \|Tt_n - u_n\| + \|(u_n - z_n) - (x - z)\| + \|(z_n - y_n) - (z - y)\| \\
&\quad + \|(y_n - t_n) + (x - y)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{3.22}$$

Step4. We claim that $\limsup_n \langle v - \bar{x}, x_n - \bar{x} \rangle \leq 0$, where $\bar{x} = P_\Omega v$. Indeed, since $\{t_n\}$ and $\{Tt_n\}$ are two bounded sequences in C , we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_n \langle v - \bar{x}, Tt_n - \bar{x} \rangle = \lim_i \langle v - \bar{x}, Tt_{n_i} - \bar{x} \rangle.$$

Since $\lim_n \|Tt_n - t_n\| = 0$, we obtain that $Tt_{n_i} \rightharpoonup z$ as $i \rightarrow \infty$.

Next, we show that $z \in \Omega$.

Since $t_{n_i} \rightharpoonup z$ and $\|Tt_n - t_n\| \rightarrow 0$, we obtain by Lemma 2.6 that $z \in F(T)$.

From (3.22) and (3.10), we obtain

$$\|t_n - x_n\| \leq \|Tt_n - t_n\| + \|Tt_n - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Furthermore, by Lemma 3.1, we have $G : C \rightarrow C$ is nonexpansive. Then, we have

$$\begin{aligned}
\|t_n - G(t_n)\| &= \|P_C(y_n - \lambda_1 A_1 y_n) - G(t_n)\| \\
&= \|P_C[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)] - G(t_n)\|
\end{aligned}$$

$$\begin{aligned}
&= \|P_C[P_C(P_C(u_n - \lambda_3 A_3 u_n) - \lambda_2 A_2 P_C(u_n - \lambda_3 A_3 u_n)) \\
&\quad - \lambda_1 A_1 P_C(P_C(u_n - \lambda_3 A_3 u_n) - \lambda_2 A_2 P_C(u_n - \lambda_3 A_3 u_n))] - G(t_n)\| \\
&= \|G(u_n) - G(t_n)\| \leq \|u_n - t_n\| \\
&\leq \|u_n - x_n\| + \|x_n - t_n\|,
\end{aligned}$$

hence $\lim_n \|t_n - G(t_n)\| = 0$. Again by Lemma 2.6, we have $z \in GVI(C, A_1, A_2, A_3)$.

Since $t_{n_i} \rightharpoonup z$ and $\|x_n - t_n\| \rightarrow 0$, we obtain that $x_{n_i} \rightharpoonup z$. From $\|u_n - x_n\| \rightarrow 0$, we also obtain that $u_{n_i} \rightharpoonup z$. By using the same argument as that in the proof of [11, Theorem 3.1, pp. 1825], we can show that $z \in MEP(F, \varphi)$. Therefore $z \in \Omega$.

On the other hand, it follows from (2.8), (3.10) and $Tt_{n_i} \rightharpoonup z$ as $i \rightarrow \infty$ that

$$\begin{aligned}
\limsup_n \langle v - \bar{x}, x_n - \bar{x} \rangle &= \limsup_n \langle v - \bar{x}, Tt_n - \bar{x} \rangle = \lim_i \langle v - \bar{x}, Tt_{n_i} - \bar{x} \rangle \\
&= \langle v - \bar{x}, z - \bar{x} \rangle \leq 0.
\end{aligned} \tag{3.23}$$

Step 5. We claim that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$.

Since

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \langle a_n v + b_n x_n + (1 - a_n - b_n)Tt_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\
&= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + b_n \langle x_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\
&\quad + (1 - a_n - b_n) \langle Tt_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\
&\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} b_n (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\
&\quad + \frac{1}{2} (1 - a_n - b_n) (\|t_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\
&\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} b_n (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\
&\quad + \frac{1}{2} (1 - a_n - b_n) (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\
&= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} (1 - a_n) (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2),
\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - a_n) \|x_n - \bar{x}\|^2 + 2a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle.$$

This together with (C1) and (3.23), we have by Lemma 2.4 that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. $\square$

If $\varphi = 0$ in Theorem 3.2, then, we obtain the following result.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H and F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5). Let the mappings $A_i : C \rightarrow H$ be α_i -inverse-strongly monotone, for all $i = 1, 2, 3$ and T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap GVI(C, A_1, A_2, A_3) \cap EP(F) \neq \emptyset$. Let $v, x_1 \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ z_n = P_C(x_n - \lambda_3 A_3 x_n), \\ y_n = P_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T P_C(y_n - \lambda_1 A_1 y_n), & n \geq 1. \end{cases}$$

If $\lambda_i \in (0, 2\alpha_i)$, for all $i = 1, 2, 3$ and the sequences $\{a_n\}, \{b_n\}$ and $\{r_n\}$ are as in Theorem 3.2, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y}, \bar{z})$ is a solution of problem (2.1), where $\bar{y} = P_C(\bar{z} - \lambda_2 A_2 \bar{z})$ and $\bar{z} = P_C(\bar{x} - \lambda_3 A_3 \bar{x})$.

If $A_3 = 0$, $\varphi = 0$, $F(x, y) = 0$ and $r_n = 1$ for all $x, y \in C$ and all $n \in \mathbb{N}$ in Theorem 3.2, then $z_n = x_n$. By Theorem 3.2, we obtain the following result.

Corollary 3.4. [6, Theorem 3.1] *Let C be a nonempty closed and convex subset of a real Hilbert space H . Let the mappings $A_1, A_2 : C \rightarrow H$ be α_1 -inverse-strongly monotone and α_2 -inverse-strongly monotone, respectively. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap GVI(C, A_1, A_2) \neq \emptyset$. Assume that v is an arbitrary point in C . Let $x_1 \in C$ and $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_2 A_2 x_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T P_C(y_n - \lambda_1 A_1 y_n), & n \geq 1. \end{cases}$$

If $\lambda_1 \in (0, 2\alpha_1)$, $\lambda_2 \in (0, 2\alpha_2)$ and the sequences $\{a_n\}, \{b_n\}$ are as in Theorem 3.2, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of problem (2.2), where $\bar{y} = P_C(\bar{x} - \lambda_2 A_2 \bar{x})$.

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