***w*-SEMIOPEN SETS
AND *w*-SEMICONTINUITY IN WEAK SPACES**

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Abstract: The purpose of this short note is to generalize *w*-open sets in *w*-spaces, say *w*-semiopen sets. First, we introduce the notion of *w*-semiopen sets and some basic properties of such the notion. Second, we introduce *W*-semicontinuous functions defined by *w*-semiopen sets and investigate characterizations of them by using the several types of operators.

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1. Introduction

Siwiec [16] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [13]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure, *w*-space, *W*-continuity and *W* -continuity were investigated in [14]. In fact, the set of all *g*-closed subsets [5] in a topological space is a kind of weak structure.

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The one purpose of our research is to generalize w -open sets in w -spaces. In 1963, Levine [6] introduced the concept of semi-open set in topological spaces and used this to define other new concepts such as semi-closed set and semi-continuity of a function. In this paper, in the same way as Levine did, we are going to study the notion of w -semiopen set in a weak space.

First, we investigate the notion of w -semiopen sets and some basic properties of such the notion. Second, we introduce W -semicontinuous functions defined by w -semiopen sets, and study characterizations of them by using the operators wC , wI , wsC and wsI .

2. Preliminaries

Definition 1 ([14]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

- (1) $\emptyset \in w_X$ and $X \in w_X$.
- (2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X . Then $V \in w_X$ is called a *w-open* set and the complement of a *w-open* set is a *w-closed* set.

The collection of all w -open sets (resp., w -closed sets) in a w -space X will be denoted by $WO(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$.

Let S be a subset of a topological space X . The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [10] (resp., α -open set [15], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S)))$, $S \subset cl(int(S))$). The complement of a preopen set (resp., α -open set, *semi-open*) is called a *preclosed* set (resp., α -closed set, *semi-closed*). The family of all preopen sets (resp., α -open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X)$, $SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X .

A subset A of a topological space (X, τ) is said to be:

- (a) g -closed [5] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X ;
- (b) gp -closed [7] if $pCl(A) \subset U$ whenever $A \subset U$ and U is open in X ;
- (c) gs -closed [1, 3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is open in X ;
- (d) $g\alpha$ -closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α -open in X where $\tau^\alpha = \alpha(X)$;
- (e) $g\alpha$ -closed [8] if $\tau^\alpha Cl(A) \subset int(U)$ whenever $A \subset U$ and U is α -open in X ;
- (f) $g\alpha$ -closed [8] if $\tau^\alpha Cl(A) \subset int(cl(U))$ whenever $A \subset U$ and U is α -open in X ;

- (g) αg -closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is open in X ;
- (h) α g -closed [9] if $\tau^\alpha Cl(A) \subset int(cl(U))$ whenever $A \subset U$ and U is open in X .

Then the family τ , $GO(X)$, $g\alpha O(X)$, $g\alpha O(X)$, $g\alpha O(X)$, $\alpha gO(X)$ and $\alpha gO(X)$ are all weak structures on X . But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X . A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [10] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

Definition 2 ([14]). Let (X, w_X) be a w -space. For a subset A of X , the w -closure of A and the w -interior of A are defined as follows:

- (1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.
- (2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 3 ([14]). Let (X, w_X) be a w -space and $A \subseteq X$.

- (1) $x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.
- (2) $x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.
- (3) If $A \subset B$, then $wI(A) \subset wI(B)$; $wC(A) \subset wC(B)$.
- (4) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.
- (5) If A is w -closed (resp., w -open), then $wC(A) = A$ (resp., $wI(A) = A$).

Definition 4 ([14]). Let $f : X \rightarrow Y$ be a function on w -spaces (X, w_X) and (Y, w_Y) . Then f is said to be W -continuous [14] if for $x \in X$ and $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subseteq V$.

3. w -Semiopen Sets, W -Semicontinuity

Definition 5. Let (X, w_X) be a w -space and $S \subseteq X$. Then S is called a w -semiopen set if $S \subseteq wC(wI(S))$. The complement of a w -semiopen set is called a w -semiclosed set.

The family of all w -semiopen sets in X will be denoted by $WSO(X)$.

Lemma 6. Let (X, w_X) be a w -space and $A \subseteq X$. Then the following things hold:

- (1) A is a w -semiclosed set if and only if $wI(wC(A)) \subseteq A$.
- (2) Every w -open (resp., w -closed) set is w -semiopen (resp., w -semiclosed).

We can easily show that a w -semiopen set may not be w -open as the next example:

Example 7. Let $X = \{a, b, c\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, X\}$ be a w -structure in X . Consider $A = \{a, b\}$; then $wC(wI(A)) = X$. So A is w -semiopen but not w -open.

Theorem 8. Let (X, w_X) be a w -space. Any union of w -semiopen sets is also w -semiopen.

Proof. Let A_i be a w -semiopen set for $i \in J$. From (3) of Theorem 2.3, $A_i \subseteq wC(wI(A_i)) \subseteq wC(wI(\cup A_i))$. Clearly, this implies $\cup A_i \subseteq wC(wI(\cup A_i))$ and so $\cup A_i$ is w -semiopen. $\square$

In general, the intersection of any two w -semiopen sets is not w -semiopen. See the next example:

Example 9. In example 3.3, consider $A = \{a, c\}$ and $B = \{b, c\}$. Note that $wC(wI(A)) = A$ and $wC(wI(B)) = B$. So both A and B are w -semiopen, but $A \cap B = \{c\}$ is not w -semiopen.

Definition 10. Let (X, w_X) be a w -space and $A \subseteq X$. Then the w -semi-closure of A and the w -semi-interior of A , denoted by $wsC(A)$ and $wsI(A)$, respectively, are defined as:

$$wsC(A) = \cap \{F : A \subseteq F, F \text{ is } w\text{-semiclosed in } X\};$$

$$wsI(A) = \cup \{U : U \subseteq A, U \text{ is } w\text{-semiopen in } X\}.$$

Theorem 11. Let (X, w_X) be a w -space and $A, B, F \subseteq X$. Then the following things hold:

- (1) $wsI(A) \subseteq A$; $A \subseteq wsC(A)$.
- (2) If $A \subseteq B$, then $wsI(A) \subseteq wsI(B)$ and $wsC(A) \subseteq wsC(B)$.
- (3) A is w -semiopen iff $wsI(A) = A$.
- (4) F is w -semiclosed iff $wsC(F) = F$.
- (5) $wsI(wsI(A)) = wsI(A)$; $wsC(wsC(A)) = wsC(A)$.
- (6) $wsC(X - A) = X - wsI(A)$; $wsI(X - A) = X - wsC(A)$.

Proof. (1) and (2) Obvious.

(3) and (4) It follows from Theorem 3.5.

(5) It follows from (3) and (4).

(6) For $A \subseteq X$,

$$\begin{aligned} X - wsI(A) &= X - \cup \{U : U \subseteq A, U \text{ is } w\text{-semiopen}\} \\ &= \cap \{X - U : U \subseteq A, U \text{ is } w\text{-semiopen}\} \\ &= \cap \{X - U : X - A \subseteq X - U, U \text{ is } w\text{-semiopen}\} \\ &= wsC(X - A). \end{aligned}$$

Similarly, we have $wsI(X - A) = X - wsC(A)$. $\square$

Theorem 12. Let (X, w_X) be a w -space and $A \subseteq X$. Then

- (1) $x \in wsC(A)$ if and only if $A \cap V \neq \emptyset$ for every w -semiopen set V containing x .
- (2) $x \in wsI(A)$ if and only if there exists a w -semiopen set U such that $U \subseteq A$.

Proof. (1) Suppose that there is a w -semiopen set V containing x such that $A \cap V = \emptyset$. Then $X - V$ is a w -semiclosed set such that $A \subseteq X - V$ and $x \notin X - V$. This implies $x \notin wsC(A)$.

The reverse relation is obvious.

- (2) Obvious. $\square$

Now, we introduce the notion of W -semicontinuous functions defined by w -semiopen sets, and study characterizations of them by using the operators wC, wI, wsC and wsI .

Definition 13. Let $f : (X, w_X) \rightarrow (Y, w_Y)$ be a function on w -spaces X and Y . Then f is said to be W -semicontinuous if for each x and each w -open set V containing $f(x)$, there exists a w -semiopen set U containing x such that $f(U) \subseteq V$.

Since every w -open set is w -semiopen, obviously, a W -continuous function is W -semicontinuous, but the converse is not true in general.

$$W\text{-continuity} \Rightarrow W\text{-semicontinuity}$$

Example 14. Let $X = \{a, b, c, d\}$. Consider two weak structures w_1 and w_2 , respectively, as the following:

$$w_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}; \quad w_2 = \{\emptyset, \{a, b\}, \{a\}, \{a, c\}, X\}.$$

Let $f : (X, w_1) \rightarrow (X, w_2)$ be the identity function. Then f is W -semicontinuous but not W -continuous.

Theorem 15. Let $f : (X, w_X) \rightarrow (Y, w_Y)$ be a function on w -spaces X and Y . Then the following statements are equivalent:

- (1) f is W -semicontinuous.
- (2) For each w -open set V in Y , $f^{-1}(V)$ is w -semiopen.
- (3) For each w -closed set B in Y , $f^{-1}(B)$ is w -semiclosed.
- (4) $f(wsC(A)) \subseteq wC(f(A))$ for $A \subseteq X$.
- (5) $wsC(f^{-1}(B)) \subseteq f^{-1}(wC(B))$ for $B \subseteq Y$.
- (6) $f^{-1}(wI(B)) \subseteq wsI(f^{-1}(B))$ for $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2) Let V be a w -open set in Y . Then for each $x \in f^{-1}(V)$, by the hypothesis, there exists a w -semiopen set U_x containing x such that $f(U_x) \subseteq V$. It implies that for each $x \in f^{-1}(V)$, $x \in U_x \subseteq f^{-1}(V)$. Hence by Theorem 3.4, $f^{-1}(V)$ is w -semiopen.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) For $A \subseteq X$,

$$\begin{aligned} f^{-1}(wC(f(A))) &= f^{-1}(\cap\{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is } w\text{-closed}\}) \\ &= \cap\{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is } w\text{-semiclosed}\} \\ &\supseteq \cap\{K \subseteq X : A \subseteq K \text{ and } K \text{ is } w\text{-semiclosed}\} \\ &= wsC(A) \end{aligned}$$

It implies that $f(wsC(A)) \subseteq wC(f(A))$.

(4) $\Rightarrow$ (5) For $B \subseteq Y$, from (4), it follows

$$f(wsC(f^{-1}(B))) \subseteq wC(f(f^{-1}(B))) \subseteq wC(B).$$

So, $wsC(f^{-1}(B)) \subseteq f^{-1}(wC(B))$.

(5) $\Rightarrow$ (6) For $B \subseteq Y$, from $wI(B) = Y - wC(Y - B)$ and (5), it follows

$$\begin{aligned} f^{-1}(wI(B)) &= f^{-1}(Y - wC(Y - B)) \\ &= X - (f^{-1}(wC(Y - B))) \\ &\subseteq X - wsC(f^{-1}(Y - B)) \\ &= wsI(f^{-1}(B)). \end{aligned}$$

So, $f^{-1}(wI(B)) \subseteq wsI(f^{-1}(B))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and V be a w -open set containing $f(x)$. Then, by Theorem 2.3 and (6), $x \in f^{-1}(V) = f^{-1}(wI(V)) \subseteq wsI(f^{-1}(V))$. So, there exists a w -semiopen set U containing x such that $x \in U \subseteq f^{-1}(V)$. It implies that f is W -semicontinuous. $\square$

Lemma 16. Let (X, w_X) be a w -space and $A \subseteq X$. Then

- (1) $wI(wC(A)) \subseteq wI(wC(wsC(A))) \subseteq wsC(A)$.
- (2) $wsI(A) \subseteq wCl(wI(wsI(A))) \subseteq wI(wC(A))$.

Proof. (1) For $A \subseteq X$, from Theorem 3.7, $wsC(A)$ is a w -semiclosed set. Hence, $wI(wC(A)) \subseteq wI(wC(wsC(A))) \subseteq wsC(A)$.

(2) It is similar to the proof of (1).

□

Finally, we characterize W -semicontinuity in w -spaces in terms of the operators wC , wI , wsC and wsI as the following theorem:

Theorem 17. *Let $f : (X, w_X) \rightarrow (Y, w_Y)$ be a function on w -spaces X and Y . Then the following statements are equivalent:*

- (1) f is W -semicontinuous.
- (2) $f^{-1}(V) \subseteq wC(wI(f^{-1}(V)))$ for each w -open set V in Y .
- (3) $wI(wC(f^{-1}(F))) \subseteq f^{-1}(F)$ for each w -closed set F in Y .
- (4) $f(wI(wC(A))) \subseteq wC(f(A))$ for $A \subseteq X$.
- (5) $wI(wC(f^{-1}(B))) \subseteq f^{-1}(wC(B))$ for $B \subseteq Y$.
- (6) $f^{-1}(wI(B)) \subseteq wC(wI(f^{-1}(B)))$ for $B \subseteq Y$.

Proof. (1) $\Leftrightarrow$ (2) It follows from Theorem 3.11 and definition of w -semiopen sets.

(1) $\Leftrightarrow$ (3) It follows from Theorem 3.11 and Lemma 3.2.

(3) $\Rightarrow$ (4) Let $A \subseteq X$. Then from Theorem 3.11 and Lemma 3.12, it follows $wI(wC(A)) \subseteq wsC(A) \subseteq f^{-1}(wC(f(A)))$. Hence, $f(wI(wC(A))) \subseteq wC(f(A))$.

(4) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (6) From (5) and Theorem 2.3, it follows

$$\begin{aligned}
 f^{-1}(wI(B)) &= f^{-1}(Y - wC(Y - B)) \\
 &= X - (f^{-1}(wC(Y - B))) \\
 &\subseteq X - wI(wC(f^{-1}(Y - B))) \\
 &= wC(wI(f^{-1}(B))).
 \end{aligned}$$

Hence, $f^{-1}(wI(B)) \subseteq wC(wI(f^{-1}(B)))$.

(6) $\Rightarrow$ (1) Let V be a w -open set in Y . Then by (6) and Theorem 2.3, we have $f^{-1}(V) = f^{-1}(wI(V)) \subseteq wC(wI(f^{-1}(V)))$. This implies that $f^{-1}(V)$ is a w -semiopen set. Hence, by (2) of Theorem 3.11, f is W -semicontinuous. □

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